Univerzita Karlova
Matematicko-fyzikální fakulta

Habilitační práce

New Techniques in Coloring Embedded Graphs

Zdeněk Dvořák

Obor: Informatika
Zaměření: Teoretická informatika

Preface

The fact that every map can be colored by four colors is probably the best known result of graph theory among the general public. It is one of the problems that shaped the graph theory as we know it today; the attempts to prove the Four Color Theorem inspired many notions that became important on their own. It also motivated the study of colorings of near-planar graphs, including the graphs embedded in the surfaces of bounded genus. Even though a computer-assisted proof of the Four Color Theorem was eventually found, many natural problems motivated by it remain unsolved and the study of colorings of planar graphs and of graphs on surfaces is one of the most active areas of research in modern graph theory.

In addition to general graph coloring methods, there is a number of techniques that were developed specifically to deal with embedded graphs, such as the method of reducible configurations and discharging, precoloring extension technique, and recoloring arguments made possible by the specific structure of cuts in embedded graphs. This habilitation thesis outlines some of recently developed refinements of these techniques. Their applications are demonstrated by giving several detailed examples based on my recent papers:

- Z. Dvorak, B. Lidicky, R. Skrekovski: 3-*choosability of triangle-free planar graphs with constraints on* 4-*cycles*, SIAM Journal on Discrete Mathematics **24** (2010), 934-945.

- Z. Dvorak, K. Kawarabayashi, R. Thomas: *Three-coloring triangle-free planar graphs in linear time*, ACM Transactions on Algorithms **7** (2011), article no. 41.

- Z. Dvorak, D. Kral, R. Thomas: *Coloring triangle-free graphs on surfaces*, SODA 2009, Proceedings of the twentieth Annual ACM-SIAM Symposium on Discrete Algorithms.

- Z. Dvorak, B. Lidicky, B. Mohar: *5-choosability of graphs with crossings far apart*, submitted.

- Z. Dvorak: 3-*choosability of planar graphs with* (≤ 4)-*cycles far apart*, submitted.

- Z. Dvorak, D. Kral, R. Thomas: *Three-coloring triangle-free graphs on surfaces I. Extending a coloring to a disk with one triangle*, submitted.
- Z. Dvorak, D. Kral, R. Thomas: *Three-coloring triangle-free graphs on surfaces II.* 4*-critical graphs in a disk*, in preparation.
- Z. Dvorak, D. Kral, R. Thomas: *Three-coloring triangle-free graphs on surfaces III. Graphs of girth five*, in preparation.
- Z. Dvorak, B. Lidicky, B. Mohar, L. Postle: 5*-list-coloring planar graphs with distant precolored vertices*, in preparation.

Prague, May 3, 2012

Zdeněk Dvořák

Prohlašuji, že jsem tuto práci vypracoval samostatně a že jsem použil pouze prameny uvedené v seznamu. Souhlasím se zapůjčováním této práce.

Contents

Chapter 1

Introduction

We assume that the reader is familiar with the basics of graph theory to the extent covered e.g. by Diestel [19]. Of a particular interest to us is the notion of a proper graph coloring, which is a function assigning to each vertex of an undirected graph a color in such a way that no two adjacent vertices have the same color. Let us note that a graph containing a loop has no proper coloring and that parallel edges can be suppressed without affecting the colorings; hence, throughout the thesis, we assume that the graphs are undirected and simple. Furthermore, by a "coloring" we always mean a proper coloring (improper colorings will be declared explicitly).

In 1852, Guthrie proposed to De Morgan a conjecture, which can be stated in the modern terms as follows: every planar graph can be properly colored using only four colors. In 1879, Kempe published a proof of this conjecture; the proof was however shown invalid 11 years later by Heawood. Still, this attempt contained an important idea of "Kempe chains", which became one of the most important tools in graph coloring theory. Using Kempe chains, Heawood showed that planar graphs are 5-colorable. Another contribution of Heawood is the generalization of the problem to the surfaces of higher genus, showing that every graph embedded in a surface of Euler genus $g > 0$ can be colored by at most

$$\left\lfloor \frac{7 + \sqrt{24g + 1}}{2} \right\rfloor \tag{1.1}$$

colors. This bound was shown to be tight by Ringel and Youngs [58], except for the case of Klein bottle where the correct bound is 6 by Franklin [38].

Another incorrect proof of 4-colorability of planar graphs was given by Tait in 1880. The error in this proof was pointed out by Petersen in 1891. In this attempt, Tait proved an important fact: the Four Color Theorem is equivalent to the claim that every bridgeless cubic planar graph is 3-edge-colorable. This sparked interest in non-3-edge-colorable graphs (snarks), which also turn out to be important for many other graph problems including the Cycle double cover conjecture and the 5-Flow Conjecture. Furthermore, this motivated the study of

Hamiltonicity of planar graphs, since every Hamiltonian cubic graph is 3-edge-colorable.

Further concepts and problems inspired by the attempts to prove the Four Color Theorem include the nowhere-zero flows (and in particular, the 4-Flow Conjecture), chromatic polynomial, and Hadwiger's conjecture.

The Four Color Theorem was finally proved by Appel and Haken [6, 7] in 1976, using a computer. The proof is based on the ideas of *reducible configurations* and *discharging*, which we explore in a greater detail in Section 2.1. A simpler proof was later given by Robertson et al. [59]. Since then, several variations of these proofs appeared ([42, 62]), however all of them are computer assisted and involve analysis of a large number of cases. Furthermore, there are significant difficulties in generalizing these proofs to other, more complicated situations; for example, it is unclear whether non-4-colorable graphs with one crossing can be described efficiently, and the existing proofs of the Four Color Theorem give a little guidance. For this reason, many mathematicians still are not satisfied with the solution of the Four Color Conjecture.

The Four Color Theorem also served as an inspiration for many other coloring results regarding planar and near-planar graphs. One of the best known results is the theorem of Grötzsch [43] claiming that every triangle-free planar graph is 3-colorable. Unlike the Four Color Theorem, this result has many relatively simple proofs ([67, 69, 72, 23]; see also Chapter 3) based on two distinct ideas: the method of reducible configurations, which we already mentioned, and the *precoloring extension* argument, which we explore in more detail in Section 2.2. Unsurprisingly, there are many possible strengthenings and generalizations of Grötzsch's theorem.

A natural question is whether we can allow some triangles while still preserving the 3-colorability. Grünbaum [44] gave a proof that every planar graph with at most three triangles is 3-colorable, however later an error was found in his proof. A correct proof of this statement was given by Aksionov [2]. Extending this theorem further is nontrivial, since there exist infinitely many 4-critical (a graph G is *k-critical* if all its proper subgraphs are $(k-1)$-colorable, but G itself is not $(k-1)$-colorable) planar graphs with four triangles. In his alternative proof of Aksionov's result, Borodin [11] claims to have described all such graphs, forming 15 infinite families, however this characterization has not not been published yet.

However, all such known examples contain a pair of triangles that are close to each other. This motivated Havel to ask whether every planar graphs with triangles sufficiently far apart is 3-colorable. In Dvořák et al. [24], we answered this question in affirmative. In addition to deep structural results on 4-critical plane graphs, we use another common technique of *recoloring*, described in details in Section 2.3. However, the bound on the distance between the triangles that we give is much larger than the known lower bound of 4 given by Aksionov and Mel'nikov [3].

Another approach is to allow unrestricted triangles, but instead forbid cycles of other lengths. It is easy to see that it is not sufficient to forbid only 4-cycles or only 5-cycles. However, Steinberg [61] conjectured that every planar graph without both 4- and 5-cycles is 3-colorable, and this question is still open. A number of improvements were made with regards to the following relaxed version: What is the smallest k such that every planar graph without cycles of length between 4 and k is 3-colorable? The current best result of Borodin et al. [13] shows that $k \leq 7$. Furthermore, many authors consider other combinations of forbidden cycles [12, 15, 16, 17]; a more complete list can be found in the on-line survey of Montassier [56].

Most proofs in graph colorings employ the following idea in some form: suppose that we already colored part of the graph and we would like to extend the coloring to the rest. This means that in the rest of the graph, only some colors (those not used on already colored neighbors) are available. This motivates the notion of *list coloring* [35]: a *list assignment* L is a function that assigns a set of colors (a *list*) to each vertex of a graph G. An *L-coloring* of G is a proper coloring such that for each $v \in V(G)$, the color of v belongs to $L(v)$. We say that G is *k-list-colorable* if it has an L-coloring for every assignment L of lists of size at least k. Does the list coloring version of the Four Color Theorem hold, i.e., is it true that every planar graph G is 4-list-colorable? This was shown to be false by Voigt [75]. Similarly, there exists a triangle-free planar graph not colorable from all lists of size three [76]. On the other hand, Thomassen [65, 69] proved that every planar graph is 5-list-colorable and every planar graph of girth five is 3-list-colorable. There are also many other sufficient conditions implying 3- or 4-list-colorability of subclasses of planar graphs; again, we refer the reader to the survey of Montassier [56].

Finally, let us consider graphs that are non-planar, but close to being planar in some sense. One possibility is to draw graphs in plane with a limited number of crossings (the *crossing number* of a graph). There are infinitely many 5-critical graphs with crossing number one, and characterizing those graphs is a challenging open problem. On the other hand, it is easy to see that all graphs with crossing number at most two are 5-colorable. Furthermore, any graph with crossing number at most four that does not contain K_6 as a subgraph is 5-colorable (Erman et al. [36]). More generally, Albertson conjectured that if a graph has chromatic number at least n, then its crossing number is greater or equal to the crossing number of K_n; Barát and Tóth [10] proved that this holds for $n \leq 16$. Analogically to Havel's problem, one can also consider the situation that the crossings are far apart. Král' and Stacho [50] proved that if no two crossings are incident with the same vertex, then the graph is 5-colorable. For list colorings, Dvořák et al. [33] and independently Campos and Havet [14] proved that every graph with crossing number at most two is 5-list-colorable. Furthermore, the same holds if the distance between every two crossings is at least 15 (Dvořák et al. [30]; see also Chapter 9).

Historically older and more developed approach to near-planar graphs is through embeddings in surfaces. Clearly, a graph is planar if and only if it has an embedding in sphere; and asking about the chromatic number of graphs in other surfaces is natural. As we mentioned before, a tight bound (1.1) on the chromatic number was given by Heawood. However, this bound turns out to be insufficiently detailed—only a small fraction of graphs achieve or approach this bound. Indeed, observe that every sufficiently large graph embedded in a fixed surface Σ contains a vertex of degree at most 6; hence, the size of every k-critical graph embedded in Σ is bounded by a function of the genus of Σ, for each $k \geq 8$. In particular, there are only finitely many such graphs. Similarly, it is easy to see that there are only finitely many 7-critical graphs embedded in Σ. Using a much more involved argument, Thomassen [70] proved that the number of 6-critical graphs in any fixed surface is also finite. His proof gives a bound on the size of such graphs which is double exponential in the genus of the surface. This bound was improved to linear by Postle and Thomas [57]. Algorithmically, these results give a linear-time algorithm to test whether a graph G embedded in Σ is 5-colorable—for each 6-critical graph embeddable in Σ, test whether it is a subgraph of G. This test can be carried out in linear time using the algorithm of Eppstein [34]. The lists of 6-critical graphs are explicitly known for the projective plane [4], the torus [66] and the Klein bottle [18, 45].

Note that 3-critical graphs are just odd cycles. Furthermore, 3-colorability is NP-complete even for planar graphs [40], hence it is likely not possible to characterize embedded 4-critical graphs. The remaining open case is that of 5-critical graphs. By the Four Color Theorem, there are no planar 5-critical graphs. On the other hand, Fisk [37] gave examples of infinite families of 5-critical graphs for every surface other than the sphere. Both characterization of such graphs and the corresponding algorithmic question of 4-colorability of graphs of bounded genus are open (and likely rather difficult) problems.

Let us now consider graphs of girth at least five. It is easy to see that there is only a finite number of such k-critical graphs embeddable in any fixed surface, for $k \geq 5$. Thomassen [71] proved that this claim holds even for $k = 4$, and in Dvořák et al. [27], we show that every 4-critical graph of girth at least five embedded in a surface of genus g has at most $O(g)$ vertices (see Chapter 7). Let us remark that there are no 4-critical graphs of girth at least five and genus at most two by Thomassen [68] and Thomas and Walls [63].

The situation is more complicated for triangle-free graphs. It is still fairly easy to argue that the number of k-critical triangle-free graphs embedded in a fixed surface is finite for $k \geq 5$. There are infinitely many 4-critical triangle-free graphs embedded in any surface other than the sphere. However, Gimbel and Thomassen [41] proved that a projective planar triangle-free graph is 4-critical if and only if it is a non-bipartite quadrangulation without separating 4-cycles. This gave a hope that the 4-critical triangle-free graphs have some structure that is easy to describe. In Dvořák et al. [28], we gave such a description (subject to some further constraints) and used it to give a linear-time algorithm to decide 3-colorability of triangle-free graphs embedded in a fixed surface. Furthermore, we proved that for every orientable surface Σ, there exists a constant d such that every triangle-free graph embedded in Σ with edge-width at least d is 3-colorable.

Essentially all of the results mentioned in this introduction were proved using a few basic techniques—reducible configurations, discharging, precoloring extension and recoloring. We give their more detailed description in the following chapter. The main focus of this thesis is the new refinements of these techniques that we developed. The first of them is the *weight technique*. It deals with the situation where we need to establish a bound on the size of a critical graph. Using various reductions, we can relate any critical graph to critical graphs with smaller numbers of vertices. However, this relationship does not make it possible to prove a bound on the size by induction, since the reduction could decrease the size of the graph below the considered bound. Instead, we assign weights to the vertices and faces of the considered graph and exhibit reductions that do not decrease the sum of weights, leading to a natural inductive argument. Part II is devoted to describing the theory of this technique in details, as well as giving some heuristic ideas on the choice of appropriate weight function and other concerns in practical applications.

The second technique is a refinement to the precoloring extension arguments and we study it in more details in Part III. We consider the situation that some claim holds for graphs in some prescribed class (e.g., that all planar graphs of girth five are 3-list-colorable). We would like to show that this claim is still true even if we allow distant perturbations of the graphs in this class (e.g., that all planar graphs such that the distance between every two (≤ 4)-cycles is at least 26 are 3-list-colorable). The key observation here is that the proof of the original statement still works, as long as the perturbations are far enough from the precolored path whose neighborhood we reduce. Once one of the perturbations appears close to the precolored path, we know that all other perturbations must be far from it, and typically do not affect the colorability. Therefore, it suffices to focus on the situation that there is exactly one perturbation (formally, we of course need to be more careful). Another important idea is to use the symmetry of the precolored path (trying to apply the reductions of the original proof on the other side of the precolored path) to obtain another short path from the perturbation to the boundary of the outer face. The two paths from the perturbation to the boundary then enable us to eliminate the perturbation similarly to the way the standard precoloring extension method deals with chords.

Chapter 2

Techniques

Let us now give an overview of basic techniques and tools used for coloring embedded graphs:

- The method of *reducible configurations and discharging* was first formalized in the context of attempts at proof of the Four Color Theorem. Nowadays, it is the most common method used to prove existence of various kinds of colorings for embedded graphs. It consists of showing that some set of configurations is both *reducible* (i.e., cannot appear in a smallest counterexample to the claim in question) and *unavoidable*, i.e., must appear in every graph from the considered class. The existence of such a set clearly implies that there exists no smallest counterexample, and consequently the claim is true. The unavoidability part of the argument is usually carried out using the discharging method, which is essentially a double-counting argument using the properties of planar (or surface) embedding. For more details, see Section 2.1.

- The *precoloring extension* technique was developed by Thomassen in his papers [65, 69] dealing with list-coloring of planar graphs, and it is the most powerful known tool in this context. The idea is to color small pieces of the graph carefully so that yet uncolored vertices have enough available colors left. This involves proving a stronger claim specifying exactly what "enough" means, and the choice of this stronger claim is a rather delicate part of the argument. For details and a deeper discussion of these issues, see Section 2.2.

- The *recoloring* method is used to deal with "cylinder-like" graphs, that is graphs where interesting parts are separated by many short cycles. In such case, it is often possible to show that the cylinder-like part is in fact irrelevant, in the sense that every precoloring of its boundary can be extended to a coloring of the whole cylindrical section, by carefully altering the coloring between consecutive short cycles. More details can be found in Section 2.3.

Finally, in Section 2.4 we discuss critical graphs. While the notion of criticality is not a proof technique per se, it makes it possible to treat properties of hypothetical minimal counterexamples to various coloring claims in a uniform way. Consequently, it is a convenient idea to use in many of the proofs. After discussing some basic properties of critical graphs, we explore the generalization of criticality to the situation where some vertices are precolored. This turns out to be useful especially in Part II.

2.1 Reducible configurations and discharging

Consider a k-critical graph G. It is easy to see that G does not contain a vertex v of degree at most $k-2$: Otherwise, by the k-criticality of G, there exists a coloring of $G-v$ by $k-1$ colors, and we can choose one of these $k-1$ colors for v, distinct from the colors of its neighbors. In this way, we would obtain a coloring of G by $k-1$ colors, contrary to the assumption that G is k-critical.

A generalization of this observation leads to the method of reducible configurations: suppose that we want to prove that every graph in some class $\mathcal{G}$ can be colored in some prescribed way. We exhibit a set $\mathcal{C}$ of *configurations* (typically, subgraphs with prescribed degrees of vertices, although other more complicated definitions are also common) and a partial well-ordering $\prec$ of $\mathcal{G}$ with the following properties:

- **reducibility:** if all graphs $G' \in \mathcal{G}$ with $G' \prec G$ are colorable and G contains a configuration belonging to $\mathcal{C}$, then G is colorable.
- **unavoidability:** every graph in $\mathcal{G}$ contains one of the configurations in $\mathcal{C}$.

Of course, if there existed a graph in $\mathcal{G}$ that would not be colorable, we could choose such a graph G that is minimal with respect to $\prec$. By unavoidability, G contains a configuration of $\mathcal{C}$, and by reducibility, G is colorable. This is a contradiction showing that every graph in $\mathcal{G}$ is colorable.

For the reducibility condition, the usual argument consists of removing the configuration or its part, thus obtaining a graph $G' \prec G$. This graph G' is colorable by the assumptions, and we then extend the coloring to the removed part, thus obtaining a coloring of G. Commonly, variations of this simple scheme are required; for example, instead of removing the configuration, we replace it by some smaller configuration (a *reducent*). This restricts the coloring of the resulting graph, which can be used to exclude the colorings that do not extend to G. Of course, introduction of a nontrivial reducent makes it harder to argue that the resulting graph still belongs to the class $\mathcal{G}$, and arguments regarding critical subgraphs as outlined in Section 2.4 are often an important tool to deal with this difficulty.

Another common idea is to alter the coloring of G' somehow before trying to extend it to a coloring of G. This may involve local recolorings of the vertices near the reduced configuration, or global changes using e.g. Kempe chains. A rather involved variant of this idea is used in the proof of the Four Color Theorem.

For the unavoidability part of the argument, one typically uses the *discharging technique*. Using Euler's formula, it is easy to see that every planar graph contains a vertex of degree at most 5. The discharging technique is a generalization of this observation, obtained using a double-counting argument over specific subgraphs of a (usually embedded) graph. If G has a 2-cell embedding in a surface of Euler genus g, then the number of its edges can be expressed as $m = n+s-2+g$, where n is the number of vertices of G and s is its number of faces. Let $0 < \alpha < 2\beta$ be arbitrary constants, and let us assign charge $c(v) = \alpha \deg(v) - 2\beta$ to each vertex v and $c(f) = (\beta - \alpha)\ell(f) - 2\beta$ to each face (where $\ell(f)$ denotes the length of the face). Then, the sum of the charges is

$$\begin{aligned}\sum_{v\in V(G)} c(v) + \sum_{f\in F(G)} c(f) &= -2\beta(n+s) + \alpha \sum_{v\in V(G)} \deg(v) + (\beta-\alpha) \sum_{f\in F(G)} \ell(f)\\ &= 2\beta(g-2-m) + 2\alpha m + 2(\beta-\alpha)m\\ &= 2\beta(g-2).\end{aligned}$$

Next, we redistribute the charge according to a set of rules (typically locally, e.g., moving it from vertices to incident faces or adjacent vertices) so that the sum of the charges is unchanged. Finally, assuming that no configuration of $\mathcal{C}$ appears in G, we argue that the resulting charge of all vertices and faces is nonnegative. If $\mathcal{G}$ consists of planar or projective planar graphs, then $g \le 1$ and this directly gives a contradiction, since the sum of the original charges is negative. In the case of surfaces of greater genus, we need further arguments showing that the sum of the charges exceeds the bound of $2\beta(g-2)$ in order to obtain the contradiction and show that some of the configurations of $\mathcal{C}$ appear in G.

Let us demonstrate some of these ideas by a simple example (more involved applications can be found in Chapters 3 and 7). By the Four Color Theorem, every planar graph is 4-colorable. Can we prescribe coloring of some vertices? E.g., suppose that we precolor an induced 4-cycle—can this precoloring always be extended to a 4-coloring of the whole graph? The answer to this question is negative, and even the algorithmic problem of deciding whether this extension exists is open. Here, we consider a special case of this question.

Theorem 1. *Let G be a plane graph with outer face F of length at most 5 bounded by an induced cycle. If no two triangles in G share a vertex, then every proper precoloring φ of F by at most four colors extends to a proper 4-coloring of G.*

Proof. For a contradiction, suppose that G is a counterexample to the claim with the smallest number of vertices. Clearly, every vertex in $V(G) \setminus V(F)$ has degree at least four. Furthermore, we can assume that F has length at least four, as if F is a triangle, we can color G by the Four Color Theorem and permute the colors to match φ.

Note that G is 2-connected: otherwise, we have $G = G_1 \cup G_2$, where F belongs to G_1, G_1 and G_2 intersect in at most one vertex, and both of them have fewer vertices than G. By the minimality of G, the precoloring φ of F extends to G_1. Furthermore, G_2 has a 4-coloring by the Four Color Theorem, and by permuting the colors in the coloring of G_2, we can assume that these colorings match on $V(G_1) \cap V(G_2)$. This would give a 4-coloring of G extending φ.

Consider now a cycle $C \neq F$ in G of length at most 5, and suppose that no chord of C is drawn inside it. We claim that C bounds a face. Indeed, otherwise we have $G = G_1 \cup G_2$, where F belongs to G_1, $G_1 \cap G_2 = C$ and C is the outer face of G_2, and both G_1 and G_2 have fewer vertices than G. By the minimality of G, we first extend φ to a 4-coloring of G_1. This gives a proper 4-coloring of C, which we then extend to G_2. This results in a proper 4-coloring of G that extends φ, which is a contradiction. Consequently, no (≤ 5)-cycle is separating. It follows that no vertex $v \in V(G) \setminus V(F)$ has two non-adjacent neighbors in F; otherwise, we would have $V(G) = V(F) \cup \{v\}$ and since no two triangles intersect, v would have degree at most three.

Consider now a triangle $xyz \subset G$ such that $x, y, z \notin V(F)$ and $\deg(x) = \deg(y) = \deg(z) = 4$. Suppose furthermore that the edge xy is incident with a 4-face $xyuv$, and let w be the neighbor of x distinct from y, z and v. Observe that the distance between u and w is at least three: otherwise, there exists a cycle C of length at most 5 containing path $uyxw$. Since xyz shares no vertex with another triangle, we have $vw \notin V(G)$; hence, $v \notin V(C)$, and similarly $z \notin V(C)$. But then C separates v from z, which is a contradiction. Since the distance between u and w is at least three, at most one of u and w belongs to F. Let G' be the graph obtained from $G - \{x, y, z\}$ by adding the edge uw. Observe that F is the outer face of G' and it is bounded by an induced cycle, and that no triangle contains the edge uw. Consequently, G' satisfies the assumptions of the theorem, and by the minimality of G, there exists a proper 4-coloring ψ of G' extending φ. Let t be the neighbor of y distinct from x, z and u. Note that since uv and uw are edges of G', we have either $\psi(v) = \psi(w)$ or $\{\psi(v), \psi(w)\} \neq \{\psi(u), \psi(t)\}$. In both of the cases, ψ can be extended to a 4-coloring of G, which is a contradiction. It follows that all faces sharing edges with xyz have length at least five.

Let us now proceed with the discharging part of the proof. We assign to each vertex v charge $\deg(v) - 4$ and to each face f charge $\ell(f) - 4$; the sum of these charges is -8. We then redistribute the charge according to the following rules:

R1 Each vertex of degree at least 5 incident with a triangle T sends 1 to the face bounded by T.

R2 Each face of length at least 5 distinct from F sends 1/3 to each triangle with that it shares an edge whose endvertices do not belong to F.

R3 Each face of length at least 5 distinct from F sends 2/3 to each vertex of degree two (belonging to F) with that it is incident.

First, let us analyze the final charge of vertices. All vertices not in $V(F)$ have degree at least four, thus their initial charge is nonnegative. Vertices only send charge by rule R1, which only applies when the initial charge of a vertex is at least one. We conclude that the final charge of each vertex in $V(G) \setminus V(F)$ is nonnegative.

Consider a vertex $v \in V(F)$ of degree two, and let f be the face incident with v distinct from F. Since F is an induced cycle and no vertex of $V(G) \setminus V(F)$ has two non-adjacent neighbors in F, it follows that f has length at least five, and thus v receives $2/3$ by R3. Therefore, the final charge of v is $-4/3$. We conclude that the sum of the final charges of the vertices is at least $-4n_2/3 - n_3$, where n_2 and n_3 are the numbers of vertices of G of degree 2 and 3, respectively.

Let us now consider a face $f \neq F$. If $\ell(f) = 4$, then both the initial and the final charge of f is 0. Suppose that $\ell(f) \geq 5$, let t be the number of triangles to that f sends charge by R2 and d the number of incident vertices of degree two. If $d = 0$, then note that $2t \leq \ell(f)$, since no two triangles share a vertex. Therefore, the final charge of f is $(\ell(f)-4)-t/2 \geq \ell(f) - \lfloor \ell(f)/2 \rfloor/3 - 4 > 0$. If $d > 0$, then $d \leq \ell(f)-4$, since F is an induced cycle and if $|V(f) \setminus V(F)| = 1$, then the vertex of $V(f) \setminus V(F)$ would have degree two. Furthermore, we have $d+2+2t \leq \ell(f)$. Therefore, the final charge of f is $(\ell(f)-4)-2d/3-t/3 \geq (\ell(f)-4)/3-1/3 \geq 0$.

Finally, suppose that f is a triangle. Its initial charge is -1. If f shares no vertex with F, then either it is incident with a vertex of degree at least five, or all incident faces have length at least five, and thus f receives total charge at least 1 by R1 and R2. We conclude that the sum of the final charges of the faces is at least $\ell(F) - 4 - k$, where k is the number of triangles that share a vertex with F.

Therefore, the sum of the final charges of G is at least $-4n_2/3 - n_3 + \ell(F) - 4 - k$. On the other hand, the sum of final charges is equal to -8, the sum of initial charges. We obtain $4n_2/3 + n_3 + k \geq \ell(F) + 4$. Let k_1 be the number of triangles that share one vertex with F and k_2 the number of triangles sharing an edge with F. We have $k = k_1 + k_2$. Furthermore, if a triangle shares only one vertex with F, then this vertex has degree at least four. Consequently, we have $n_2 + n_3 + k_1 \leq \ell(F)$. Combining these inequalities, we have $n_2/3 + k_2 \geq 4$; however, $n_2 \leq 5$ and $k_2 \leq 2$, which is a contradiction. □

At first, determining the components of the proof—the choice of initial charge, reducible configurations and discharging rules—may seem a somewhat daunting task. Nevertheless, with a little experience this becomes rather routine. In principle, the choice of the initial charge does not matter, as one charge can be changed into another by a simple discharging rule (although choosing the initial charge properly may simplify the arguments and the presentation of the proof). The common choices are

- $c(v) = \deg(v)-6$ for vertices (and the corresponding charge $c(f) = 2\ell(f)-6$ for faces) in the problems for triangulations or graphs with many triangles;

the choice ensures that triangles have zero charge, making them easier to deal with.

- $c(v) = \deg(v) - 4$ for vertices and $c(f) = \ell(f) - 4$ for faces in problems for triangle-free graphs (for similar reasons).
- $c(v) = \deg(v) - d$ for vertices (and the corresponding charge for faces) in problems where we know that the minimal degree of a vertex is at least d.

Once the initial charge is chosen, we should ideally have only a few kinds of objects with negative charge (triangles in the example Theorem 1), and we need to find a charge to move to them from somewhere (nearby vertices of big degree and large faces). Usually, we can come up with examples of parts of the graphs where not enough charge is available in the neighborhood of an object with negative charge (in our case, triangle with all vertices of degree four and incident with a 4-face). In such a part, we need to find some reducible configuration, excluding its existence. Several rounds of this process (attempt to discharge and find a reducible configuration if that is not possible) often lead to a proof. Even the proofs of the Four Color Theorem were found by a similar procedure, although computers were used both to test the reducibility of the configurations and to check where a negative charge appears after application of proposed discharging rules.

2.2 Precoloring extension

The idea of the precoloring extension technique is as follows: suppose that we want to color a planar graph. We will maintain lists of allowed colors for each vertex. We pick a few vertices incident with the outer face, color them and remove their colors from the list of their neighbors. We repeat this operation until the whole graph is colored.

Of course, in order for this to work, we need to ensure that the lists of the vertices incident with the outer face do not shrink too much, so that the coloring still exists. To do so, we need to pick the vertices to be colored in such a way that not too many of them have a common neighbor, and select their coloring carefully to avoid decreasing the lists of the vertices incident with the outer face below a certain bound.

Before we discuss the technique in more detail, let us give an example. We prove that every planar graph is 5-list-colorable. The presented proof is a slight variation of the well known proof of this fact by Thomassen [65]. The main difference in our proof is the choice of the vertices to color—Thomassen's proof uses a different idea of only picking one vertex, but instead of coloring it directly, he "reserves" two possible colors for the vertex by removing them from the lists of its neighbors that are not incident with the outer face, and only chooses one of the two colors when the rest of the graph is colored depending on the colors chosen for its neighbors incident with the outer face.

Theorem 2. *Let G be a plane graph with outer face F and let $P \subseteq F$ be a path (possibly empty) of length at most one. Let L be a list assignment such that $|L(v)| \geq 5$ for $v \in V(G) \setminus V(F)$, $|L(v)| \geq 3$ for $v \in V(F) \setminus V(P)$ and $|L(v)| \geq 1$ for $v \in V(P)$. If P is L-colorable, then G is L-colorable.*

Proof. By removing the extra colors, we can assume that $|L(v)| = 5$ for $v \in V(G) \setminus V(F)$, $|L(v)| = 3$ for $v \in V(F) \setminus V(P)$ and $|L(v)| = 1$ for $v \in V(P)$, and that P has length one. Assume for a contradiction that G is a counterexample with the smallest number of vertices. Clearly, G is connected. Furthermore, G is 2-connected: otherwise, we have $G = G_1 \cup G_2$, where G_1 and G_2 intersect in a single vertex v and both G_1 and G_2 have fewer vertices than G. We can assume that $P \subseteq G_1$. By the minimality of G, there exists an L-coloring φ of G_1. Furthermore, there also exists a coloring of G_2 from the lists obtained from L by changing the list of v to $\{\varphi(v)\}$. This gives an L-coloring of G, which is a contradiction. A similar argument shows that F is bounded by an induced cycle (otherwise, we split G on the chord of F, color the part containing P and extend the coloring to the other part).

Let p be a vertex of P and v_1 the vertex of $V(F) \setminus V(P)$ adjacent to p, and choose a subset S of $L(v_1) \setminus L(p)$ of size two. Let $pv_1v_2 \ldots v_nq$ be the subpath of F such that $S \subseteq L(v_i)$ for $1 \leq i \leq n$ and $S \not\subseteq L(q)$ (such a path exists, since S is not a subset of the list of any vertex of P). Let ψ be a coloring of the path $v_1 \ldots v_n$ by colors in S (alternating on this path) such that $\psi(v_n) \notin L(q)$. Let $G' = G - \{v_1, \ldots, v_n\}$ and let L' be the list assignment such that $L'(v) = L(v) \setminus S$ for every $v \in V(G) \setminus V(F)$ which has a neighbor in $\{v_1, \ldots, v_n\}$ and $L'(v) = L(v)$ for every other vertex v. By the minimality of G, the graph G' is L'-colorable, and combined with ψ, this gives an L-coloring of G. This is a contradiction. □

The key part of a proof by precoloring extension is the choice of the constraints on the lists incident with the outer face. Obviously, every graph satisfying these constraints (at least without a precolored path) should be colorable—in Theorem 2, we cannot allow the vertices of F to have lists of size two, since there are many examples of graphs not colorable from such lists (in fact, the assumptions of Theorem 2 are rather tight and almost any attempt to strengthen them fails). On the other hand, the conditions need to be strong enough to be maintainable—while Theorem 2 would certainly be true if we required the vertices incident with the outer face to have lists of size four, it does not seem possible to devise reductions preserving this assumption. Finding the right conditions usually needs a lot of experimentation.

A related concern is the choice of the length of the precolored path P, which is needed to eliminate short paths between vertices incident with the outer face. In Theorem 2, we only needed to eliminate chords, and thus a path of length one sufficed. More involved situations may require considering longer precolored paths. For example, Thomassen's proof [69] of 3-list-colorability of planar graphs of girth 5 assumes that all vertices incident with the outer face have list of size at least two and that the vertices with list of size two form an independent set. Furthermore, one of the reductions in the proof colors not only vertices incident with the outer face, but also some vertices in their neighborhood. Thus, to ensure the validity of the list assignment after this reduction, one needs to exclude the existence of a path $v_1v_2v_3v_4$, where v_1 and v_4 are incident with the outer face, v_1 and v_2 are being colored and v_4 has list of size two. Therefore, one needs to be able to handle such a path of length three.

A problem that may appear with longer precolored paths is that the colorability claim may no longer hold without further assumptions. Sometimes, these assumptions can be expressed as a finite list of forbidden subgraphs with prescribed lists; for example in Chapter 10, we need to deal with a precolored path of length five, and this requires us to forbid 11 graphs depicted in Figure 10.2.

Let us also remark that the precoloring extension technique naturally gives proofs for list coloring statements, as the lists are in any case needed to record the effect of the already colored part of the graph. In particular, all known proofs of the 5-list-colorability of planar graphs and 3-list-colorability of planar graphs of girth 5 use this technique, and it is an open problem whether these claims can be proved say by the method of reducible configurations (which would be preferable for some generalizations). On the other hand, this seems to be a limitation of the technique in the cases where the graphs from the considered class are k-colorable, but not necessarily k-list-colorable. While it still may be possible to get a precoloring extension proof of such a statement, the reductions must somehow use the fact that all the lists are subsets of a fixed set of k colors.

For examples of more involved applications of the precoloring extension technique, see Chapters 8, 9 and 10.

2.3 Recoloring

The basic difficulty in generalizing claims from sphere to the other surfaces is the presence of short non-contractible cycles. For example, if we tried to prove Theorem 1 in the projective plane, we would run into problems at the point where we attempt to reduce the triangle xyz by adding the edge uw—together with a path of length two between u and w, this could result in a non-contractible triangle, which would not be excluded by the preceding argument.

One possible way to deal with this problem is as follows. First, we show that the hypothetical minimal counterexample G actually has to contain many disjoint short non-contractible cycles (e.g., by a modification of a discharging argument showing existence of many configurations that are reducible unless they are incident with a non-contractible cycle). We then argue that many of these cycles have the same homotopy. Consequently, there exists a part of the surface homeomorphic to a cylinder containing a subgraph G with disjoint short cycles C_1, C_2, …, C_m (for m large enough) going around the cylinder. We can assume that for any $i < j < k$, C_j separates C_i from C_k; let G_i denote the subgraph of G drawn between C_i and C_{i+1}, and let G' denote the subgraph of G between C_1 and C_m. The graphs G_i are planar, and we show that G_i has several colorings that differ on $V(C_i) \cup V(C_{i+1})$ (for this, we can use the other discussed methods, or possibly find a coloring of G_i by fewer colors and use the additional available colors to obtain further colorings). We then combine the available colorings and prove that every precoloring of $C_1 \cup C_m$ extends to a coloring of G'. This gives a contradiction, since $G - [V(G') \setminus (V(C_1) \cup V(C_m))]$ is colorable by the minimality of G, and the extension would give a coloring of G.

Let us give an example, which can be useful as a part of a proof that the number of 7-critical graphs embedded in a fixed surface is finite (although, let us note that there exists a much easier way of proving this using results of Gallai [39] on the structure of critical graphs).

Lemma 3. *Let G be a plane graph and A and B two faces of G bounded by triangles. Let φ be a coloring of $A \cup B$ by 6 colors. Suppose that φ does not extend to a 6-coloring of G, but it does extend to a 6-coloring of every proper subgraph of G that includes $A \cup B$. Then G contains at most 18 triangles separating A from B, including the cycles A and B themself.*

Proof. Note that every non-facial triangle in G separates A from B and that every vertex not in $A \cup B$ has degree at least 6. Otherwise, let $G = G_1 \cup G_2$, where $A \cup B \subseteq G_1$ and G_2 consists of the part of G drawn inside the triangle in the former case and of the vertex of degree at most five and its neighborhood in the latter case. By the assumptions of the lemma, φ extends to a 6-coloring of G_1, and we can further extend it to G_2. This gives a 6-coloring of G extending φ, which is a contradiction.

Let C_1, …, C_N be the triangles in G separating A from B ordered in such a way that C_i separates A from C_{i+1} for $1 \le i \le N-1$; note that $C_1 = A$ and $C_N = B$. We claim that

(1) *C_i is vertex-disjoint with C_{i+3} for $1 \le i \le N-3$.*

Proof. Suppose on the contrary that C_i intersects with C_{i+3} for some i; then $C_i \cup C_{i+3}$ contains a closed walk of length 4 or 6 bounding an open disk that is disjoint with A and B, but contains at least one vertex of G. Furthermore, if the length of the closed walk is 6, then it passes through one vertex twice. Let us choose a closed walk W (not necessarily contained in $C_i \cup C_{i+3}$) with these properties so that the open disk Δ bounded by W is as small as possible. If no edge joining vertices of W is drawn inside Δ, then let $Q = W$ and $\Lambda = \Delta$. Otherwise, let Q be a separating induced cycle with $V(Q) \subseteq V(W)$ such that the open disk Λ bounded by Q is disjoint with A and B. Let d be the number of vertices of Q, and note that $4 \leq d \leq 5$ (all non-facial triangles separate A from B, and if W has length 6, then it passes through one vertex twice). Let G' be subgraph of G drawn in the closure of Λ, let n be the number of its vertices and m the number of its edges. Since all vertices of G not in $A \cup B$ have degree at least 6, we have $2m = \sum_{v \in V(G')} \deg_{G'}(v) \geq 6(n-d) + \sum_{v \in V(Q)} \deg_{G'}(v)$, and thus $m \geq 3n - 3d + \frac{1}{2} \sum_{v \in V(Q)} \deg_{G'}(v)$. On the other hand, the number of edges of G' is $3n - 6 - \sum_{f \in F(G')}(\ell(f) - 3)$ by Euler's formula. Therefore,

$$6d - 12 \geq 2 \sum_{f \in F(G')} (\ell(f) - 3) + \sum_{v \in V(Q)} \deg_{G'}(v).$$

If Q is a cycle, then it bounds a face of length d, hence

$$4d - 6 \geq 2 \sum_{f \in F(G'), f \neq Q} (\ell(f) - 3) + \sum_{v \in V(Q)} \deg_{G'}(v).$$

Since Q is an induced cycle, observe that the number of vertices of degree two in G' is bounded by $\sum_{f \in F(G'), f \neq Q}(\ell(f) - 3)$. Therefore, $4d - 6 \geq 3d$, which is a contradiction. Let us now consider the case that $Q = W = vv_1v_2vw_1w_2$ is a walk of length 6 passing twice through a vertex v. Suppose that v_1, v_2 and v have a common neighbor u. By the minimality of Δ and the absence of separating triangles, we conclude that u is the only vertex drawn inside Δ; but then u has degree at most 5, which is a contradiction. Therefore, v, v_1 and v_2 have no common neighbor. It follows that v_1 either has degree at least four or it is incident with a face of length at least four. By symmetry, the same holds for v_2, w_1 and w_2. Note that v has degree at least four and that a 4-face cannot be incident with three of v_1, v_2, w_1 and w_2, since no edge between vertices of Q is drawn inside Λ. We conclude that $2\sum_{f \in F(G')}(\ell(f) - 3) + \sum_{v \in V(Q)} \deg_{G'}(v) \geq 20$, which gives a contradiction. □

Suppose now that that $N \geq 19$. Then by (1), there exist pairwise vertex-disjoint cycles $K_1, \ldots, K_7$ in G separating A from B, where $K_1 = A$, $K_7 = B$ and K_i separates A from K_{i+1} for $1 \leq i \leq 6$. Let G_i denote the subgraph of G drawn between K_i and K_{i+1}. By the Four Color Theorem, G_i is 4-colorable; let us fix such a 4-coloring ψ_i. Let vertices of K_i be denoted by v_1^i, v_2^i and v_3^i, where the labels are chosen so that for every $i \in \{1, \ldots, 6\}$, there exist at least two values $j \in \{1, 2, 3\}$ satisfying $\psi_i(v_j^i) = \psi_i(v_j^{i+1})$. We say that G_i is copying if the equality holds for all three values of j, and that it is j-changing if $\psi_i(v_j^i) \neq \psi_i(v_j^{i+1})$ for some $j \in \{1, 2, 3\}$.

Let c_1, c_2 and c_3 be distinct colors and let α be a coloring of K_2 using at least one of these colors. We claim that there exists a 6-coloring β of $G_2 \cup G_3$ extending α such that $\{\beta(v_1^4), \beta(v_2^4), \beta(v_3^4)\} = \{c_1, c_2, c_3\}$. The coloring β is constructed as follows: let β_1 be a coloring obtained from ψ_2 by permuting the colors so that β_1

matches α on K_2. We can assume that $\alpha(v_1^2) = 1$, $\alpha(v_2^2) = 2$ and $\alpha(v_3^2) = 3$, and that β_1 does not use colors 5 and 6.

Suppose first that G_3 is copying. If $\{\beta_1(v_1^3), \beta_1(v_2^3), \beta_1(v_3^3)\} \cap \{c_1, c_2, c_3\} = \emptyset$, then let β_1' be obtained from β_1 by changing the color of v_1^3 to 5, otherwise let $\beta_1' = \beta_1$. Let $S = \{\beta_1'(v_1^3), \beta_1'(v_2^3), \beta_1'(v_3^3)\}$ and note that we can assume that $c_1 \in S$. Let β_2 be the coloring obtained from ψ_3 by permuting its colors so that β_2 matches β_1' on K_2, such that β_2 does not use any color in $\{c_2, c_3\} \setminus S$. Let β_2' be obtained from β_2 by recoloring at most two of vertices of K_4 by colors c_2 and c_3 if necessary, so that $\{\beta_2'(v_1^4), \beta_2'(v_2^4), \beta_2'(v_3^4)\} = \{c_1, c_2, c_3\}$. We let β be the combination of β_1' and β_2'.

If G_3 is not copying, then by symmetry, we can assume that it is 3-changing. Since $\{c_1, c_2, c_3\} \neq \{4, 5, 6\}$, we can obtain a coloring β_1' of G_2 such that $\beta_1'(v_3^3) \notin \{c_1, c_2, c_3\}$ from β_1 by recoloring v_3^3 if needed. Then, we choose β_2 by permuting the colors of ψ_3 so that it matches β_1' on K_3 and so that $\beta_2(v_3^4) \in \{c_1, c_2, c_3\}$. Finally, we obtain β_2' by recoloring v_1^4 and v_2^4 if necessary so that $\{\beta_2'(v_1^4), \beta_2'(v_2^4), \beta_2'(v_3^4)\} = \{c_1, c_2, c_3\}$, and let β be the combination of β_1' and β_2'.

Let $\{c_1, c_2, c_3\} = \{1, \ldots, 6\} \setminus \{\varphi(v_1^7), \varphi(v_2^7), \varphi(v_3^7)\}$. Observe that we can permute the colors of ψ_1 and possibly recolor one vertex of K_2 so that the resulting coloring α of G_1 matches φ on K_1 and uses at least one of the colors c_1, c_2 and c_3 on K_2. By the preceding claim, we can then extend α to a 6-coloring ψ of $G_1 \cup G_2 \cup G_3$ that uses exactly the colors c_1, c_2 and c_3 on K_4. Symmetrically, there exists a 6-coloring ψ' of $G_4 \cup G_5 \cup G_6$ matching φ on K_7 such that ψ' uses exactly the colors c_1, c_2 and c_3 on K_4. Furthermore, by permuting the colors c_1, c_2 and c_3 in ψ' (which do not appear on K_7), we can ensure that ψ and ψ' match on K_4. Therefore, the combination of ψ and ψ' is a 6-coloring of G extending φ. This is a contradiction. $\square$

The previous lemma can be used to deal with non-facial triangles, as indicated by the following example.

Theorem 4. *Let G be a plane graph and A and B two faces of G bounded by triangles. Let φ be any coloring of $A \cup B$ by 6 colors. If the distance between A and B is at least 80, then φ extends to a 6-coloring of G.*

Proof. For contradiction, assume that G is a counterexample with the smallest number of vertices. Clearly, every vertex not in $A \cup B$ has degree at least 6. Furthermore, since every planar graph is 6-colorable, every triangle that does not separate A from B bounds a face (otherwise we would first color the part of the graph outside the triangle, then extend the coloring inside). Similarly, G is connected.

Suppose that G contains a vertex $v \notin V(A) \cup V(B)$ of degree 6 not incident with a triangle separating A from B, such that all incident faces are triangles. Let v_1, v_2, ..., v_6 be the neighbors of v in order around v according to the drawing of G. For $1 \leq i \leq 6$, let d_i be the distance between A and v_i. Since all faces incident with v are triangles, we have $|d_{i+1} - d_i| \leq 1$ for each i (where $d_7 = d_1$). We can assume that d_1 is the smallest of these distances. A straightforward case analysis shows that $d_a = d_b$, where $\{a, b\}$ is equal to $\{2, 6\}$ or $\{1, 3\}$ or $\{3, 6\}$ or $\{1, 5\}$ or $\{2, 5\}$. Note that at most one of v_a and v_b belongs to B, as otherwise $v_a v v_b$ would be a triangle separating A from B. Similarly, neither v_a nor v_b belongs to A, as if say v_a did, we would have $d_b = d_a = 0$ and v_b would belong to A as well, and $v_a v v_b$ would be a triangle separating A from B.

Let G' be the graph obtained from $G - v$ by identifying v_a with v_b. This does not create a loop, since v does not belong to a triangle separating A from B. Since $d_a = d_b$, the distance between A and B in G' is greater or equal to the distance between A and B in G. By the minimality of G, there exists a 6-coloring of G' extending φ. Since the neighbors of v have at most five different colors in this coloring (v_a and v_b get the same color), we conclude that this coloring extends to a 6-coloring of G. This is a contradiction, hence each vertex of degree 6 not belonging to $A \cup B$ is either incident with a face of length at least four or with a triangle separating A from B.

Let us now give each vertex v charge $\deg(v) - 6$ and each face f the charge $2\ell(f) - 6$. The sum of these charges is -12. Each face of length at least four now sends charge $1/2$ to each incident vertex of degree 6. Furthermore, for each triangle separating A from B, we increase the charge of all its incident vertices by $1/2$. By this adjustment, the charge of each triangular face is 0, the charge of any other face is at least $2\ell(f) - 6 - \ell(f)/2 = 3\ell(f)/2 - 6 \geq 0$ and the charge of each vertex not in $A \cup B$ is at least $1/2$. Note that all vertices of $A \cup B$ have final charge at least -4. Let $n = |V(G)|$ and let N be the number of triangles separating A from B. We have $-12 + 3N/2 \geq -24 + (n-6)/2$, and thus $N \geq n/3 - 10$. Since the distance between A and B is at least 80 and G is connected, it follows that G has at least 85 vertices and we have $N \geq 19$. Note that by the minimality of G, φ extends to every proper subgraph of G including $A \cup B$. This gives a contradiction with Lemma 3. $\square$

A rather more difficult example of an application of the recoloring technique can be found in Chapter 6.

2.4 Critical graphs

A technical difficulty that often appears in the proofs of the coloring results is that the considered graph may contain "irrelevant" parts. For example, when proving k-colorability, we usually do not care about vertices of degree less than k (as long as the class of graphs in question is closed under induced subgraphs). To avoid this difficulty, one typically considers a hypothetical minimal counterexample G, which has the property that every proper subgraph of G is k-colorable, while G itself is not k-colorable. Equivalently, G is $(k+1)$-critical.

This motivates the study of the properties of k-critical graphs. For example, every k-critical graph has minimum degree at least $k-1$ (more generally, Dirac [20] proved that they are $(k-1)$-edge-connected). Gallai [39] proved that every 2-connected subgraph induced by vertices of degree $k-1$ in a k-critical graph is either complete or an odd cycle, and this result is important in many coloring proofs. For instance, it gives a lower bound on the density of 7-critical graphs that implies that for any g, only finitely many of them have genus at most g, as observed by Mohar [54] and independently by Thomassen [64]. For a survey on other properties of k-critical graphs, see [47].

In proofs using the method of reducible configurations, one is often required to deal separately with short non-facial cycles. For instance, suppose that we would like to prove Grötzsch's theorem and 3-color a triangle-free graph G, and that G has a face $v_1v_2v_3v_4$ of length four. We can attempt identifying v_1 with v_3, eliminating the face and suppressing the parallel edges. If the resulting graph is 3-colorable, the original graph G clearly is 3-colorable as well. Of course, a possible problem is that the identification can create a triangle. This happens exactly when v_1 and v_3 are joined by a path v_1xyv_3 of length three in G. Note that $v_2 \notin \{x, y\}$, since G is triangle-free. It follows that $C = v_1v_2v_3yx$ is a 5-cycle in G. If C bounded a face, then v_2 would have degree two, and we could remove v_2 instead. Therefore, only the case that C is non-facial is problematic.

One possible way how to deal with this difficulty is the following: we will prove a more general claim that in every triangle-free planar graph G, every precoloring of a 5-cycle K by three colors can be extended to a 3-coloring of G. In a hypothetical minimal counterexample to this claim, every 5-cycle bounds a face; and consequently, we can always reduce 4-faces as outlined in the preceding paragraph (assuming that at least one of v_1 and v_3 does not belong to K). Indeed, if C were a non-facial 5-cycle, then we could express G as $G_1 \cup G_2$, where $K \subseteq G_1$, $G_1 \cap G_2 = C$ and both G_1 and G_2 are proper subgraphs of G. By the minimality of G, there would exist a 3-coloring of G_1 extending the precoloring of K. This coloring induces a 3-coloring of C, and again by the minimality of G, we could extend it to a 3-coloring of G_2. This would give a 3-coloring of G extending the precoloring of K. The proof of Theorem 1 demonstrated this idea.

Sometimes, more complicated variants of the argument are needed. The available reducible configurations may force us to deal with cycles of length k, but it may happen that the claim that every precoloring of a k-cycle extends is false. For example, in his proof of Grötzsch's theorem, Thomassen [68] needed to deal with cycles of length 9 in graphs of girth five, and it is not true that every precoloring of a 9-cycle in such a graph extends. To handle this issue, he proved the following claim.

Theorem 5. *Let G be a plane graph of girth at least 5 and let $C \subseteq G$ be a facial cycle of length at most 9. Let φ be a proper coloring of C by 3-colors. There exists a 3-coloring of G extending φ, unless either $\ell(C) \geq 8$ and there exists a chord of*

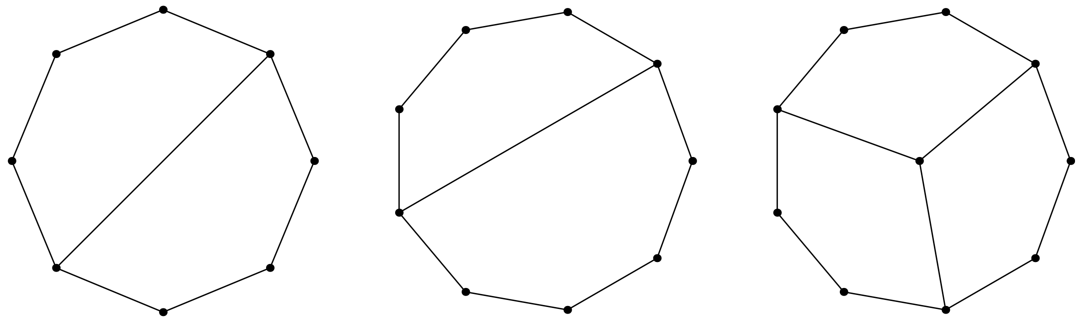


Figure 2.1: 4-critical graphs of girth five and a precolored (≤ 9)-face

C joining two vertices u and v with $\varphi(u) = \varphi(v)$, or $\ell(C) = 9$, three vertices u, v and w of C have a common neighbor and $\varphi(u) \neq \varphi(v) \neq \varphi(w) \neq \varphi(u)$.

This does not enable us to eliminate non-facial 9-cycles entirely; however, we can assume that the interior of every 9-cycle contains at most one vertex, which in turn may be sufficient to argue the reducibility of a configuration. Examples of similar applications can be found throughout the thesis; in this section, we develop some of the necessary theory and present ideas on how results similar to Theorem 5 can be proved.

Notably, Theorem 5 identifies what could be considered to be "4-critical graphs with precolored cycle C" (which are planar and of girth five, and where C bounds a face)— 8- or 9-cycle C with a chord, and a 9-cycle C together with a vertex with three neighbors in C, see Figure 2.1. However, it is not entirely clear what we mean by a critical graph with a precolored subgraph. There are at least two basic ways of defining this concept; for a graph G and its proper subgraph T and a fixed number of colors k, we will say that

- G is *strongly T-critical (for k-colorability)* if there exists a proper k-coloring φ of T that extends to a k-coloring of every proper subgraph of G that includes T, but not to a k-coloring of G.

- G is *T-critical (for k-colorability)* if for every proper subgraph G' of G including T, there exists a proper k-coloring of T that extends to a k-coloring of G', but not to a k-coloring of G.

Let us remark that T itself is not considered to be T-critical. This choice is essentially arbitrary—there are several occasions where including this trivial case would simplify statements of results, but many other cases where we would need to exclude it.

Clearly, if G is strongly T-critical, then it is also T-critical. However, the converse does not hold. For example, if G consists of a 5-cycle C with two chords, then G is C-critical for 3-colorability, but not strongly C-critical (while a cycle C with one chord would be both C-critical and strongly C-critical). In

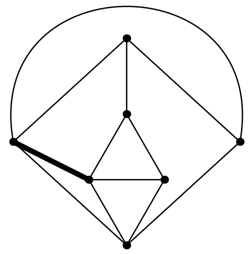


Figure 2.2: Subgraph versus subset distinction

the special case that T is empty, both $\emptyset$-critical and strongly $\emptyset$-critical graphs for k-colorability coincide with $(k+1)$-critical graphs.

We define the criticality with respect to a subgraph, rather than a subset of vertices. This is mostly for a notational convenience (most often, we will study graphs critical with respect to a facial cycle C, and it is easier to write C-critical than $V(C)$-critical). Consider a graph T and let T' be a graph with the same set of vertices, but with no edges. Note that a T-critical graph is not necessarily T'-critical—the graph G in Figure 2.2 is T-critical for 3-coloring, where T consists of the thick edge e, however $G-e$ has no 3-coloring, and thus G is not T'-critical. However, each T'-critical graph is also T-critical; more generally:

Lemma 6. *If G is T-critical and $T \subseteq S \subset G$, then G is S-critical.*

Proof. If G' is a proper subgraph of G including S, then by the criticality of G, there exists a coloring of T that extends to a coloring of G', but not to a coloring of G. This gives a coloring of S that extends to a coloring of G', but not to G. Consequently, G is S-critical. □

While the definition of strongly T-critical graphs may seem more natural, there is a good motivation for the definition of a T-critical graphs—they capture the information about which precolorings of T extend to the whole graph, as the following lemma shows. For a graph G and its subgraph T, let $c_k(T,G)$ denote the set of proper k-colorings of T that extend to a proper k-coloring of G.

Lemma 7. *Let T be a proper subgraph of G and let k be an integer. There exists $G' \subseteq G$ with $T \subseteq G'$ such that $c_k(T,G') = c_k(T,G)$ and either G' is T-critical or $G' = T$.*

Proof. We let G' be a minimal subgraph of G such that $T \subseteq G'$ and $c_k(T,G') = c_k(T,G)$. For every proper subgraph $G'' \subset G'$ with $T \subseteq G''$, the minimality of G' implies that $c_k(T,G'') \neq c_k(T,G) = c_k(T,G')$, hence there exists a k-coloring of T that extends to G'', but not G'. Consequently, either $G' = T$ or G' is T-critical. □

The analogous claim does not hold for strongly T-critical graphs, as the aforementioned example of a cycle with two chords shows.

Let us now demonstrate that the notion of graphs critical with respect to a subgraph can be used to restrict non-facial cycles in embedded critical graphs; i.e., that if G is a critical graph drawn on a surface and W is a non-facial contractible cycle in G, then the subgraph of G drawn inside W is W-critical. We state the result in a little more general way which allows W to be a closed walk with repeated vertices and edges.

Theorem 8. *Let G be a graph embedded in a surface Σ and let $\Lambda \subset \Sigma$ be an open disk bounded by a closed walk $W \subseteq G$ such that Λ is not a face of G. Let T be a subgraph of G disjoint from Λ. Let f' be a homeomorphism from the interior of a closed disk Δ to Λ, and let $f : \Delta \to \Sigma$ be the continuous extension of f'. Let G' be the graph drawn in Δ such that $f(G')$ is equal to the subgraph of G drawn in the closure of Λ and the boundary of Δ is formed by a cycle $C \subset G'$. If G is T-critical, then G' is C-critical.*

Proof. Let G_1 be a proper subgraph of G' that includes C, and let $G_2 = G - f(G' - G_1)$. Note that G_2 is a proper subgraph of G including T. If G is T-critical, then there exists a coloring ψ of T that extends to a coloring φ of G_2, but not to a coloring of G. Let ψ' be the coloring of C such that $\psi'(v) = \varphi(f(v))$ for every $v \in V(C)$. Observe that ψ' extends to a coloring of G_1 (given as the preimage of φ), but does not extend to a coloring of G'. Since this holds for every proper subgraph of G' including C, we conclude that G' is C-critical. □

More generally, a claim analogous to Theorem 8 can be applied to any cut in a graph. Another useful operation is cutting the surface along a path.

Lemma 9. *Let G be a graph embedded in a surface Σ, let T be a subgraph of G and let c be a simple (possibly closed) curve such that for every edge $e \in E(G)$, if the interior of e intersects c, then e is completely contained in c. Let Σ' be the (possibly disconnected) surface with holes obtained from Σ by cutting along c and let $f : \Sigma' \to \Sigma$ be the corresponding continuous mapping. Let G' be the graph drawn in Σ' such that $f(G')$ is equal to G, each edge $e \in G \cap c$ corresponds to two edges of G' and each vertex $v \in G \cap c$ distinct from the endpoints of c corresponds to two vertices of G'. Let T' consist of the subgraph of G' contained in $f^{-1}(T \cup c)$. If G is T-critical, then $G' = T'$ or G' is T'-critical.*

Proof. Suppose that $G' \neq T'$, and let G_1 be a proper subgraph of G' including T'. Let $G_2 = f(G_1)$ and note that G_2 is a proper subgraph of G including T. If G is T-critical, then there exists a coloring ψ of T that extends to a coloring φ of G_2, but not to G. Let ψ' be the coloring of T' defined by $\psi'(v) = \varphi(f(v))$ for $v \in V(T')$. Observe that ψ' extends to a coloring of G_1, but not of G'. This implies that G' is T'-critical. □

Let us remark that these claims do not hold for strongly T-critical graphs. In the view of these results, one could question whether the strong T-criticality is actually useful. One reason for introducing it is historical. There have been quite a few papers dealing with the issue, especially with the case of extending a precoloring of a facial cycle in a planar graph. Walls [77] and Thomassen [71] independently listed plane strongly C-critical graphs of girth at least five, where C is a facial cycle of length at most 11 (the latter paper in fact proves a more general claim about list coloring), and Dvořák and Kawarabayashi [22] extended this for facial 12-cycles. For triangle-free graphs, the problem was considered by Gimbel and Thomassen [41] for 6-cycles and by Aksenov et al. [1] for 7-cycles. Similar results for extension of a coloring of a (≤ 6)-cycle to a 5-coloring of a planar graph were obtained by Thomassen [66]. All these results are essentially formulated in terms of strong criticality.

A more substantial reason is that in some situations, it is more convenient to work with strong reducibility. If G is strongly T-critical, then we can fix a precoloring φ of T that does not extend to G, but extends to every proper subgraph (for T-criticality, there is no such single precoloring). This may simplify some arguments; for example, if we use the method of reducible configurations and we need to reduce a configuration appearing close to T, we can use several different reductions depending on the colors of the vertices of T according to φ.

Fortunately, it is typically easy to switch between T-criticality and strong T-criticality as needed. One direction is trivial, since every strongly T-critical graph is also T-critical. For the other direction, note that if G is T-critical, then there exists a proper coloring φ of T that does not extend to a coloring of G. Now, if $G' \subseteq G$ is a minimal subgraph of G including T such that φ does not extend to a coloring of G', then G' is strongly T-critical and $G' \neq T$. Theorem 8 can then be used to obtain information about the difference between G and G'.

Indeed, it is not hard to prove the following claims (see e.g. [28, 31]): Let G be a plane graph and C a facial cycle in G. Suppose that G is C-critical for k-colorability and that $G \neq C$. Then every face C' of G distinct from C has length at most

- $\ell(C) - 1$ if $k = 5$,
- $\ell(C) - 2$ if $k = 3$ and G is triangle-free, and
- $\ell(C) - 3$ if $k = 3$ and G has girth at least five.

Consequently in these cases, one can generate C-critical graphs from strongly C-critical ones by combining them with C'-critical graphs, where $\ell(C') < \ell(C)$ (which in turn can be recursively obtained from strongly C'-critical graphs). And since the size of the precolored subgraph decreases, this algorithm is guaranteed to terminate. Using it, one can easily verify that although Walls [77] and Thomassen [71] explicitly only determine strongly C-critical graphs (plane of girth at least five, with respect to a facial cycle C of length at most 11), they in fact list all such C-critical graphs except for cycles with two chords. Let us remark that this observation does not hold for 4-colorability of plane graphs—for every t there exists a C-critical plane graph G for 4-colorability, where C is a 4-face and G has t other 4-faces. This illustrates difficulties with extending the Four Color Theorem.

How can we prove what the T-critical graphs (for a prescribed graph T) from some class are? Both the technique of reducible configurations and of precoloring extension can be useful. In both cases, assume that we already know the list $\mathcal{L}$ of all such graphs (which may be finite, or at least belong to some easily described infinite class of graphs). Now, for a contradiction we assume that there exists a T-critical graph G not belonging to $\mathcal{L}$, and let us consider the smallest such graph. If G is not strongly T-critical, then it contains a proper subgraph $G' \supset T$ that is (strongly) T-critical, and by the minimality of G, we have $G' \in \mathcal{L}$. Therefore, G can be obtained from some graph in $\mathcal{L}$ by filling its faces with smaller critical graphs. Thus, it is easy to generate all such candidates for G and prove that they either are not T-critical or belong to $\mathcal{L}$.

Suppose now that G is strongly T-critical, and thus there exists a coloring φ of T that extends to every proper subgraph of G, but not to G. Let us now discuss the specifics of the two methods.

- In order to apply the reducible configurations technique, we first find a reducible configuration in G that is far enough from T such that the coloring of T does not interfere with the reducibility (for this, it is often useful that in a plane graph, the charges do not sum to zero, but to a strictly negative number, and thus we can somewhat increase the charge of the vertices of T while still keeping the sum of charges negative). Let G' be the graph obtained from G by performing the reduction; we have that φ does not extend to a coloring of G', and thus G' has a strongly T-critical subgraph G''. By the minimality of G, we have $G'' \in \mathcal{L}$. Thus, G can be obtained from a graph in $\mathcal{L}$ by first "unreducing" a configuration, and then filling in further critical graphs in the faces of the resulting graph $G''' \subseteq G$. We again need to argue that all graphs obtained in that way either are not T-critical or belong to $\mathcal{L}$.

  There is a slight difficulty with the last step. Suppose that we are considering say the case that T is a facial cycle in a plane graph and that we are using 5 colors. As mentioned before, in this case each face of G'' has length at most $\ell(T)-1$. However, the faces of the graph G''' could be longer, since unreducing the configuration usually extends the lengths of cycles. If the faces of G''' have length at most $\ell(T)$, we can still proceed with the proof by induction. However, if some face has length greater than $\ell(T)$, some more involved argument to deal with the case must be found.

  An example of this approach can be found in Chapter 4. A more involved application of the technique combined with other ideas is also used in Chapter 7.

- If we use the precoloring extension method, we instead remove vertices of T from G and remove their colors according to φ from the lists of their neighbors. Since φ does not extend to a coloring of G, it follows that the resulting list assignment does not satisfy the assumptions of the particular result that we are using (e.g., if T is a facial cycle, we use 5 colors and we apply Theorem 2, there must exist a vertex v with list of size at most two). This may give information necessary for the inductive proof of the characterization of critical graphs—continuing our example, v has at least three neighbors in T. Let G' be the subgraph of G induced by $V(T) \cup v$, and note that all faces of G' distinct from T have length at most $\ell(T)$. Using Theorem 8 and the inductive assumption that all graphs critical with respect to a facial $\ell(T)$-cycle belong to $\mathcal{L}$, we can again argue that either G is not T-critical or it belongs to $\mathcal{L}$.

  A more detailed version of this argument can be found in Chapter 8. An analogous approach for 3-list-colorability of plane graphs of girth at least 5 was investigated by Dvořák and Kawarabayashi [22].

Both approaches can also be naturally used to determine the elements of $\mathcal{L}$: initially, set $\mathcal{L} = \emptyset$ and attempt to proceed with the proof. If the attempt fails, we obtain a counterexample—a T-critical graph from the considered class that does not belong to $\mathcal{L}$. We include this graph and repeat the whole process. Of course, this only leads to a result when $\mathcal{L}$ is finite, otherwise we at some point need to guess a generalization of the obtained examples of T-critical graphs that enables us to describe an infinite set $\mathcal{L}$ in a finite way.

Almost all other examples given in the rest of the thesis deal with critical graphs in some form. In Chapter 7, we give bounds on the size of embedded graphs of girth 5 critical for 3-colorability, instead of an exact characterization. In Chapters 8 and 9, we deal with graphs critical for 5-list-colorability that are embedded in plane with distant perturbations (crossings, precolored vertices, ...), and in Chapter 10, we deal with similar ideas for 3-list-colorability of plane graphs with distant (≤ 4)-cycles.

Part I

Applications of the basic techniques

In this part, we give several more realistic examples of the methods outlined in Chapter 2. The first of them (Chapter 3) uses the method of reducible configurations to give a new proof of Grötzsch's theorem with the property that the reducibility of the configurations can be checked by inspecting a constant number of edges, which is useful in the algorithmic setting. The second one (Chapter 4) demonstrates the use of discharging method to find a list of graphs critical with respect to a precolored facial cycle. The third one (Chapter 5) uses the precoloring extension method to prove 3-list-colorability of plane graphs without triangles and with restrictions on 4-cycles, generalizing a result of Thomassen [72]. The final example (Chapter 6) combines the method of reducible configurations and the recoloring technique to derive a rather technical result regarding graphs critical with respect to two facial cycles (this result is later used in Chapter 7).

Chapter 3

Three-coloring triangle-free planar graphs in linear time[1]

The following is a classical theorem of Grötzsch [43].

Theorem 10. *Every triangle-free planar graph is* 3*-colorable.*

This result has been the subject of extensive research. Thomassen [67, 72] found two short proofs and extended the result in many ways. We return to the various extensions later, but let us discuss algorithmic aspects of Theorem 10 first. It is easy to convert either of Thomassen's proofs into a quadratic-time algorithm to find a 3-coloring, but it is not clear how to do so in linear time. A serious problem appears very early in the algorithm. Given a facial cycle C of length four, one would like to identify a pair of diagonally opposite vertices of C and apply recursion to the smaller graph. It is easy to see that at least one pair of diagonally opposite vertices on C can be identified without creating a triangle, but how can we efficiently decide which pair? If we could test in (amortized) constant time whether given two vertices are joined by a path of length at most three, then that would take care of this issue. This can, in fact, be done, using a data structure of Kowalik and Kurowski [49] *provided* the graph does not change. In our application, however, we need to repeatedly identify vertices, and it is not clear how to maintain the data structure of Kowalik and Kurowski in overall linear time. Kowalik [48] developed a sophisticated enhancement of this data structure that supports edge addition and deletion in amortized $O(\log n)$ time. Furthermore, he found a variant of the proof of Grötzsch's theorem that can be turned into an $O(n \log n)$ algorithm to 3-color a triangle-free planar graph on n vertices using this data structure. We improve this to a linear-time algorithm, as follows.

Theorem 11. *There is a linear-time algorithm to* 3*-color an input triangle-free planar graph.*

[1]The results of this chapter are based on Dvořák et al. [23].

To describe the algorithm we exhibit a specific list of five reducible configurations, called "multigrams", and show that every triangle-free planar graph contains one of those reducible configurations. Proving this is the only step that requires some effort; the rest of the algorithm is entirely straightforward, and the algorithm is very easy to implement. Given a triangle-free planar graph G we look for one of the reducible configurations in G, and upon finding one we modify G to a smaller graph G', and apply the algorithm recursively to G'. It is easy to see that every 3-coloring of G' can be converted to a 3-coloring of G in constant time. Furthermore, each reducible configuration has a vertex of degree at most three, and, conversely, given a vertex of G of degree at most three it can be checked in constant time whether it belongs to a reducible configuration. Thus at every step a reducible configuration can be found in amortized constant time by maintaining a list of candidates for such vertices. As a by-product of the proof of correctness of our algorithm we give a short proof of Grötzsch's theorem.

We work with simple graphs embedded in plane. On several occasions we will be identifying vertices, but when we do, we will remove the resulting parallel edges. When this will be done by the algorithm we will make sure that the only parallel edges that arise will form faces of length two. The detection and removal of such parallel edges can be done in constant time.

3.1 Short proof of Grötzsch's theorem

Let G be a plane graph. Somewhat nonstandardly, we call a cycle F in G *facial* if it bounds a face in a connected component of G, regardless of whether F is a face or not (another component of G might lie in the disk bounded by F). This technicality makes no difference in this section, because here we may assume that all graphs are connected. However, it will be needed in the description of the algorithm, because the graph may become disconnected during the course of the algorithm, and we cannot afford to decompose it into connected components.

By a *tetragram* in G we mean a sequence (v_1, v_2, v_3, v_4) of vertices of G such that they form a facial cycle in G in the order listed. We define a *hexagram* $(v_1, v_2, \ldots, v_6)$ similarly. By a *pentagram* in G we mean a sequence $(v_1, v_2, v_3, v_4, v_5)$ of vertices of G such that they form a facial cycle in G in the order listed and v_1, v_2, v_3, v_4 all have degree exactly three. We will show that every triangle-free planar graph of minimum degree at least three has a tetra-, penta- or hexagram with certain additional properties that will allow an inductive argument. But first we need the following lemma.

Lemma 12. *Let G be a connected triangle-free plane graph and let f_0 be the unbounded face of G. Assume that the boundary of f_0 is a cycle C of length at most six, and that every vertex of G not on C has degree at least three. If $G \neq C$, then G has either a tetragram, or a pentagram $(v_1, v_2, v_3, v_4, v_5)$ such that $v_1, v_2, v_3, v_4 \notin V(C)$.*

Proof. We define the charge of a vertex v to be $3\deg(v) - 12$, the charge of the face f_0 to be $3|V(C)| + 11$ and the charge of a face $f \neq f_0$ of length ℓ to be $3\ell - 12$. It follows from Euler's formula that the sum of the charges of all vertices and faces is -1.

We now redistribute the charges according to the following rules. Every vertex not on C of degree three will receive one unit of charge from each incident face, each vertex on C of degree three will receive three units from f_0, and each vertex of degree two on C will receive five units from f_0 and one unit from the other incident face. Thus the final charge of every vertex is non-negative.

We now show that the final charge of f_0 is also non-negative. Let ℓ denote the length of C. Then f_0 has initial charge of $3\ell+11$. By hypothesis at least one vertex of C has degree at least three, and hence f_0 sends a total of at most $5(\ell - 1) + 3$ units of charge, leaving it at the end with charge of at least $3\ell+11-5(\ell-1)-3 \geq 1$.

Since no charge is lost or created, there is a face $f \neq f_0$ whose final charge is negative. Since f sends at most one unit to each incident vertex, we see that f has length at most five. Furthermore, if f has length exactly five, then it sends one unit to at least four incident vertices. None of those could be a degree two vertex on C, for then f would not be sending anything to the ends of the common subpath of the boundaries of f and f_0. Thus the vertices of f form the desired tetragram or pentagram. □

Let $k = 4, 5, 6$, and let $(v_1, v_2, \ldots, v_k)$ be a tetragram, pentagram or hexagram in a triangle-free plane graph G. If $k = 4$ or $k = 6$, then we say that $(v_1, v_2, \ldots, v_k)$ is *safe* if every path in G of length at most three with ends v_1 and v_3 is a subgraph of the cycle $v_1 v_2 \cdots v_k$. For $k = 5$ we define safety as follows. For $i = 1, 2, 3, 4$ let x_i be the neighbor of v_i distinct from v_{i-1} and v_{i+1} (where $v_0 = v_5$). Then $x_i \notin \{v_1, \ldots, v_5\}$, because G is triangle-free. Assume that

- the vertices x_1, x_2, x_3, x_4 are pairwise distinct and pairwise non-adjacent, and
- there is no path in $G \setminus \{v_1, v_2, v_3, v_4\}$ of length at most three from x_2 to v_5, and
- every path in $G \setminus \{v_1, v_2, v_3, v_4\}$ of length at most three from x_3 to x_4 has length exactly two, and its completion via the path $x_3 v_3 v_4 x_4$ results in a facial cycle of length five in G (in particular, there is at most one such path).

In those circumstances we say that the pentagram $(v_1, v_2, \ldots, v_5)$ is *safe.*

Lemma 13. *Every triangle-free plane graph G of minimum degree at least three has a safe tetragram, a safe pentagram, or a safe hexagram.*

Proof. Let G be as stated. If (v_1, v_2, v_3, v_4) is a tetragram in G, then one of the tetragrams (v_1, v_2, v_3, v_4), (v_2, v_3, v_4, v_1) is safe, as G is planar and triangle-free. Thus we may assume that G has no 4-faces, and hence every 4-cycle in G is separating.

Let us define an induced subgraph G_1 of G and a facial cycle C_1 of G_1 in the following way: If G has a separating cycle of length at most five, then let us select such a cycle C_1 so that the disk it bounds is as small as possible, and let G_1 be the subgraph of G consisting of all vertices and edges drawn in the closed disk bounded by C_1. If G has no separating cycle of length at most five, then let $G_1 := G$ and let C_1 be a facial cycle of G of length at most five. Such a facial cycle exists, because the minimum degree of G is at least three. In the latter case, we also redraw G so that C_1 becomes the outer face; thus G_1 is always drawn in the closed disk bounded by C_1. Note that G_1 does not contain any separating cycle of length at most five, and thus G_1 does not contain any 4-cycle except possibly C_1.

Next, we define a subgraph G_2 of G_1 and its facial cycle C_2 as follows. If G_1 contains a separating cycle of length six, then choose such a cycle C_2 so that the disk it bounds contains as few vertices as possible, and let G_2 be the subgraph of G_1 consisting of all vertices and edges drawn in the closed disk bounded by C_2. Otherwise, let $G_2 := G_1$ and $C_2 := C_1$. Note that G_2 does not contain any separating cycle of length at most six. As G has no 4-faces, it follows that any cycle of length at most six in G_2 bounds a face.

The cycle C_2 is induced in G, for if it had a chord, then the chord would belong to G_1 (because G_1 is an induced subgraph of G), and hence $V(C_2)$ would include the vertex-sets of two distinct cycles of length at most (and hence exactly) four in G_1, a contradiction.

From Lemma 12 applied to the graph G_2 and facial cycle C_2 we deduce that G_2 has a pentagram $(v_1, v_2, v_3, v_4, v_5)$ such that $v_1, v_2, v_3, v_4 \notin V(C_2)$. We may assume that neither this pentagram nor the pentagram $(v_4, v_3, v_2, v_1, v_5)$ is safe in G, for otherwise the lemma holds. Let x_i be the neighbor of v_i outside of the pentagram, for $1 \leq i \leq 4$. Note that all of these neighbors belong to G_2, and as G_2 is triangle-free and contains no 4-cycles other than C_2 and no separating cycles of length at most 5, they are distinct and mutually non-adjacent. It follows that $|\{x_1, x_2, x_3, x_4\} \cap V(C_2)| \leq 3$, and by symmetry we may assume that at least one of x_3 and x_4 does not lie on C_2. Furthermore, as each cycle of length at most six in G_2 is facial, if $v_5 \in V(C_2)$, then $\{x_1, x_2, x_3, x_4\} \cap V(C_2) = \emptyset$.

Since the pentagram $(v_1, v_2, v_3, v_4, v_5)$ is not safe in G, there exists a pair of vertices x, y such that either $\{x, y\} = \{x_2, v_5\}$ or $\{x, y\} = \{x_3, x_4\}$, and there exists a path P in $G \backslash \{v_1, v_2, v_3, v_4\}$ with ends x and y such that P has length at most three, and if $\{x, y\} = \{x_3, x_4\}$, then either P has length exactly three, or its completion via the path $x_3v_3v_4x_4$ does not result in a facial cycle in G. If $\{x, y\} = \{x_2, v_5\}$ then let Q denote the path $x_2v_2v_1v_5$; otherwise let Q denote the path $x_3v_3v_4x_4$. Suppose first that $P \cup Q$ bounds a face in G. Then it follows that $\{x, y\} = \{x_3, x_4\}$, and hence P has length exactly three. Let the vertices of $P \cup Q$ be $x_3v_3v_4x_4ab$ in order. Let us argue that $(x_4, v_4, v_3, x_3, a, b)$ is a safe hexagram. If that were not the case, then there would exist a path $x_4u_1v_3$ or $x_4u_1u_2v_3$ for some $u_1, u_2 \neq v_4$. Since v_2 and v_3 have degree three and the vertices x_1, x_2, x_3 and x_4 are distinct and mutually non-adjacent, the former case is not possible, and in the latter case $u_2 = x_3$. However, since at most one of x_3 and x_4 lies on C_2, $x_4u_1x_3v_3v_4$ would be a separating 5-cycle in G_2, and hence in G_1, a contradiction.

Thus we may assume that $P \cup Q$ does not bound a face in G, and so $P \cup Q$ is a separating cycle in G. It follows from the choice of C_2 that $P \cup Q$ is not a subgraph of G_2. But not both x, y belong to C_2 and C_2 is induced; thus a subpath R of $P \cup Q$ of length four joins two vertices w_1, w_4 of C_2, and a vertex w of $(P \cup Q) \backslash V(G_2)$ is adjacent to both w_1 and w_4. If $w \notin V(G_1)$, then $w_1, w_4 \in V(C_1)$, because they belong to C_2. But C_1 has length at most five, and w_1, w_4 are not adjacent, because G is triangle-free. Thus w_1, w_4 have a common neighbor in C_1, and this neighbor can replace w. Thus we may assume that $w \in V(G_1)$.

If w_1 and w_4 have a common neighbor in C_2, then R can be completed using this neighbor to a cycle that contradicts the choice of C_2. It follows that w_1, w_4 are at distance three on C_2, and so we may assume that the vertices of C_2 are $w_1, w_2, \ldots, w_6$, in order. From the symmetry we may assume that $w_1w_2w_3w_4w$ bounds a face, by the minimality of C_1. Thus the closed disk bounded by $P \cup Q$ does not include w_5, w_6, and it includes no vertex of $V(G) - V(G_2)$, except w. Thus $P \cup Q$ contradicts the choice of C_2. □

Proof of Theorem 10. Let G be a triangle-free plane graph. We proceed by induction on $|V(G)|$. We may assume that every vertex v of G has degree at least three, for otherwise the theorem follows by induction applied to $G \setminus v$. By Lemma 13 there is a safe tetra-, penta-, or hexagram $(v_1, v_2, \ldots, v_k)$. If $k = 4$ or $k = 6$, then we apply induction to the graph obtained from G by identifying v_1 and v_3. It follows from the definition of safety that the new graph has no triangles, and clearly every 3-coloring of the new graph extends to a 3-coloring of G.

Thus we may assume that $(v_1, v_2, \ldots, v_5)$ is a safe pentagram in G. Let G' be obtained from $G \setminus \{v_1, v_2, v_3, v_4\}$ by identifying v_5 with x_2, and x_3 with x_4. It follows from the definition of safety that G' is triangle-free, and hence it is 3-colorable by the induction hypothesis. Any 3-coloring of G' can be extended to a 3-coloring of G: let c_1 be the color of x_1, c_2 the color of x_2 and v_5, and c_3 the color of x_3 and x_4. If $c_1 = c_2$, then we color the vertices v_4, v_3, v_2 and v_1 in this order. Note that when v_i $(i = 1, 2, 3, 4)$ is colored, it is adjacent to vertices of at most two different colors, and hence we can choose the third color for it. Similarly, if $c_2 = c_3$, then we color the vertices in the following order: v_1, v_2, v_3 and v_4. Let us now consider the case that $c_1 \neq c_2 \neq c_3$. We color v_2 with c_1, v_3 with c_2, and choose a color different from c_1 and c_2 for v_1 and a color different from c_2 and c_3 for v_4. Thus G is 3-colorable, as desired. $\square$

Let us note that the essential ideas of the proof came from Thomassen's work [67]. For graphs of girth at least five Thomassen actually proves a stronger statement, namely that every 3-coloring of an induced facial cycle of length at most nine extends to a 3-coloring of the entire triangle-free plane graph, unless some vertex of G has three distinct neighbors on C (and those neighbors received three different colors). By restricting ourselves to Theorem 10 we were able to somewhat streamline the argument. Another variation of the same technique is presented in [48].

3.2 Graph representation

For the purpose of our algorithm, graphs will be represented by means of doubly linked adjacency lists. More precisely, the neighbors of each vertex v will be listed in the clockwise cyclic order in which they appear around v, and the two occurrences of the same edge will be linked to each other. The facial walks of the graph can be read off from this representation using the standard face tracing algorithm ([55], page 93). Thus all vertices and edges incident with a facial cycle of length k can be listed in time $O(k)$. Here we make use of our non-standard definition of facial cycle.

Suppose that D is a fixed constant (in our algorithm, $D = 59$). We can perform the following operations with graphs represented in the described way in constant time:

- remove an edge when a corresponding entry of the adjacency list is given
- add an edge with ends u, v into a face f, assuming that the edges preceding and following u, v in the facial boundary of f are specified
- remove an isolated vertex
- determine the degree of a vertex v if $\deg(v) \leq D$, or prove that $\deg(v) > D$
- check whether two vertices u and v such that $\min(\deg(u), \deg(v)) \leq D$ are adjacent
- check whether the distance between two vertices u and v such that $\max(\deg(u), \deg(v)) \leq D$ is at most two
- given an edge e incident with a face f, output all vertices whose distance from e in the facial walk of f is at most two, and determine whether the length of the component of the boundary of f that contains e has length at most 6

- output the subgraph consisting of vertices reachable from a vertex v_0 through a path v_0, v_1, $\ldots$, v_t of length $t \le D$, such that $\deg(v_i) \le D$ for $0 \le i < t$ (but the degree of v_t may be arbitrary).

All the transformations and queries executed in the algorithm can be expressed in terms of these simple operations.

3.3 The algorithm

The idea of our algorithm is to find a safe tetragram, pentagram or hexagram γ in G and use it to reduce the size of the graph as in the proof of Theorem 10 above. Finding γ is easy, but the difficulty lies in testing safety. To resolve this problem we prove a variant of Lemma 13 that will guarantee the existence of such γ with an additional property that will allow testing safety in constant time. The additional property, called security, is merely that enough vertices in and around γ have bounded degree. Unfortunately, the additional property we require necessitates the introduction of two more configurations, a variation of tetragram called "octagram" and a variation of pentagram called "decagram". For the sake of consistency, we say that a *monogram* in a graph G is the one-vertex sequence (v) comprised of a vertex $v \in V(G)$ of degree at most two.

Now let G be a plane graph, let $k \in \{1, 4, 5, 6\}$ and let $\gamma = (v_1, v_2, \ldots, v_k)$ be a mono-, tetra-, penta-, or hexagram in G. Let C be a subgraph of G. (For the purpose of this section the reader may assume that C is the null graph, but in the next section we will need C to be a facial cycle of G.) A vertex of G is *big* if it has degree at least 60, and *small* otherwise. A vertex $v \in V(G)$ is *C-admissible* if it is small and does not belong to C; otherwise it is *C-forbidden*. A pentagram $(v_1, v_2, \ldots, v_5)$ is called a *decagram* if v_5 has degree exactly three (and hence $v_1, \ldots, v_5$ all have degree three). A tetragram is called an *octagram* if all its vertices have degree exactly three. A *multigram* is a monogram, tetragram, pentagram, hexagram, octagram or a decagram. The vertex v_1 will be called the *pivot* of the multigram $(v_1, v_2, \ldots, v_k)$. In the following γ will be a multigram, and we will define (or recall) what it means for γ to be safe and C-secure. We will also define a smaller graph G', which will be called the *γ-reduction of G*.

If γ is a monogram, then we define it to be always *safe*, and we say that it is *C-secure* if $v_1 \notin V(C)$. We define $G' := G \setminus v_1$.

Now let γ be a tetragram. Let us recall that γ is safe if the only paths in G of length at most three with ends v_1 and v_3 are subgraphs of the facial cycle $v_1v_2v_3v_4$. We say that γ is *C-secure* if

- it is safe, and

- v_1 is C-admissible and has degree exactly three, and

- letting x denote the neighbor of v_1 other than v_2 and v_4, the vertex x is C-admissible, and
- either
  - v_3 is C-admissible, or
  - every neighbor w of x is C-admissible or belongs to a 4-face incident with the edge v_1x (either v_1v_2wx or v_1v_4wx).

We define G' to be the graph obtained from G by identifying the vertices v_1 and v_3 and deleting one edge from each of the two pairs of parallel edges that result.

If γ is an octagram, then it is always *safe*, and it is *C-secure* if v_1, v_2, v_3, v_4 are all are C-admissible. We define $G' := G \setminus \{v_1, v_2, v_3, v_4\}$.

Now let γ be a decagram, and for $i = 1, 2, 3, 4$ let x_i be the neighbor of v_i other than v_{i-1} or v_{i+1}, where v_0 means v_5. We say that the decagram γ is *safe* if x_1, x_3 are distinct, non-adjacent and there is no path of length two between them. We say that γ is *C-secure* if it is safe and the vertices $v_1, v_2, \ldots, v_5, x_1, x_3$ are all C-admissible. We define G' to be the graph obtained from $G \setminus \{v_1, v_2, \ldots, v_5\}$ by adding the edge x_1x_3.

Now let γ be a pentagram, and for $i = 1, 2, 3, 4$ let x_i be as in the previous paragraph. Let us recall that the safety of γ was defined prior to Lemma 13. We say that γ is *C-secure* if it is safe, the vertices $v_1, v_2, \ldots, v_5, x_1, x_2, x_3, x_4$ are all C-admissible, either v_5 or x_2 has no C-forbidden neighbor, and either x_3 or x_4 has no C-forbidden neighbor. We define G' as in the proof of Theorem 10: G' is obtained from $G \setminus \{v_1, v_2, v_3, v_4\}$ by identifying x_2 and v_5; identifying x_3 and x_4; and deleting one of the parallel edges should x_3 and x_4 have a common neighbor.

Finally, let γ be a hexagram. Let us recall that γ is safe if every path of length at most three in G between v_1 and v_3 is the path $v_1v_2v_3$. We say that γ is *C-secure* if v_1, v_3, v_6 are C-admissible, v_1 has degree exactly three, and the neighbor of v_1 other than v_2 or v_6 is C-admissible. We define G' to be the graph obtained from G by identifying the vertices v_1 and v_3 and deleting one of the parallel edges that result.

We say that a multigram γ is *secure* if it is K_0-secure, where K_0 denotes the null graph. This completes the definition of safe and secure multigrams.

Lemma 14. *Let G be a triangle-free plane graph, let γ be a safe multigram in G, and let G' be the γ-reduction of G. Then G' is triangle-free, and every 3-coloring of G' can be converted to a 3-coloring of G in constant time. Moreover, if γ is secure, then G' can be regarded as having been obtained from G by deleting at most 126 edges, adding at most 116 edges, and deleting at least one isolated vertex.*

Proof. The graph G' is triangle-free, because γ is safe. Unless γ is an octagram or a decagram, we argue that every 3-coloring of G' can be extended to a 3-coloring of G as in the proof of Theorem 10. If γ is an octagram or a decagram, then every 3-coloring of G' extends to a 3-coloring G, because each vertex of $V(G) - V(G')$ has a list of at least two available colors, and, in the case of a decagram, the lists are not all equal by the construction of G'.

If γ is secure, then every time vertices u and v are identified in the construction of G', one of u, v is small. Thus the identification of u and v can be seen as a deletion of at most 59 edges and addition of at most 59 edges. The lemma follows by a more careful examination of the construction of G'. □

Let G and C be as above. We say that two small vertices $u, v \in V(G)$ are *close* if either there is a path of length at most four between u and v consisting of small vertices, or a facial cycle of length at most six contains both u and v. A vertex u is close to an edge e if both u and e belong to the facial walk of the same face and the distance between u and and one end of e in this facial walk is at most two. Thus for every vertex v there are at most $1 + 4 \cdot 59 + 59^2 + 59^3 + 59^4$ vertices that are close to v, and for every edge e, there are at most 10 vertices that are close to e.

Lemma 15. *Given a triangle-free plane graph G and a vertex $v \in V(G)$, it can be decided in constant time whether G has a secure multigram with pivot v.*

Proof. This follows by inspecting the subgraph of G induced by vertices and edges that are close to v and testing the security of all multigrams with pivot v that lie in this subgraph. Given such multigram, the only non-trivial part of testing security is testing safety. Thus we may assume that the multigram satisfies all conditions in the definition of security, except safety. To test safety we need to check the existence of certain paths P of bounded length with prescribed ends. We claim that whenever such a test is needed every vertex of P, except possibly one, is small. The claim follows easily, except in the case of a tetragram $vv_2v_3v_4$, where v has degree three, the vertex v_3 is big, and letting x denote the neighbor of v_1 other than v_2 and v_4, x is small, but has a big neighbor w. In this case the straightforward check whether w and x_3 are adjacent would take more than constant time, but it actually follows that w and x_3 are not adjacent: the vertex w belongs to a 4-face incident with the edge vx, for otherwise the tetragram is not secure; but then it follows that w and x_3 are not adjacent, for otherwise wv_3v_2 would be a triangle. This proves our claim that in the course of testing safety it suffices to examine paths with all but one vertex small.

It follows from the claim that security can be tested in constant time, as desired. □

Lemma 16. *Let G and G' be triangle-free plane graphs, such that for some pair of non-adjacent vertices $u, v \in V(G)$ the graph G' is obtained from G by adding the edge uv. Let γ be a secure multigram in exactly one of the graphs G, G'. Then the pivot of γ is close to u or v in G, or to the edge uv in G'.*

Proof. Let v_1 be the pivot of γ. The claim is obvious if $v_1 \in \{u, v\}$, and thus assume this is not the case. In particular, γ is not a monogram or an octagram, and γ corresponds to a facial cycle F in G or G'. If F does not exist in G or F is not facial in G or G', then v_1 is close to the edge uv in G'. Let us now consider the case that F is a facial cycle both in G and G'. As $v_1 \notin \{u, v\}$, the degree of v_1 is three both in G and G'. Let x_1 be the neighbor of v_1 distinct from its neighbors on F. Note that x_1 is small in G.

Suppose first that γ is a tetragram or a hexagram. Observe that the removal of the edge uv from G' must decrease the degree of some of the vertices affecting the security of γ, change the length of one of the faces incident with the edge v_1x_1 affecting the security of γ, or destroy a path affecting its safety. Therefore, if $\{u, v\} \cap (V(F) \cup \{x_1\}) = \emptyset$ and v_1 is not close to the edge uv in G', then u or v is a small neighbor of x_1 in G that is big in G'. We conclude that v_1 is close to u or v in G.

Let us now consider the case that $\gamma = (v_1, v_2, \ldots, v_5)$ is a decagram or a pentagram. As γ is secure in G or G', all the vertices of γ are small in G. If $\{u, v\} \cap V(F) \neq \emptyset$, then v_1 is close to u or v in G, and thus assume that this is not the case. It follows that the degree of v_i is the same in G and G', for $1 \leq i \leq 5$; in particular, $\deg(v_i) = 3$ for $1 \leq i \leq 4$. Let x_i be the neighbor of v_i not incident with F, for $1 \leq i \leq 4$. Similarly, we conclude that x_1 and x_3 are small in G, and if γ is a pentagram, then x_2 and x_4 are small in G. If $\{u, v\} \cap \{x_1, x_3\} \neq \emptyset$, or γ is a pentagram and $\{u, v\} \cap \{x_2, x_4\} \neq \emptyset$, then u or v is close to v_1 in G. If this is not the case, then the removal or addition of uv cannot affect the security of γ if γ is a decagram.

We are left with the case when γ is a pentagram, and $\{u, v\} \cap \{x_1, x_2, x_3, x_4\} = \emptyset$. It follows that the neighborhoods of x_2, x_3, x_4 and v_5 are the same in G and in G'. As γ is secure in G or G', all neighbors of v_5 or x_2, and all neighbors of x_3 or x_4 are small in G. As γ is not secure both in G and G', the removal of uv

- destroys a path of length at most three between x_2 and v_5 or between x_3 and x_4, or
- removes an edge incident with the common neighbor y of x_3 and x_4, thus making the 5-cycle $x_3v_3v_4x_4y$ facial, or
- decreases the degree of a neighbor of x_2, x_3, x_4 or v_5, making it small in G.

In all the cases, u or v is a small neighbor of x_2, x_3, x_4 or v_5, and hence it is close to v_1 in G. □

The next theorem will serve as the basis for the proof of correctness of our algorithm. We defer its proof until the next section.

Theorem 17. *Every non-null triangle-free planar graph has a secure multigram.*

We are now ready to prove Theorem 11, assuming Theorem 17.

Algorithm 18. *There is an algorithm with the following specifications:*
Input: *A triangle-free planar graph.*
Output: *A proper* 3*-coloring of* G*.*
Running time: $O(|V(G)|)$.

Description. Using a linear-time planarity algorithm that actually outputs an embedding, such as [60] or [78], we can assume that G is a plane graph. The algorithm is recursive. Throughout the execution of the algorithm we will maintain a list L that will include the pivots of all secure multigrams in G, and possibly other vertices as well. We initialize the list L to consist of all vertices of G of degree at most three.

At a general step of the algorithm we remove a vertex v from L. There is such a vertex by Theorem 17 and the requirement that L include the pivots of all secure multigrams. We check if G has a secure multigram with pivot v. This can be performed in constant time by Lemma 15. If no such multigram exists, then we go to the next iteration. Otherwise, we let γ be one such multigram, and let G' be the γ-reduction of G. By Lemma 14 the graph G' is triangle-free and can be constructed in constant time by adding and deleting bounded number of edges, and removing a bounded number of isolated vertices. For every edge uv that was deleted or added during the construction of G' we add to L all vertices that are close to u or v, or to the edge uv in G or G'. By Lemma 16 this will guarantee that L will include the pivots of all secure multigrams in G'. We apply the algorithm recursively to G', and convert the resulting 3-coloring of G' to one of G using Lemma 14. Since the number of vertices added to L is proportional to the number of vertices removed from G we deduce that the number of vertices added to L (counting multiplicity) is at most linear in the number of vertices of G. Thus the running time is $O(|V(G)|)$, as claimed.

Algorithm 18 has the following extension.

Algorithm 19. *There is an algorithm with the following specifications:*
Input: *A triangle-free plane graph* G*, a facial cycle* C *in* G *of length at most five, and a proper* 3*-coloring* ϕ *of* C*.*
Output: *A proper* 3*-coloring of* G *whose restriction to* $V(C)$ *is equal to* ϕ*.*
Running time: $O(|V(G)|)$.

Description. The description is exactly the same, except that we replace "secure" by "C-secure" and appeal to Lemma 20 rather than Theorem 17.

3.4 Proof of correctness

In this section we prove Theorem 17, thereby completing the proof of correctness of the algorithm from the previous section. The theorem will follow from the next lemma. If xy is an edge in a plane graph, and f is a face of G incident with y but not with the edge xy, then we say that f *is opposite to* xy. Let us emphasize that this notion is not symmetric in x, y.

Lemma 20. *Let G be a connected triangle-free plane graph and let f_0 be its outer face. Assume that f_0 is bounded by a cycle C of length at most six, $V(G) \neq V(C)$, and if C has length six, then $|V(G) - V(C)| \geq 2$. Then G contains a C-secure multigram.*

Proof. Suppose for a contradiction that the lemma is false, and let G be a counterexample with $|E(G)|$ minimum. We first establish the following claim.

(1) *If $K \neq C$ is a cycle in G of length at most six, then K bounds a face, or K has length six and the open disk bounded by K contains at most one vertex.*

To prove (1) let K be as stated, and let G' be the subgraph of G consisting of all vertices and edges that belong to the closed disk bounded by K. If K does not satisfy the conclusion of (1), then G' and K satisfy assumptions of Lemma 20. From the induction hypothesis applied to G' and K we deduce that G' has a K-secure multigram. However, every K-secure multigram in G' is a C-secure multigram in G.

It follows from (1) that C is an induced cycle and that every tetragram in G is safe.

We assign charges to vertices and faces of G as follows. Initially, a vertex v will receive a charge of $9\deg(v) - 36$ if $v \notin V(C)$, and $8\deg(v) - 19$ otherwise. The outer face f_0 will receive a charge of zero, and every other face f of length ℓ will receive a charge of $9\ell - 36$. By Euler's formula the sum of the charges is equal to

$$\begin{aligned}
& \sum_{v \notin V(C)} 9(\deg(v) - 4) + \sum_{v \in V(C)} (8\deg(v) - 19) + \sum_{f \neq f_0} 9(\operatorname{size}(f) - 4) \\
= \; & \sum_{v \in V(G)} 9(\deg(v) - 4) + \sum_{f} 9(\operatorname{size}(f) - 4) - \sum_{v \in V(C)} \deg(v) + 8|V(C)| + 36 \\
= \; & 8|V(C)| - \sum_{v \in V(C)} \deg(v) - 36 \leq -1,
\end{aligned}$$

because all vertices of C have degree at least two, and at least one has degree at least three by hypothesis. Furthermore,

(2) *if at least k vertices of C have degree at least three, then the sum of the charges is at most $-k$.*

We now redistribute the charges according to the following rules. The new charge thus obtained will be referred to as the *final* charge. We need a definition first. Let $f \neq f_0$ be a face of G incident with a vertex $v \in V(C)$. If there exist two consecutive edges in the boundary of f such that both are incident with v and neither belongs to C, then we say that f is a *v-interior face*. The rules are:

(A) every face other than f_0 sends three units of charge to every incident vertex v such that either $v \in V(C)$ and v has degree two in G, or $v \notin V(C)$ and v has degree exactly three,

(B) every big vertex not on C sends three units to each incident face, and four units to each 4-face that shares an edge with C,

(C) every vertex $v \in V(C)$ sends three units to every v-interior face,

(D) if $x \in V(G)$ is C-forbidden, and y is a C-admissible neighbor of x of degree three, then x sends three units to the unique face opposite to xy, and one unit to the face opposite to yz for every C-admissible neighbor z of y of degree three,

(E) every C-forbidden vertex sends five units to every C-admissible neighbor of degree at least four,

(F) for every C-admissible vertex y of degree at least four that has a C-forbidden neighbor we select a C-forbidden neighbor x of y and let y send one unit to each face opposite to xy, and one unit to the face opposite to yz for every C-admissible neighbor z of y of degree three.

Since G does not satisfy the conclusion of the theorem, it follows that every vertex of G has degree at least two, and every vertex of degree exactly two belongs to C. With these facts in mind we now show that every vertex has non-negative charge. To that end let $v \in V(G)$ have degree d, and assume first that v is C-admissible. If $d = 3$, then it starts out with a charge of -9 and receives three from each incident face by rule (A) for a final total of zero. If $d \geq 4$, then v starts out with a charge of $9d - 36 \geq 0$. If v has no C-forbidden neighbor, then it sends no charge and the claim holds. Thus we may assume that v has a C-forbidden neighbor, and let x be such neighbor selected by rule (F). Then v receives at least five units by rule (E), and sends at most $2d - 3$ by rule (F) for a total of at least $9d - 36 + 5 - (2d - 3) = 7d - 28 \geq 0$. Thus every C-admissible vertex has non-negative final charge. If v is big, but does not belong to C, then it sends only by rules (B), (D) or (E). It sends at most $3d$ using the first clause of rule (B), at most 24 using the second clause of rule (B) and at most $5d$ using rules (D) or (E) for a total final charge of at least $9d - 36 - 3d - 24 - 5d \geq 0$, because $d \geq 60$. Thus we may assume that $v \in V(C)$. Then v starts out with a charge of $8d - 19$ and sends a net total of $3(d - 3)$ using rules (A) or (C) (if $d = 2$, then v

receives 3 by rule (A); and otherwise it sends $3(d-3)$ by rule (C)) and it sends $5(d-2)$ using rule (D) or (E) for a total of $8d-19-3(d-3)-5(d-2)=0$. This proves our claim that the final charge of every vertex is non-negative.

It also follows that every face of length $\ell \geq 6$ has non-negative final charge, for every face sends at most three units to each incident vertex and only to those vertices by rule (A); thus the final charge is at most $9\ell - 36 - 3\ell \geq 0$.

We have thus shown that G has a face f of length at most five with strictly negative final charge. Clearly f is not the outer face.

(3) *No vertex incident with f has degree two.*

To prove (3) suppose for a contradiction that a vertex v of degree two is incident with f. Thus v and the two edges incident with v and f belong to C. Since $G \neq C$ and f has length at most five we deduce that at least two vertices incident with f are incident with C and have degree at least three. Those two vertices do not receive any charge from f, and hence f has length four, because it has negative charge.

We deduce that f is bounded by a cycle $u_1u_2u_3u_4$, where u_1, u_2, u_3 are consecutive vertices of C, and u_2 has degree two. It follows that $u_4 \notin V(C)$, because C is induced. Since f has negative charge it does not receive charge by rule (B), and hence u_4 is small and C-admissible. Let C' be the cycle obtained from C by replacing the vertex u_2 by u_4; note that $|V(C')| = |V(C)| \leq 6$. As u_4 has degree greater than two, C' does not bound a face, hence it follows from (1) that $|V(C')| = 6$ and the open disk bounded by C' contains at most one vertex. Therefore, it contains exactly one, because $|V(G)| - V(C)| \geq 2$. Let that vertex be v_4; then the remaining vertices of C can be numbered v_1, v_2, v_3 so that the cycle C is $u_1u_2u_3v_1v_2v_3$ and v_4 is adjacent to v_1, v_3 and u_4. Then (u_4, u_1, u_2, u_3) is a C-secure tetragram, contrary to the assumption that G is a counterexample to the theorem. This proves (3).

Let uv be an edge of G such that f is opposite to uv. Let us say that v is a *sink* if v has degree three and both u and v are C-admissible. Let us say that v is a *source* if either $v \notin V(C)$ and v is big, or $v \in V(C)$ and f is v-interior. Since v does not have degree two by (3) we deduce that v is a sink if and only if it has degree three and receives three units of charge from f by rule (A) and f does not receive three units by rule (D) from u. Likewise, the vertex v is a source if and only if it sends three units to f by the first clause of rule (B) or by rule (C). Let s be the number of sources, and t the number of sinks. Thus the charge of f is at least $9+3s-3t$ if f has length five and at least $3s-3t$ if f has length four.

Let us assume now that f has length five, and let $v_1, v_2, \ldots, v_5$ be the incident vertices, listed in order. Since f has negative charge, at least four of the five incident vertices are sinks, and so we may assume that v_1, v_2, v_3, v_4 are sinks. Thus $\gamma = (v_1, v_2, \ldots, v_5)$ is a pentagram. For $i = 1, 2, 3, 4$ let x_i be the neighbor of v_i distinct from v_{i-1} and v_{i+1} (where $v_0 = v_5$). From (1) and the fact that G

has no C-secure tetragram we deduce that the vertices x_1, x_2, x_3, x_4 are distinct and pairwise non-adjacent. If v_5 is a C-admissible vertex of degree three, then it follows from (1) that γ is C-secure decagram—otherwise, if there is a path of length two between x_1 and x_3, then consider the 6-cycle $K = x_1v_1v_2v_3x_3y$. By (1) the open disk bounded by K includes at most one vertex of G. It follows that v_4 and v_5 are not inside the disk; thus either $y = x_2$ or x_2 is inside the disk. In either case, it follows that x_2 is adjacent to x_1 and x_3, a contradiction. Thus v_5 is either not C-admissible, or has degree at least four.

Therefore, v_5 is not a sink, and hence the final charge of f is at least -3. It follows that v_5 is not a source, which in turn implies that v_5 is C-admissible (because v_1 and v_4 are C-admissible), and hence has degree at least four. We claim that γ is a safe pentagram. If there exists a path P in $G \setminus \{v_1, v_2, v_3, v_4\}$ of length at most three with ends x_2 and v_5, then P can be completed to a cycle K using the path $v_5v_1v_2x_2$. By (1) we conclude that this cycle bounds an open disk that contains at most one vertex, and it follows that x_1 is adjacent to x_2, which is a contradiction. In order to complete the proof that γ is safe it suffices to consider a path in $G \setminus \{v_1, v_2, v_3, v_4\}$ of length at most three with ends x_3 and x_4. This path can be completed via the path $x_4v_4v_3x_3$ to a cycle K'. Since v_3 and v_4 have degree three, and x_3 is not adjacent to x_4, we deduce from (1) that K' is a facial cycle. Since x_3 is not adjacent to x_4 we may assume for a contradiction that K' has length six; let its vertices in order be $x_3v_3v_4x_4ab$. Then $(v_4, v_3, x_3, b, a, x_4)$ is a C-secure hexagram in G, a contradiction. This proves our claim that γ is a safe pentagram. By symmetry the pentagram $(v_4, v_3, v_2, v_1, v_5)$ is also safe. We have already established that the vertices $v_1, v_2, \ldots, v_5, x_1, x_2, x_3, x_4$ are C-admissible. If x_i has a C-forbidden neighbor for some $i \in \{1, 2, 3, 4\}$, then f receives one unit of charge either from that neighbor by rule (D) if x_i has degree three, or from x_i by rule (F) otherwise. Since the degree of v_5 is greater than three, if v_5 has a C-forbidden neighbor, then it sends one unit of charge to f by rule (F). Thus at most two vertices among v_5, x_1, x_2, x_3, x_4 have a C-forbidden neighbor, and hence it follows that either γ, or $(v_4, v_3, v_2, v_1, v_5)$ is a C-secure pentagram, a contradiction.

Thus we have shown that f has length four. Let v_1, v_2, v_3, v_4 be the incident vertices listed in order. Let us recall that every tetragram is safe. Since f has negative charge at least $3s - 3t$, we may assume that v_1 is a sink and v_3 is not a source. Let $\gamma = (v_1, v_2, v_3, v_4)$. Since v_3 is not a source and γ is not a C-secure tetragram, $v_3 \in V(C)$ and f is not v_3-interior. Then, (3) implies that exactly one of v_2v_3, v_3v_4 is an edge of C, and hence we may assume the latter. In particular, $v_2 \notin V(C)$. If v_2 is a sink, then the charge of f is at least -6, otherwise it is at least -3.

Let v be the neighbor of v_1 other than v_2 and v_4. Since v_1 is a sink, v is C-admissible. If v has no C-forbidden neighbor, then γ is a C-secure tetragram, a contradiction. Thus v has a C-forbidden neighbor u. Suppose first that $u \notin V(C)$; hence u is big and f receives 4 units of charge from u by rule (B). As the charge of

f is negative, we conclude that v_2 is a sink. Let v' be the neighbor of v_2 distinct from v_1 and v_3. Since γ is not a C-secure tetragram, v' has a C-forbidden neighbor u'. However, by rules (D) and (F), f receives one unit of charge from each of u and u' (or twice one unit of charge from u, if $u = u'$), making its final charge nonnegative.

We conclude that every C-forbidden neighbor of v belongs to C. Since rules (D) or (F) still apply, we obtain

(4) *each 4-face f' that shares an edge with C has final charge at least $-2t$, where $t \in \{1, 2\}$ is the number of sinks of f'.*

As γ is not a C-secure tetragram, at least one C-forbidden neighbor u of v is adjacent to neither v_2 nor v_4. Let C, C_1, C_2 be the three cycles in the graph consisting of C and the path uvv_1v_4, numbered so that v_3 belongs to C_2. We claim that C_2 has length at least seven. Note that v_2 lies in the open disk bounded by C_2; thus by (1) the cycle C_2 has length at least six. Assume that C_2 has length exactly six. By (1), the open disk it bounds contains v_2 and no other vertex of G. It follows that v_2 has degree three and is adjacent to u, which contradicts the choice of u.

It follows that C_2 has length at least seven, and hence C_1 has length at most five, and by the choice of u, it has length exactly five. By (1), C_1 bounds a face. Thus u and v_4 have a common neighbor of degree two on C, say z. Let $f(\gamma)$ denote the face bounded by C_1. Let us call each tetragram γ' for which $f(\gamma')$ is defined *bad*. Note that a face f may correspond to two bad tetragrams (if v_2 is a sink, then (v_2, v_1, v_4, v_3) is a bad tetragram as well). At this point, we have proved that only the faces of G that correspond to bad tetragrams have negative final charge. Additionally, if f corresponds to one bad tetragram, then its final charge is at least -2 and if it corresponds to two bad tetragrams, then its final charge is at least -4. Let b be the number of bad tetragrams in G.

Let us consider the bad tetragram γ. The face $f(\gamma)$ starts out with a charge of 9, sends three units to each of v_1, v, z by rule (A), and receives one either from v_3 by rule (D), or from v_2 by rule (F) for a total of $+1$ (v_2 is small, as γ contains more sinks than sources). Also, if there exists a tetragram γ' distinct from γ such that $f(\gamma) = f(\gamma')$, then the final charge of $f(\gamma)$ is at least $+2$. It follows that the total charge of G is at least $-b$.

Since v_3, v_4 and u have degree at least three, by (2) the total charge of G is at most -3, and so $b \geq 3$. However, since $b > 2$, there must be a bad tetragram other than γ and (v_2, v_1, v_4, v_3), giving at least one more vertex of C of degree at least three. Therefore, the final charge of G is at most -4 by (2), and hence $b \geq 4$. Let u' be the unique neighbor of u in $C \backslash z$. Since $b \geq 4$ it follows by inspection that v_3v_4 and uu' are the only edges of C that belong to a bad tetragram, and that G has a vertex v' of degree three with neighbors v, v_2, u'. It follows that (v, v', v_2, v_1) is a C-secure octagram, as desired. $\square$

Proof of Theorem 17. Let G be a triangle-free planar graph. We may assume that G is actually drawn in the plane. If G has a vertex of degree two or less, then it has a secure monogram, and so we may assume that G has minimum degree at least three. It follows that G has a facial cycle C of length at most five. Let H be the component of G containing C. We may assume that C bounds the outer face of H. Since H has minimum degree at least three it follows that $V(H) - V(C) \neq \emptyset$. By Lemma 20 H has a C-secure multigram; but any C-secure multigram in H is a secure multigram in G, as desired. $\square$

Chapter 4

Coloring planar graphs with one (≤ 4)-cycle[1]

Thomassen [71] proved that the number of vertices of every 4-critical graph of girth five embedded in a surface Σ is bounded by a constant C depending only on the genus g of Σ. However, the dependence is double exponential. In Dvořák et al. [27], we improve this dependence to linear. In order to deal with degenerate cases of the claim, we need to consider R-critical (for 3-colorability) plane graphs that contain at most one cycle T of length at most four different from R, and where R is a facial cycle of length at most 6. Grötzsch's theorem admits the following strengthening, which follows from almost all known proofs of the theorem.

Theorem 21. *There is no R-critical triangle-free plane graph G, where R is a cycle in G of length at most five.*

A strengthening of this result admitting one triangle was obtained by Aksionov [2]. Combined with [41], the result can be stated as follows.

Theorem 22. *Let G be a plane graph with a facial cycle R of length at most five and at most one triangle T distinct from R. If G is R-critical, then R has length exactly five, T shares at least one edge with R and all faces of G distinct from T and R have length exactly four.*

In particular, if G is a plane graph with a facial cycle R of length at most six and at most one (≤ 4)-cycle T and G is R-critical, then R has length exactly 6. Furthermore, the characterization of 3-colorability of triangle-free planar graphs with a precolored 6-face by Gimbel and Thomassen [41] implies that T exists and it is a triangle. The main result of this chapter considers this remaining case. The two outcomes are illustrated in Figure 4.1.

[1]The results of this chapter are based on Dvořák et al. [25].

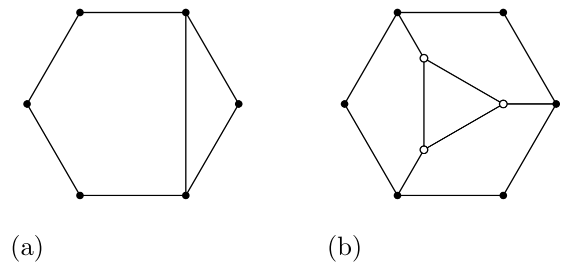


Figure 4.1: Critical graphs with a precolored 6-cycle and one triangle.

Theorem 23. *Let G be a plane graph with a facial cycle R of length at most six, let T be a triangle in G, and assume that every cycle in G other than T and R has length at least five. Let ϕ be a 3-coloring of R that does not extend to a 3-coloring of G. Then C has length exactly six and either*

(a) $\phi(u) = \phi(v)$ *for two distinct vertices* $u, v \in V(C)$ *that are adjacent in* G, *or*

(b) $\phi(u_1) = \phi(u_2) = \phi(u_3)$ *for three pairwise distinct vertices* $u_1, u_2, u_3 \in V(C)$, *where each* u_i *is adjacent to a different vertex of* T.

We will need a characterization of plane graphs of girth at least five with a precolored face of length at most 11, proved by Thomassen [71] and Walls [77].

Theorem 24. *Let H be a plane graph of girth at least five, and let C be a facial cycle in H of length $k \leq 11$. If H is C-critical, then*

(a) $k \geq 8$, $V(H) = V(C)$ *and* C *is not induced, or*

(b) $k \geq 9$, $H - V(C)$ *is a tree with at most* $k - 8$ *vertices, and every vertex of* $V(H) - V(C)$ *has degree three in* G, *or*

(c) $k \geq 10$ *and* $H - V(C)$ *is a connected graph with at most* $k - 5$ *vertices containing exactly one cycle, and the length of this cycle is five.*

4.1 Graphs with one triangle

To prove Theorem 23 we prove, for the sake of the inductive argument, the following slightly more general result. Theorem 23 will be an immediate corollary.

Theorem 25. *Let G be a plane graph with outer cycle R of length at most six and assume that*

(∗) *there exists a face* f_0 *of* G *such that every cycle in* G *of length at most four bounds a closed disk containing* f_0.

If G is R-critical, then R has length exactly six and G is isomorphic to one of the graphs depicted in Figure 4.1.

Proof. Let G be as stated, and suppose for a contradiction that it is not isomorphic to either of the two graphs depicted in Figure 4.1. By Gimbel and Thomassen [41], the graph G has a triangle T. We may assume that G is minimal in the sense that the theorem holds for every graph with fewer vertices. A vertex $v \in V(G) - V(R)$ will be called *internal*. The R-criticality of G implies that

(2) *every internal vertex of G has degree at least three.*

If C is a cycle in G, then by $\mathrm{ins}(C)$ we denote the subgraph of G consisting of all vertices and edges drawn in the closed disk bounded by C. Let us recall that by Theorem 8,

(3) *for every non-facial cycle C in G, $\mathrm{ins}(C)$ is a C-critical graph.*

It follows from $(*)$, (3) and Theorem 22 that

(4) *T is the only cycle in G of length at most four.*

Next we constrain cycles in G of length at most seven:

(5) *Let $C \neq R$ be a cycle in G of length at most seven that does not bound a face. Then C has length at least six, and the closed disk bounded by C includes T.*

To prove (5) let C be as stated. By the minimality of G and (3) we deduce that C has length at least six. If T is not contained in the closed disk Δ bounded by C, then $(*)$ implies that $\mathrm{ins}(C)$ has girth at least five, contrary to (3) and Theorem 24. Thus T is contained in Δ, and (5) follows.

It follows that T bounds the face f_0. The same argument implies the following two claims. To prove the second one, Theorem 24 is applied to a graph obtained from G by splitting repeated vertices of C so that C will become a cycle in the new graph.

(6) *Let $C \neq R$ be a cycle in G of length six that does not bound a face. Then $\mathrm{ins}(C)$ is isomorphic to one of the graphs depicted in Figure 4.1.*

(7) *Let $C \neq R$ be a closed walk in G of length $k \leq 11$ bounding an open disk Δ disjoint from T, and let H be the subgraph of G consisting of vertices and edges drawn in the closure of Δ. Then H satisfies the conclusion of Theorem 24.*

From (4) and Theorem 22 it follows that

(8) *R has length six,*

and since every cycle of length at most five bounds a face by (5) we deduce that

(9) *the graph G has no subgraph H with outer face R such that H is isomorphic to either of the two graphs depicted in Figure 4.1; in particular, R is induced.*

Next we claim that

(10) *every internal vertex has at most one neighbor in $V(R)$.*

To prove (10) suppose for a contradiction that an internal vertex v_2 has two neighbors $v_1, v_3 \in V(R)$. Let P denote the path $v_1v_2v_3$, and let R, C_1, C_2 be the three cycles of $R \cup P$. By (4) either one of C_1, C_2 is T and the other has length seven, or C_1, C_2 both have length five. In either case it follows from (5) that C_1, C_2 both bound faces of G, and hence v_2 has degree two, contrary to (2). This proves (10).

(11) *The cycle T is disjoint from R.*

To prove (11) suppose for a contradiction that $v \in V(T) \cap V(R)$. By (9) and (10) v is the only vertex of $T \cap R$. The graph $T \cup R$ has a face bounded by a walk C of length nine. By (7) at least one of the vertices of $V(T) \setminus V(R)$ has degree two, contrary to (2). This proves (11).

Let us fix an orientation of the plane, and let $T = t_1t_2t_3$ and $R = r_1r_2 \dots r_6$ be numbered in clockwise cyclic order according to the drawing of G.

(12) *G has at most one edge joining T to R.*

To prove (12) suppose that say $t_1r_1, t_2r_i \in E(G)$ for some $i \in \{1, \dots, 6\}$. By (4) we have $3 \leq i \leq 5$. Let $C_2 = r_1t_1t_3t_2r_ir_{i+1} \dots r_6$. As t_3 has degree at least three, C_2 does not bound a face; thus C_2 has length at least eight by (5), and we conclude that $i = 3$. Thus C_2 has length exactly eight, and hence by (7) $\text{ins}(C_2)$ consists of C_2 and at most one chord. Since t_3 has degree at least three, this chord exists and joins t_3 with r_5, and hence G has a subgraph isomorphic to the graph depicted in Figure 4.1(b), contrary to (9). This proves (12).

(13) *G does not contain a 5-face incident only with internal vertices of degree three.*

To prove (13) suppose for a contradiction that G contains such a 5-face $C = v_1v_2v_3v_4v_5$. For $1 \leq i \leq 5$, let x_i be the neighbor of v_i not belonging to C (each v_i has such a neighbor, because T and C bound faces by (5) and each vertex of C has degree three). Since T is disjoint from R by (11) and G contains no 4-cycles by (4), it follows that at most three of the vertices $x_1, \dots, x_5$ belong to R. Without loss of generality we may assume that x_1 is internal. Note also that $x_1 \notin \{x_3, x_4\}$, as G does not contain a 4-cycle. By the symmetry between x_3 and x_4 we may assume that if x_3 is adjacent to a vertex of R, then so is x_4. Let G' be the graph obtained from $G - V(C)$ by adding the edge x_1x_3. Observe that every coloring of G' extends to a coloring of G: given a coloring of G', every vertex in C has a list of two available colors, and the lists of v_1 and v_3 are different. We conclude that ϕ does not extend to a 3-coloring of G'.

Our next objective is to show that G' satisfies $(*)$. To that end let $K' \neq T$ be a cycle in G' of length at most four. Then K' includes the edge x_1x_3 by (4). Consider the cycle K in G obtained from K' by replacing the edge x_1x_3 by the path $P = x_1v_1v_2v_3x_3$. Note that K has length at most seven, and that it does not bound a face (since v_1 and v_2 have neighbors drawn on opposite sides of P). Thus by (5) T is a subgraph of $\mathrm{ins}(K)$. We conclude that G' satisfies $(*)$, as desired.

Let G'' be a minimal subgraph of G' such that R is a subgraph of G'' and every 3-coloring of R that extends to a 3-coloring of G'' extends to a 3-coloring of G'. Note that $G'' \neq R$, since ϕ does not extend to a 3-coloring of G''. Then G'' is R-critical, and the minimality of G implies that G'' is isomorphic to one of the graphs depicted in Figure 4.1. But R is an induced subgraph of G by (9) and x_1 is internal, and hence G'' is isomorphic to the graph of Figure 4.1(b). Let L' be the triangle of G''. By (12) we have $L' \neq T$, and hence x_1x_3 is an edge of L'. Let t be the third vertex of L'. We may assume that x_1 is adjacent to r_1, x_3 is adjacent to r_3 and t is adjacent to r_5, where the adjacencies take place in G, G' and G''. Let D' be the face boundary of the 5-face of G'' incident with the edge x_1x_3, and let D be the 8-cycle of G obtained from D' by replacing the edge x_1x_3 by the path P. Let L be the 6-cycle in G obtained from L' by replacing the edge x_1x_3 by the path P. By (6) T lies in the closed disk bounded by L, and since t is adjacent to r_5 it follows that $\mathrm{ins}(D)$ includes no cycle of length at most four. By (7) no vertex of G lies in the open disk bounded by D, and hence v_4 and v_5 lie in the open disk bounded by L. Since G has no 4-cycles we deduce that $x_4 \notin \{x_1, t\}$, and $x_3 \neq x_4$, for otherwise $T = x_3v_4v_3$ and the cycle $x_3tx_1v_1v_5v_4$ includes the edge v_5x_5 in its inside but not T, contrary to (5). Since x_3 is adjacent to r_3, the choice of x_3 implies that x_4 is adjacent to a vertex of R, contrary to the planarity of G. This proves (13).

(14) *The distance between R and T is at least two.*

To prove (14) suppose for a contradiction that the distance between R and T is at most one. Then it is exactly one by (11), and so we may assume that say $t_1r_1 \in E(G)$. Let C be the closed walk in G of length 11 obtained by traversing $R \cup T$ and the edge t_1r_1 twice, and let H be the subgraph of G consisting of all vertices and edges of G drawn in the closure of open disk bounded by C. By (7) the graph H satisfies (a), (b) or (c) of Theorem 24. If it satisfies (a), then by (2) applied to t_2 and t_3 the graph H consists of $R \cup T$ and two edges, one incident with t_2 and the other with t_3. It follows that the graph of Figure 4.1(b) is isomorphic to a subgraph of G, contrary to (9). If H satisfies (b), then $H - V(C)$ is a tree X with at most three vertices, each of degree three. Both t_2 and t_3 have a neighbor in X, and hence $X \cup \{t_2, t_3\}$ includes the vertex-set of a 5-cycle, contrary to (13). Finally, H cannot satisfy (c) by (13). This proves (14).

(15) *No two vertices of degree two are adjacent in G*

.

To prove (15) suppose for a contradiction that G has two adjacent vertices of degree two. By (2) they belong to R, and so we may assume that say r_2 and r_3

have degree two. The edge r_2r_3 is not contained in any 5-cycle, as otherwise R would have a chord or an internal vertex would have two neighbors in R, contrary to (9) and (10). Let G' be the graph obtained from G by contracting the edge r_2r_3, and let R' be the corresponding outer cycle of G'. Then G' has no cycle of length at most 4 distinct from T. Furthermore, every 3-coloring ψ of R can be modified to a 3-coloring ψ' of R' such that $\psi(r_i) = \psi(r'_i)$ for $i \in \{1, 4, 5, 6\}$, and ψ extends to G if and only if ψ' extends to a 3-coloring of G'. It follows that G' is R'-critical, contrary to Theorem 22. This proves (15).

(16) *For every path $v_1v_2v_3v_4$ with v_2 and v_3 internal and $v_1, v_4 \in V(R)$ there exists $r \in V(R)$ such that $v_1v_2v_3v_4r$ bounds a 5-face.*

To prove (16) consider a path $P = v_1v_2v_3v_4$ with v_2 and v_3 internal and $v_1, v_4 \in V(R)$, and let C_1 and C_2 be the cycles of $R \cup P$ other than R such that T lies in the closed disk bounded by C_1. Since T is disjoint from R, C_1 does not bound a face, and hence it has length at least six by (5). Thus C_2 has length at most six, and hence bounds a face by (5), and therefore has length at most five by (15). This proves (16).

(17) *All faces of G distinct from R and T have length exactly five.*

To prove (17) consider a face $v_1v_2 \dots v_k$ of length $k \geq 6$ in G. By (10) we may assume without loss of generality that v_2 and v_3 are internal. Furthermore, if $k = 6$, then not all of v_1, v_4, v_5 and v_6 may belong to R, by (15), and hence, by symmetry, we may assume that either v_4 or v_6 is internal. Let $W = \{v_2, v_k\}$ if $k > 6$ and $W = \{v_2, v_4, v_6\}$ if $k = 6$. Let G' be the graph obtained from G by identifying the vertices of W to a new vertex w and deleting all resulting parallel edges. Thus $E(G') \subseteq E(G)$. By (2) and (5) the vertices of W are pairwise non-adjacent, thus the identifications created no loops. Observe that every 3-coloring ψ of G' gives rise to a 3-coloring of G (color the vertices of W using $\psi(w)$). It follows that some 3-coloring of R does not extend to a 3-coloring of G'. Let G'' be a minimal subgraph of G' such that R is a subgraph of G'' and every 3-coloring of R that extends to a 3-coloring of G'' also extends to a 3-coloring of G'; then G'' is R-critical.

Next we show that G'' satisfies $(*)$. As a first step we prove that G'' does not have a triangle other than T. To that end let $K' \neq T$ be a triangle in G''. Recall that $E(G'') \subseteq E(G)$. Two of the edges of K' are incident in G with distinct vertices $w_1, w_2 \in W$. Let K be the corresponding 5-cycle in G, obtained from K' by replacing w with the shortest path between w_1 and w_2 in G. Observe that K does not bound a face in G, contrary to (5). Therefore, G'' does not have a triangle distinct from T. Consider now a 4-cycle L' in G''. The corresponding cycle L in G (constructed in the same way as K) has length six. As L does not bound a face we can apply (6) to the cycle L. By the first result of this paragraph it follows that T is contained in the closed disk bounded by L', and hence G'' satisfies $(*)$.

Since G'' has fewer vertices than G, G'' is one of the graphs drawn in Figure 4.1. Furthermore, the first result of the previous paragraph implies that T is the unique triangle of G''. However, this implies that the distance between T and R in G is at most one, contradicting (14). This proves (17).

(18) *Every* 5*-face incident with four internal vertices of degree three is incident with an edge of* T*.*

To prove (18) suppose for a contradiction that G contains a 5-face $f = v_1v_2v_3v_4v_5$, where v_1, v_3, v_4 and v_5 are internal vertices of degree three, and that f does not share an edge with T. By (13) the degree of v_2 is at least four. Let x_1, x_3, x_4 and x_5 be the neighbors of v_1, v_3, v_4 and v_5, respectively, outside of $\{v_1, v_2, \ldots, v_5\}$. If $v_2 \in V(R)$, then x_3 is internal since v_3 has only one neighbor in R by (10), and x_4 and x_5 are internal by (16) and (2) applied to v_1 and v_3. Also, not all of x_1, x_3, x_4 and x_5 belong to R, as T does not share an edge with f. Thus, we may assume that at least one of x_3 and x_4 and at least one of v_2 and x_5 is internal. As f does not share an edge with T, the vertices v_2, x_3, x_4 and x_5 are distinct and pairwise non-adjacent. Let G' be the graph obtained from $G - \{v_1, v_3, v_4, v_5\}$ by identifying v_2 with x_5 to a new vertex w_1 and x_3 with x_4 to a new vertex w_2. Note that any coloring ψ of G' extends to a coloring of G: Give v_2 and x_5 the color $c_1 = \psi(w_1)$ and x_3 and x_4 the color $c_2 = \psi(w_2)$. If $c_1 = c_2$, then color the vertices of $V(F) \setminus \{v_2\}$ in the order v_1, v_5, v_4 and v_3. Similarly, if $c_1 = \psi(x_1)$, then color the vertices v_3, v_4, v_5 and v_1 in order. Finally, if $\psi(x_1) \neq c_1 \neq c_2$, then color v_1 by c_2, v_3 by $\psi(x_1)$, v_4 by c_1 and choose a color for v_5 distinct from c_1 and c_2. It follows that some 3-coloring of R does not extend to a 3-coloring of G'. Let G'' be a minimal subgraph of G' such that R is a subgraph of G'' and every 3-coloring of R that extends to a 3-coloring of G'' also extends to a 3-coloring of G'. Then G'' is R-critical.

Next we show that G'' satisfies $(*)$. Consider a cycle K' of G'' of length at most four distinct from T, and let $K \subseteq G$ be the corresponding cycle obtained by replacing w_1 by $v_2v_1v_5x_5$ or w_2 by $x_4v_4v_3x_3$ or both. If we replaced both, then K has length at most 10 and it has two chords v_2v_3 and v_4v_5. Thus one of them must belong to T, contradicting the assumption that f does not share an edge with T. Therefore, we expanded only one vertex in K', and hence $6 \leq |V(K)| \leq 7$. By (17), K does not bound a face. By (5) T is a subgraph of $\mathrm{ins}(K)$, and hence G'' satisfies $(*)$, as claimed.

Since G'' has fewer vertices than G, we conclude that G'' is isomorphic to one of the graphs from Figure 4.1. Let K' be the unique triangle of G''. Using (14), we conclude that $K' \neq T$. Let K be the corresponding cycle of length six in G. By (17) the cycle K does not bound a face. By (6) $\mathrm{ins}(K)$ is isomorphic to one of the graphs drawn in Figure 4.1. By (14) we conclude that G'' cannot be the graph in Figure 4.1(a), and hence G'' is isomorphic to the graph in Figure 4.1(b). We may therefore assume that t_1, t_2, t_3 are adjacent in G'' to r_1, r_5, r_3, respectively, where t_1, t_2, t_3 are the vertices of K'. From the symmetry we may assume that the edge t_1t_3 of K' corresponds to the path t_1abct_3 in K. Now either $\{t_1, a, b, c\}$ or $\{a, b, c, t_3\}$ is equal to $\{v_2, v_1, v_5, x_5\}$ or $\{x_3, v_3, v_4, x_4\}$, and so from the symmetry we may assume that the ordered quadruple (t_1, a, b, c) is equal to one of (v_2, v_1, v_5, x_5), (x_5, v_5, v_1, v_2), (x_3, v_3, v_4, x_4), and (x_4, v_4, v_3, x_3).

We claim that t_1, r_1 and t_3, r_3 are adjacent in G. Indeed, they are adjacent in G'', and so it remains to rule out the case that the edge t_1r_1 or t_3r_3 was created during the identifications that produced G'' from G. This is done by examining the four cases listed above. Let first $(t_1, a, b, c) = (v_2, v_1, v_5, x_5)$. Since $\mathrm{ins}(K)$ is isomorphic to one of the graphs in Figure 4.1 and T shares no edge with f, we deduce that the vertices v_3, v_4 do not belong to $\mathrm{ins}(K)$. Let $i \in \{1, 3\}$, and assume for a contradiction that r_i is not adjacent to t_i in G. But they are adjacent in G'', and so either one of x_3, x_4 is equal to t_i and the other is adjacent to r_i, or one of x_3, x_4 is equal to r_i and the other is adjacent to t_i. Since G is planar, it follows that if $i = 1$, then x_3 is adjacent or equal to t_i, and if $i = 3$, then the vertex x_4 is adjacent or equal to t_i. Thus in the former case the set $\{v_2, v_3, x_3\}$ induces a triangle distinct from T (because T is a subgraph of $\mathrm{ins}(K)$). In the latter case we deduce that x_4 is adjacent to t_3, for otherwise the set $\{t_3, x_4, v_4, v_5, x_5\}$ induces a cycle of length at most four, again a contradiction. Since x_4 is adjacent to t_3, we have $x_3 = r_3$. From (7) applied to the cycle $t_3t_2r_5r_4r_3v_3v_4v_5x_5$ we deduce that $t_3x_5v_5v_4x_4$ is a cycle of internal degree three vertices, contrary to (13). This completes the first of the four cases. The second case is handled similarly. In the last two cases we deduce, using the same argument as above, that v_1, v_2, v_5 do not belong to $\mathrm{ins}(K)$. Thus $\mathrm{ins}(K)$ has two adjacent vertices v_3 and v_4 of degree two, and hence is isomorphic to the graph in Figure 4.1(a). It follows that $t_2 \in V(T)$, contrary to (14). We have thus shown that t_1, r_1 and t_3, r_3 are adjacent in G.

Let D' denote the cycle $r_1t_1abct_3r_3r_2$ of length eight. By (7) it follows that $\mathrm{ins}(D') \backslash E(D')$ includes at most one edge, contrary to the fact that all of $v_1, v_2, \ldots, v_5$ belong to $\mathrm{ins}(D')$. This proves (18).

(19) *The cycle R has no subpath $z_1z_2z_3$ with* $\deg(z_2) = 3$ *and* $\deg(z_1) = \deg(z_3) = 2$.

To prove (19) suppose for a contradiction that say r_2 and r_4 have degree two and r_3 has degree three. By (15) the vertices r_1 and r_5 have degree at least three. By (17) the face incident with r_2 distinct from the outer face is bounded by a 5-cycle, say $r_1r_2r_3yx$. Similarly, there is a face bounded by a 5-cycle $r_3r_4r_5zy$, where $x \neq z$ by (2). Let K be the 6-cycle $r_1xyzr_5r_6$. By (2) and (6) the graph $\mathrm{ins}(K)$ is isomorphic to one of the graphs in Figure 4.1, contrary to (14). This proves (19).

(20) *If R has at least two vertices of degree two, then it has at least one vertex of degree at least four.*

To prove (20) suppose for a contradiction that R has at least two vertices of degree two and the remaining vertices of degree at most three. By (15) and

(19) G has exactly two vertices of degree two, and the distance in R between them is three. We may therefore assume that r_1 and r_4 have degree two, and r_2, r_3, r_5, r_6 have degree three. By (17), G has a 6-cycle $C = x_1x_2x_3x_4x_5x_6$ such that $x_1r_2, x_3r_3, x_4r_5, x_6r_6 \in E(G)$. By (6) the graph $\mathrm{ins}(C)$ is isomorphic to one of the graphs in Figure 4.1. It follows that either x_2 or x_5 has degree two, contrary to (2). This proves (20).

We are now ready to complete the proof of Theorem 23 using the so-called discharging argument. Let us assign charges to the vertices and faces of G in the following way: Each face f of length $|f|$ not bounded by R or T gets a charge of $1 = |f| - 4$, the face bounded by T gets charge $2 = (|V(T)| - 4) + 3$, and the face bounded by R gets charge $0 = (|V(R)| - 4) - 2$. A vertex $v \in V(R)$ of degree two gets charge $-1/3 = (\deg(v) - 4) + 5/3$, a vertex $v \in V(R)$ of degree three gets charge $0 = (\deg(v) - 4) + 1$, and all other vertices v get charge $\deg(v) - 4$.

(21) *The total sum of the charges is at most $-1/3$.*

To prove (21) we deduce from Euler's formula the sum of the charges is at most $\sum_{f\in F(G)}(|f| - 4) + \sum_{v\in V(G)}(\deg(v) - 4) + n_3 + 5n_2/3 + 1 = n_3 + 5n_2/3 - 7$, where n_2 is the number of vertices of degree two and n_3 is the number of vertices of R of degree three in G. By (15) $n_2 \le 3$. By (19), if $n_2 = 3$ then $n_3 = 0$. By (20), if $n_2 = 2$, then $n_3 \le 3$. It follows that $n_3 + 5n_2/3 \le 20/3$, and hence the sum of the charges is at most $-1/3$, as desired. This proves (21).

Let us now redistribute the charge according to the following rules: every face distinct from R sends $1/3$ to each incident vertex of degree two and each incident internal vertex of degree three. The face T sends $1/3$ to each face that shares an edge with it. The final charge of each vertex and of the faces R and T is clearly non-negative. Since the sum of the final charges is equal to the sum of the initial charges, it follows from (21) that G has a face f of strictly negative final charge. The face f has length five; let v_1, v_2, v_3, v_4, v_5 be the incident vertices in order.

If say v_2 were a vertex of degree two, then by (15), v_1 and v_3 would be vertices of R of degree at least three, and hence f would send no charge to them, contrary to the fact that the final charge of f is strictly negative. It follows that all vertices of f have degree at least three, and since the final charge of f is negative, f sends charge to at least four of them. Therefore, at least four of the vertices incident with f are internal and have degree three. The fifth vertex has degree at least four by (13). By (18) f shares an edge with T. However, f sends $1/3$ to each of its incident vertices of degree three and nothing to the fifth vertex, and receives $1/3$ from T; hence the final charge of f is non-negative, a contradiction. $\square$

We are now ready to prove Theorem 23.

Proof of Theorem 23. Let G, T and ϕ be as in Theorem 23. We may assume that R bounds the outer face. Let G' be a minimal subgraph of G such that R is a subgraph of G and ϕ does not extend to a 3-coloring of G'. It follows that G' is R-critical. If T is not a subgraph of G', then we let f_0 be any face of G';

otherwise we let f_0 denote any face of G' that is contained in the closed disk bounded by T. Then G' satisfies hypothesis $(*)$ of Theorem 25. By Theorem 25 the graph G' is isomorphic to one of the graphs depicted in Figure 4.1. If neither of the two outcomes of Theorem 23 holds, then ϕ extends to a 3-coloring of G', a contradiction. $\square$

Chapter 5

3-list-colorability of planar graphs with constraints on (≤ 4)-cycles[1]

Grötzsch [43] proved that every triangle-free planar graph is 3-colorable. For some time, the question whether this result holds in the list coloring setting was open; finally, Voigt [76] found a triangle-free planar graph that is not 3-list-colorable. On the other hand, Thomassen [69] proved that every planar graph of girth at least 5 is 3-list-colorable. Numerous papers study additional conditions that force a triangle-free planar graph to be 3-list-colorable, see e.g. [51, 53, 79, 80, 81, 82].

In particular, let us point out the result of Li [52], strengthening the result of Thomassen [69]: every planar triangle-free graph such that no 4-cycle shares a **vertex** with another 4- or 5-cycle is 3-list-colorable. We further improve this result, only forbidding the 4-cycles sharing an **edge** with other 4- or 5-cycles. Cycles C_1 and C_2 in a graph are *adjacent* if they intersect in a single edge, i.e., if $V(C_1) \cap V(C_2) = \{u, v\}$ for an edge uv.

Theorem 26. *Any planar triangle-free graph without* 4*-cycles adjacent to* 4*- and* 5*-cycles is* 3*-list-colorable.*

Since the subgraph formed by a 4-cycle adjacent to a 4- or 5-cycle contains a 6- or 7-cycle, we obtain the following corollary.

Corollary 27. *Any planar graph without* 3*-,* 6*- and* 7*-cycles is* 3*-list-colorable.*

This strengthens the results of Lidický [53] that planar graphs without 3-, 6-, 7- and 8-cycles are 3-list-colorable, and of Zhang and Xu [80] that planar graphs without 3-, 6-, 7- and 9-cycles are 3-list-colorable. Theorem 26 also implies the result of Lam et al. [51] that planar graphs without 3, 5 and 6-cycles are 3-list-colorable.

[1] The results of this chapter are based on Dvořák et al. [32].

5.1 Proof of Theorem 26

A *path of length* k (or a k*-path*) is a path on $k+1$ vertices. Using the proof technique of precoloring extension developed by Thomassen [69], we show the following generalization of Theorem 26:

Theorem 28. *Let* G *be a triangle-free planar graph without* 4*-cycles adjacent to* 4*- and* 5*-cycles, with the outer face* C*, and* P *a path of length at most three such that* $V(P) \subseteq V(C)$*. The graph* G *can be* L*-colored for any list assignment* L *such that*

- $|L(v)| = 3$ *for all* $v \in V(G) \setminus V(C)$*;*
- $2 \le |L(v)| \le 3$ *for all* $v \in V(C) \setminus V(P)$*;*
- $|L(v)| = 1$ *for all* $v \in V(P)$*, and the colors in the lists give a proper coloring of the subgraph of* G *induced by* $V(P)$*;*
- *the vertices with lists of size two form an independent set; and*
- *each vertex with lists of size two has at most one neighbor in* P*.*

Note that we view the single-element lists as a precoloring of the vertices of P. Also, P does not have to be a part of the facial walk of C, as we only require $V(P) \subseteq V(C)$. If C is a cycle, then let $\ell(C)$ denote its length. Theorem 28 has the following easy consequence:

Corollary 29. *Let* G *be a triangle-free planar graph without* 4*-cycles adjacent to* 4*- and* 5*-cycles, with the outer face bounded by an induced cycle* C *of length at most* 9*. Furthermore, assume that*

- *if* $\ell(C) = 8$*, then at least one edge of* C *does not belong to a* 4*-cycle; and*
- *if* $\ell(C) = 9$*, then* C *contains two consecutive edges that do not belong to* 4*- and* 5*-cycles.*

Let L *be an assignment of lists of size* 1 *to the vertices of* C *and lists of size* 3 *to the other vertices of* G*. If* L *prescribes a proper coloring of* C*, then* G *can be* L*-colored.*

Proof. The claim follows from Theorem 28 for $\ell(C) = 4$. If $\ell(C) \in \{5, 6, 7\}$, then let $u_1w_1vw_2u_2$ be an arbitrary subpath of C. Let L' be the list assignment obtained from L by removing the color $L(v)$ from the lists of vertices adjacent to v. We also set the lists of w_1 and w_2 to 2-lists such that the precoloring of the other vertices of C forces the prescribed color $L(w_1)$ on w_1 and $L(w_2)$ on w_2, i.e., $L'(w_1) = L(w_1) \cup L(u_1)$ and $L'(w_2) = L(w_2) \cup L(u_2)$. As all the vertices x with $|L'(x)| = 2$ are neighbors of a single vertex v, the graph $G - v$ together with the list assignment L' satisfies the assumptions of Theorem 28. It follows that we can L'-color $G - v$, giving an L-coloring of G.

Let us now consider the case that $\ell(C) = 8$, and let $C = w_1 u v w_2 r_1 r_2 r_3 r_4$, where the edge uv does not belong to a 4-cycle. Let us delete vertices u and v from G, remove the color in $L(u)$ from the lists of neighbors of u and the color in $L(v)$ from the lists of neighbors of v, and change the list of w_1 to $L(w_1) \cup L(r_4)$ and the list of w_2 to $L(w_2) \cup L(r_1)$, so that the precoloring of the path $P = r_1 r_2 r_3 r_4$ forces the right colors on w_1 and w_2. As uv does not belong to a 4-cycle, the vertices with lists of size two form an independent set. As C is induced, both w_1 and w_2 have only one neighbor in the 3-path P. Let x be a neighbor of u other than v and w_1. The vertex x cannot be adjacent to both r_1 and r_4, as the 4-cycle $u x r_4 w_1$ would be adjacent to a 5-cycle $x r_1 r_2 r_3 r_4$. Similarly, x cannot be adjacent to both r_1 and r_3 or both r_2 and r_4. As G does not contain triangles, x has at most one neighbor in P. By symmetry, this is also true for the neighbors of v. Therefore, the graph satisfies assumptions of Theorem 28, and can be colored from the prescribed lists.

Finally, suppose that $\ell(C) = 9$, and let $C = w_1 u v w w_2 r_1 r_2 r_3 r_4$, where the edges uv and vw are not incident with 4- and 5-cycles. We argue similarly as in the previous case. We delete vertices u, v and w from G and remove their colors from the lists of their neighbors. We also set the list of w_1 to $L(w_1) \cup L(r_4)$ and the list of w_2 to $L(w_2) \cup L(r_1)$, so that the precoloring of the path $r_1 r_2 r_3 r_4$ forces the right colors on w_1 and w_2. Observe that the resulting graph satisfies assumptions of Theorem 28, hence it can be colored. $\square$

Before we proceed with the proof of Theorem 28, let us describe the notation that we use in figures. We mark the precolored vertices of P by full circles, the vertices with list of size three by empty circles, and the vertices with list of size two by empty squares. The vertices for that the size of the list is not uniquely determined in the situation demonstrated by the particular figure are marked by crosses.

Proof of Theorem 28. Suppose G together with lists L is a smallest counterexample, i.e., such that $|V(G)| + |E(G)|$ is minimum among all graphs that satisfy the assumptions of Theorem 28, but cannot be L-colored, and $\sum_{v \in V(G)} |L(v)|$ is minimum among all such graphs. Let C be the outer face of G and P a path with $V(P) \subseteq V(C)$ as in the statement of the theorem. We first derive several properties of this counterexample. Note that each vertex v of G has degree at least $|L(v)|$. A cycle K in G is *separating* if $K \neq C$ and the interior of K contains at least one vertex. A *chord* of a cycle K is an edge in G joining two distinct vertices of K that are not adjacent in K.

Lemma 30. *Let K be a separating cycle in G. Then, $\ell(K) \geq 8$. Furthermore, if $\ell(K) = 8$, then every edge of K belongs to a 4-cycle lying inside K; and if*

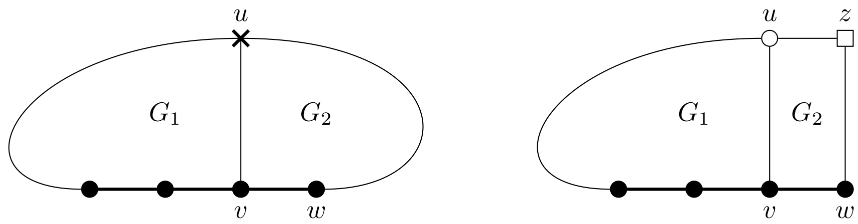


Figure 5.1: A chord of C

$\ell(K) = 9$, then at least one of any two consecutive edges of K belongs to a 4- or 5-cycle lying inside K.

Proof. We may assume that K is induced, as otherwise we could consider a shorter separating cycle of length at most 7. Let G_1 be the subgraph of G drawn inside K (including K, but excluding the chords of K drawn outside of K) and G_2 the subgraph of G drawn outside of K (including K, but excluding the chords of K drawn inside K). By the minimality of G, Theorem 28 holds for G_1 and G_2 and their subgraphs. Therefore, there exists a coloring of G_1 from the prescribed lists, and this coloring can be extended to G_2 by Corollary 29. This is a contradiction, as G cannot be colored from the lists. □

As G does not have triangles and 4-cycles adjacent to 4- and 5-cycles, a cycle of length at most 7 does not have a chord. Therefore, Lemma 30 implies that every cycle of length at most 7 bounds a face. Similarly, a cycle K of length 8 with an edge that does not belong to a 4-cycle in the interior of K either bounds an 8-face, or has a chord splitting it to a 4-face and a 6-face, or two 5-faces.

Lemma 31. *The graph G is 2-connected.*

Proof. Obviously, G is connected. Suppose now that v is a cut vertex of G and G_1 and G_2 are nontrivial induced subgraphs of G such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v\}$. Both G_1 and G_2 satisfy the assumptions of Theorem 28. If v is precolored, then by the minimality of G there exist L-colorings of G_1 and G_2, and they combine to a proper L-coloring of G. If v is not precolored, then we may assume that $P \subseteq G_1$. An L-coloring of G_1 assigns a color c to v. We change the list of v to $\{c\}$, color G_2 and combine the colorings to an L-coloring of G. □

By Lemma 31, C is a cycle. A *k-chord* of C is a path $Q = q_0q_1 \ldots q_k$ of length k joining two distinct vertices of C, such that $V(C) \cap V(Q) = \{q_0, q_k\}$ (e.g., 1-chord is just a chord).

Lemma 32. *The cycle C has no chords.*

Proof. Suppose $e = uv$ is a chord of C, separating G to two subgraphs G_1 and G_2 intersecting in e. If both u and v are precolored, then we L-color G_1 and G_2 by the minimality of G and combine their colorings. We assume by symmetry that $u \notin V(P)$, and that $|V(P) \cap V(G_1)| \geq |V(P) \cap V(G_2)|$. In particular, $|(V(P) \cap V(G_2)) \setminus \{u, v\}| \leq 1$. Furthermore, we may choose uv such that G_2 is as small as possible. Then, the outer face of G_2 does not have a chord. Let us find an L-coloring of G_1 and change the lists of u and v to the single-element sets containing the colors assigned to them. If G_2 with these new lists satisfies assumptions of Theorem 28, then we find its coloring and combine the colorings to an L-coloring of G, hence assume that this is not the case.

Let $X = (V(P) \cap V(G_2)) \setminus \{u, v\}$. As G_2 does not satisfy assumptions of Theorem 28, there exists a vertex z with list of size two adjacent to two precolored vertices. As G is triangle-free, we conclude that X is not empty, say $X = \{w\}$ (see Figure 5.1), and z is adjacent to u and w. Since G_2 contains no separating 4-cycles and the outer face of G_2 is chordless, $z \in V(C)$ implies that G_2 is equal to the cycle $uvwz$. Since $|L(z)| = 2$, the assumptions of Theorem 28 imply $|L(u)| = 3$. Let c_1 be the color of u in the coloring of G_1, and c_2 the single color in the list of w. If $L(z) \neq \{c_1, c_2\}$, then we can color z and finish the coloring of G, hence assume that $L(z) = \{c_1, c_2\}$. Let c be a color in $L(u) \setminus (\{c_1\} \cup L(v))$ (this set is nonempty, as $|L(v)| = 1$ and $|L(u)| = 3$).

Let us now color z by c_1 and set the list of u to $\{c\}$. If G_1 with this list at u satisfies assumptions of Theorem 28, then we can color G_1, and thus obtain an L-coloring of G. Since G does not have such an L-coloring, the assumptions are violated, i.e., either u is adjacent to a vertex of P other than v, or G_1 contains a vertex (with list of size two) adjacent to both u and a vertex of P. This is a contradiction, as G would in both of these cases contain either a triangle, or a 4- or 5-cycle adjacent to the 4-cycle $uvwz$. □

By the previous lemma, P is a part of the facial walk of C, and C is an induced cycle.

Lemma 33. *$\ell(C) \geq 8$.*

Proof. Suppose that $\ell(C) \leq 7$. Let φ be a proper L-coloring of C (such a coloring exists, as if $V(C) \neq V(P)$, then C contains at least one vertex with list of size three). Let L' be the list assignment defined by $L'(v) = \{\varphi(v)\}$ for $v \in V(C)$ and $L'(v) = L(v)$ for $v \in V(G) \setminus V(C)$. If $5 \leq \ell(C) \leq 7$, then the proof of Corollary 29 implies that G has an L'-coloring (the proof only uses Theorem 28 for proper subgraphs of G, which satisfy it by the assumption that G is a minimal counterexample). Such a coloring is also an L-coloring of G.

If $\ell(C) = 4$, then we delete one of the vertices of C and remove its color from the lists of its neighbors. It is easy to verify that the resulting graph satisfies the assumptions of Theorem 28, hence it has a proper coloring by the minimality of G. This coloring extends to an L-coloring of G, which is a contradiction. □

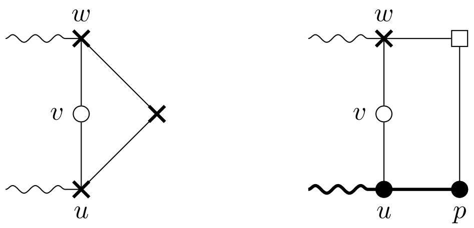


Figure 5.2: Possible 2-chords in G

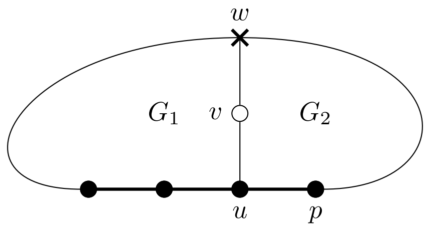


Figure 5.3: A 2-chord of C

Lemma 34. *No* 4*-cycle shares an edge with another* 4*- or* 5*-cycle.*

Proof. Suppose that $C_1 = v_1v_2v_3v_4$ and $C_2 = v_1v_2u_3 \dots u_t$ are cycles sharing the edge v_1v_2, $\ell(C_1) = 4$ and $t = \ell(C_2) \in \{4, 5\}$. Note that $C_1 \neq C$ and $C_2 \neq C$ by Lemma 33. By Lemma 30, both C_1 and C_2 bound a face. If $v_3 = u_3$, then v_2 would be a 2-vertex with list of size three. Thus, $v_3 \neq u_3$ and by symmetry, $v_4 \neq u_t$. As G does not contain triangles, $v_3 \neq u_t$ and $v_4 \neq u_3$, and in case that $t = 5$, $v_3 \neq u_4$ and $v_4 \neq u_4$. Therefore, C_1 and C_2 are adjacent, contradicting the assumptions of Theorem 28. □

Note that we can assume that $|V(P)| = 4$, as otherwise we can prescribe color for more of the vertices of C, without violating the assumptions of Theorem 28. Let $P = p_1p_2p_3p_4$. We say that a k-chord Q of C *splits off* a face F from G if $F \neq C$ is a face of both G and $C \cup Q$. See Figure 5.2 for an illustration of 2-chords splitting off a face.

Lemma 35. *Every* 2*-chord uvw of C splits off a k-face F such that*

(a) $|V(F) \cap V(P)| \leq 2$ *and* $\{u, w\} \not\subseteq V(P)$*,*

(b) $k \leq 5$*, and*

(c) *if* $|V(F) \cap V(P)| \leq 1$*, then* $k = 4$*.*

In particular, the cycle C has no 2*-chord with* $|L(w)| = 2$ *and* $u \neq p_2, p_3$*.*

Proof. Suppose first that $u, w \in V(P)$. By Lemma 30, the 2-chord uvw together with a part of P bounds a face K. Color v by a color different from the colors of u and w, and remove $V(K) \setminus \{u, v, w\}$ from G, obtaining a graph G'. Note that a path of length at most three is precolored in G'. Since G cannot be L-colored, we may assume that G' does not satisfy the assumptions of Theorem 28, i.e., there exists z with $|L(z)| = 2$ adjacent to both v and a vertex $y \in V(P) \cap V(G')$. As G is triangle-free, $y \notin \{u, w\}$. It follows that $yuvz$ or $ywvz$ is a 4-face. This is a contradiction, as K would be an adjacent 4-face. Therefore, $\{u, w\} \not\subseteq V(P)$, and by symmetry we assume that $w \notin V(P)$.

The 2-chord uvw splits G to two subgraphs G_1 and G_2 intersecting in uvw. Let us choose G_2 such that $|V(P) \cap V(G_2)| \le |V(P) \cap V(G_1)|$, see Figure 5.3. Note that $|V(P) \cap V(G_2)| \le 2$. Let us consider the 2-chord uvw such that $|V(P) \cap V(G_2)|$ is minimal, subject to the assumption that G_2 is not a face. By the minimality of G, there exists an L-coloring φ of G_1. Let L' be the list assignment for G_2 such that $L'(u) = \{\varphi(u)\}$, $L'(v) = \{\varphi(v)\}$, $L'(w) = \{\varphi(w)\}$ and $L'(x) = L(x)$ for $x \in V(G_2) \setminus \{u, v, w\}$. Let P' be the precolored path in G_2 (consisting of u, v, w, and possibly one other vertex p of P adjacent to u). As C has no chords and G_2 is not a face, P' is an induced subgraph. Since G cannot be L-colored, we conclude that G_2 cannot be L'-colored, and thus G_2 with the list assignment L' does not satisfy the assumptions of Theorem 28. Therefore, there exists a vertex z with $|L(z)| = 2$, adjacent to two vertices of P'.

Since G_2 is not a face, Lemmas 30 and 32 imply that z is not adjacent to both w and p. Similarly, z is not adjacent to both u and w. It follows that z is adjacent to v and p, and thus $|V(P) \cap V(G_2)| = 2$. Since we have chosen the 2-chord uvw so that $|V(P) \cap V(G_2)| = 2$ is minimal among the 2-chords for that G_2 is not a face, the 2-chord wvz splits off a face F' from G. Let x be the neighbor of z in F' other than v. Since $|L(z)| = 2$, it holds that $|L(x)| = 3$. As F' is a face, $\deg(x) = 2$, which is a contradiction. It follows that for every 2-chord, G_2 is a face. The choice of G_2 establishes (a).

Let $wvuv_4 \dots v_k$ be the boundary of the face G_2. Note that $V(P) \cap V(G_2) \subseteq \{u, v_4\}$, and $v_4, \dots, v_k$ have degree two. If $k > 5$, then at least one of v_5 and v_6 has list of size three, which is a contradiction, proving (b). Similarly, if $|V(F) \cap V(P)| \le 1$ and $k = 5$, then at least one of v_4 and v_5 would be a 2-vertex with list of size three, proving (c).

Consider now a 2-chord uvw such that $|L(w)| = 2$ and $u \notin \{p_2, p_3\}$, and let x be the neighbor of w in G_2 distinct from v. As $u \notin \{p_2, p_3\}$, no vertex of $V(P) \setminus \{u\}$ lies in G_2. Therefore, $|L(x)| = 3$ and $\deg(x) = 2$, a contradiction. We conclude that no such 2-chord exists. □

Let us now consider the 3-chords of C:

Lemma 36. *Every 3-chord $Q = uvwx$ of C such that $u, x \notin \{p_2, p_3\}$ splits off a 4- or 5-face.*

Proof. Suppose that Q splits G into two subgraphs G_1 and G_2 intersecting in $uvwx$, such that $V(P) \cap V(G_2) \subseteq \{u, x\}$. Let us L-color G_1 and consider the vertices u, v, w and x of G_2 as precolored according to this coloring. If ux were an edge, then Q would split off a 4-face. It follows that Q is an induced path thus this precoloring of Q is proper. Similarly, as Q does not split off a 5-face, u and x do not have a common neighbor with list of size two. Neither v nor w is adjacent to a vertex with list of size 2 by Lemma 35. Therefore, G_2 satisfies assumptions of Theorem 28, and the coloring can be extended to G_2, giving an L-coloring of G. This is a contradiction. □

Let $x_1x_2x_3x_4$ be the part of the facial walk of C such that x_1 is adjacent to p_4 and $x_2 \neq p_4$. By Lemma 33, $\{x_1, x_2, x_3, x_4\} \cap V(P) = \emptyset$. Let us now show a few properties of the vertices x_1, x_2, x_3, x_4 and their neighbors.

Lemma 37. *Let $Q = v_0v_1 \dots v_k$ be a k-chord starting and ending at a vertex of $x_1x_2x_3x_4$, or a cycle intersecting C in a single vertex $x \in \{x_1, x_2, x_3, x_4\}$. The following holds (for some $i \in \{1, 2, 3, 4\}$):*

- *If $\ell(Q) = 2$, then $Q = x_iv_1x_{i+2}$ splits off a 4-face.*
- *If $\ell(Q) = 3$, then Q splits off either a 4-face $x_ix_{i+1}v_1v_2$, or a 5-face $x_ix_{i+1}x_{i+2}v_1v_2$.*
- *If $\ell(Q) = 4$, then Q forms a boundary of a 4-face $x_iv_1v_2v_3$, or splits off a 5-face $x_ix_{i+1}v_1v_2v_3$, or splits off a 6-face $x_ix_{i+1}x_{i+2}v_1v_2v_3$.*

Proof. By a simple case analysis. The details are left to the reader. □

Note also that if Q splits off a face of form $x_ix_{i+1}x_{i+2}v_1 \dots v_{k-1}$, then $\deg(x_{i+1}) = |L(x_{i+1})| = 2$.

Lemma 38. *If Q is a k-chord with $k \leq 3$, starting at a vertex x_i (where $1 \leq i \leq 4$) and ending at a vertex with list of size two, then $k = 3$ and Q bounds a 4-face.*

Proof. Let $Q = q_0q_1 \dots q_k$, where $q_0 \in \{x_1, x_2, x_3, x_4\}$ and $|L(q_k)| = 2$. By Lemmas 32 and 35, $k > 2$. If $k = 3$, then by Lemma 36, Q splits off a 4- or 5-face. However, the latter is impossible, as $|L(q_3)| = 2$, so the remaining vertex of the 5-face, whose degree is two, would have a list of size three. □

Lemma 39. *There is no 2-chord from $\{p_1, p_2\}$ to $\{x_1, x_2, x_3, x_4\}$.*

Proof. Suppose $Q = p_ivx_j$ is such a 2-chord, and let K be the cycle formed by Q and $p_i \dots p_4x_1 \dots x_j$. Note that $\ell(K) \leq 9$. Let us choose Q such that $\ell(K)$ is minimal. By Lemma 35, Q splits off a face F such that $\ell(F) \leq 5$. Furthermore, if $\ell(K) = 9$, then $i = 1$, and hence $|V(P) \cap V(F)| = 1$. In that case, the claim (c) of Lemma 35 implies $\ell(F) = 4$. See Figure 5.4 for illustration. It follows that the edges p_iv and vx_j are not incident with a 4-cycle inside K, and if $\ell(K) = 9$,

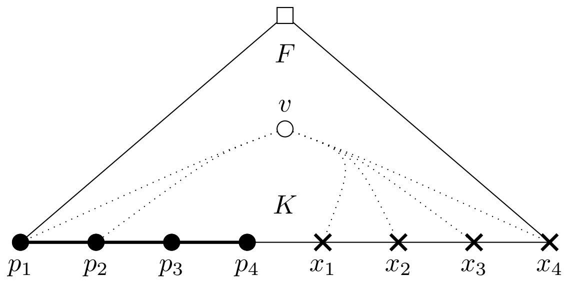


Figure 5.4: A 2-chord from p_1 or p_2 to $\{x_1, x_2, x_3, x_4\}$

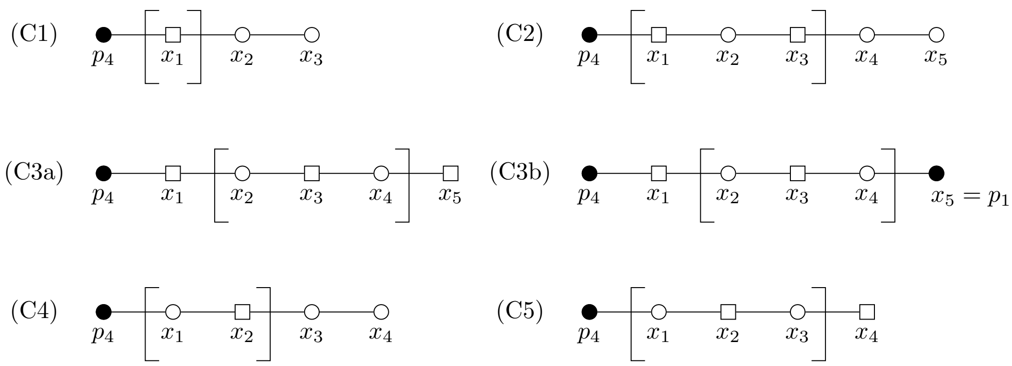


Figure 5.5: The construction of the set X_1

then they are not incident with a 5-cycle. By Lemma 30, K is not separating. If $\ell(K) \leq 7$, then K bounds a face, and $\deg(v) = 2$, which is a contradiction. Similarly, if $\ell(K) > 7$, then K has a chord incident with v. By the minimality of $\ell(K)$, v is adjacent to p_3 or p_4. However, this contradicts Lemma 35(a). $\square$

If both x_1 and x_2 have lists of size three, then we remove one color from $L(x_1)$ and find a coloring by the minimality of L (note that x_1 is not adjacent to any vertex with list of size two, and has only one neighbor in P, as C does not have chords). Therefore, exactly one of x_1 and x_2 has a list of size two. Let x_5 be the neighbor of x_4 in C distinct from x_3. We now distinguish several cases depending on the lists of vertices in $\{x_1, x_2, x_3, x_4\}$, in order to choose a set $X_1 \subseteq \{x_1, x_2, x_3, x_4\}$ of vertices that we are going to color (and remove).

(C1) If $|L(x_1)| = 2$ and $|L(x_2)| = |L(x_3)| = 3$ (see Figure 5.5(1)), then we set $X_1 = \{x_1\}$.

(C2) If $|L(x_1)| = 2$, $|L(x_2)| = 3$, $|L(x_3)| = 2$, $|L(x_4)| = 3$ and $|L(x_5)| = 3$ (see Figure 5.5(2)), then we set $X_1 = \{x_1, x_2, x_3\}$.

(C3) If $|L(x_1)| = 2$, $|L(x_2)| = 3$, $|L(x_3)| = 2$, $|L(x_4)| = 3$ and $|L(x_5)| \leq 2$ (see Figure 5.5(3)), then we set $X_1 = \{x_2, x_3, x_4\}$.

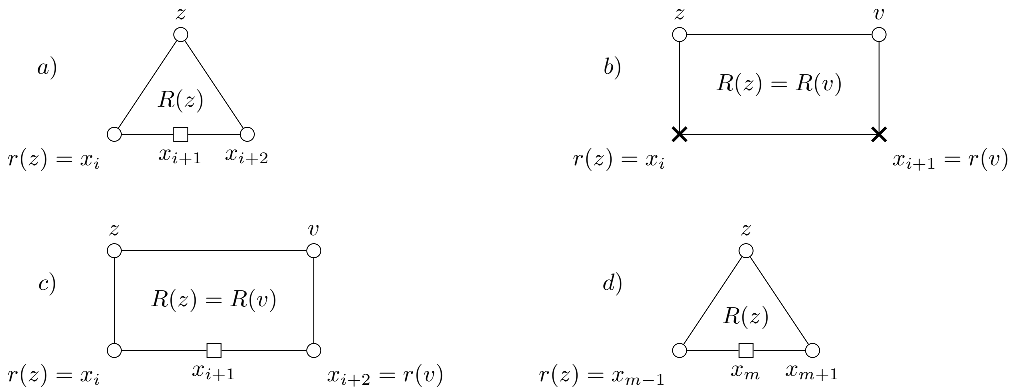


Figure 5.6: The construction of the set X_2

(C4) If $|L(x_1)| = 3$, $|L(x_2)| = 2$, $|L(x_3)| = 3$ and $|L(x_4)| = 3$ (see Figure 5.5(4)), then we set $X_1 = \{x_1, x_2\}$.

(C5) If $|L(x_1)| = 3$, $|L(x_2)| = 2$, $|L(x_3)| = 3$ and $|L(x_4)| = 2$ (see Figure 5.5(5)), then we set $X_1 = \{x_1, x_2, x_3\}$.

Let $m = \max\{i : x_i \in X_1\}$. Note the following properties of the set X_1:

- $|X_1| \leq 3$.
- If $|L(x_m)| = 2$, then $m \leq 3$ and $|L(x_{m+1})| = |L(x_{m+2})| = 3$.
- If $|L(x_m)| = 3$, then $|L(x_{m+1})| \leq 2$.

Let $\mathcal{F}$ be the set of faces of G incident with the edges of the path induced by X_1 ($\mathcal{F} = \emptyset$ in the case (C1)). We define a set $X_2 \subseteq V(G) \setminus V(C)$, together with functions $r : X_2 \to X_1$ and $R : X_2 \to \mathcal{F}$. A vertex $z \in V(G) \setminus V(C)$ belongs to X_2 if

- *z is adjacent to two vertices in X_1* (see Figure 5.6(a) as an example). By Lemma 37, z lies in a (uniquely determined) 4-face $F = x_i x_{i+1} x_{i+2} z$, where $x_i, x_{i+1}, x_{i+2} \in X_1$. We define $r(z) := x_i$ and $R(z) := F$. Or,
- *there exists a path $xzvy$ such that $x, y \in X_1$ and $v \notin \{p_1\} \cup X_1$* (see Figure 5.6(b), (c) and (d) for examples). If $v = x_{m+1}$, then by Lemma 35, the 2-chord xzv splits off a 4-face F. Otherwise the 3-chord $xzvy$ splits off a 4- or 5-face F by Lemma 37. We define $r(z) := x$ and $R(z) := F$. Note that $v \neq x_1$: otherwise, $x_1 \notin X_1$ and we are in case (C3), hence $|L(x_1)| = 2$ and the 2-chord $x_1 z x$ would contradict Lemma 35. It follows that v also belongs to X_2, unless $v = x_{m+1}$.

Let us now show that $r(z)$ and $R(z)$ are well-defined. As a 4-face cannot be adjacent to a 4- or 5-face and G is triangle-free, z does not have another neighbor in X_1. Also, if there existed another path $xzv'y'$ with $y' \in X_1$ splitting off a face F', then both F and F' would be 5-faces; however, that would imply $|X_1| \geq 5$, which is a contradiction. Therefore, r and R are defined uniquely. Furthermore, v is the only neighbor of z in X_2, and $R(v) = R(z)$ (assuming that $v \neq x_{m+1}$).

We now find an L-coloring of $X_1 \cup X_2$ that we aim to extend to a coloring of G.

Lemma 40. *Let $H = G[V(P) \cup X_1 \cup X_2]$ be the subgraph of G induced by $V(P) \cup X_1 \cup X_2$. There exist an L-coloring φ_1 of X_1 and an L-coloring φ_2 of X_2 such that*

- *the coloring of H given by φ_1, φ_2 and the precoloring of P is proper,*
- *if $|L(x_{m+1})| \leq 2$, then $\varphi_1(x_m) \notin L(x_{m+1})$,*
- *if $x_1 \notin X_1$ (i.e., in the case (C3) of the definition of X_1), then $L(x_1) \neq L(p_4) \cup \{\varphi_1(x_2)\}$, and*
- *if $z \in X_2$ is adjacent to x_{m+1}, then $|L(x_{m+1}) \setminus \{\varphi_1(x_m), \varphi_2(z)\}| \geq 2$.*

Proof. Suppose first that there exists $z \in X_2$ adjacent to x_{m+1}. Note that z is unique, $m \geq 2$ and $R(z) = x_{m-1}x_m x_{m+1}z$ is a 4-face. As G does not contain a 2-vertex with list of size three, $|L(x_m)| = 2$ and $|L(x_{m-1})| = |L(x_{m+1})| = 3$. This happens only in the cases (C2) and (C4) of the definition of X_1, thus $x_1 \in X_1$ and $m \leq 3$. Furthermore, x_{m-1} is the only neighbor of z in X_1 and z is not adjacent to any other vertex of X_2. As $R(z)$ is a 4-face and G does not contain 4-cycles adjacent to 4- or 5-cycles, z is not adjacent to p_3 and p_4. By Lemma 39, z is not adjacent to p_1 and p_2, either, thus any choice of the color for z is consistent with the precoloring of P. Let us distinguish the following cases:

- If $L(z) \cap L(x_m) \neq \emptyset$, then choose $c \in L(z) \cap L(x_m)$ and let $\varphi_1(x_m) = \varphi_2(z) = c$.
- If $L(z) \neq L(x_{m+1})$, then choose $\varphi_2(z) \in L(z) \setminus L(x_{m+1})$ and $\varphi_1(x_m) \in L(x_m)$ arbitrarily.
- Finally, consider the case that $L(z) \cap L(x_m) = \emptyset$ and $L(z) = L(x_{m+1})$, i.e., the lists of x_m and x_{m+1} are disjoint. We choose $\varphi_1(x_m) \in L(x_m)$ and $\varphi_2(z) \in L(z)$ arbitrarily.

On the other hand, suppose that no vertex of X_2 is adjacent to x_{m+1}. If $|L(x_{m+1})| = 2$, then choose $\varphi_1(x_m) \in L(x_m) \setminus L(x_{m+1})$. Otherwise, choose $\varphi_1(x_m) \in L(x_m)$ arbitrarily (in case that $m = 1$, choose a color different from the one in $L(p_4)$)

In both of these cases, the precoloring of x_m (and possibly z) can be extended to a proper coloring ψ of the subgraph induced by $\{x_1, \ldots, x_m, z\}$ consistent with the precoloring of P. We fix φ_1 as the restriction of ψ to X_1.

Let us now construct (the rest of) the coloring φ_2. Consider a vertex $u \in X_2$ that is not adjacent to x_{m+1}. As $u \notin V(C)$, it holds that $|L(u)| = 3$. If u has no neighbor in X_2, then it has two neighbors in X_1, say $r(u)$ and x, and $R(u)$ is a 4-face. We claim that u is not adjacent to any $p_i \in V(P)$. Otherwise, we obtain $i \geq 3$ by Lemma 39. By Lemma 35, the 2-chord $p_i u r(u)$ splits off a 4- or 5-face. This face shares an edge with $R(u)$, which is a contradiction. Therefore, any choice of $\varphi_2(u) \in L(u) \setminus \{\varphi_1(x), \varphi_1(r(u))\}$ is consistent with the precoloring of P.

Finally, suppose that u has a neighbor $w \in X_2$. As we argued in the definition of X_2, each of u and w has exactly one neighbor in X_1, and u and w do not have any other neighbors in X_2. Also, w is not adjacent to x_{m+1}, as otherwise G would contain a triangle or two adjacent 4-cycles. By Lemma 35(a), each of u and w has at most one neighbor in P. If one of them does not have any such neighbor, then we can easily color u and w, hence assume that $p_i u$ and $p_j w$ are edges. By Lemma 39, $i, j \geq 3$. Without loss of generality, $j = 3$ and $i = 4$. This is a contradiction, as the 4-face $p_3 p_4 u w$ shares an edge with $R(u)$. $\square$

Consider the colorings φ_1 and φ_2 constructed in Lemma 40. Let $G' = G - (X_1 \cup X_2)$ and let L' be the list assignment such that $L'(v)$ is obtained from $L(v)$ by removing the colors of the neighbors of v in X_1 and X_2 for $v \neq x_1$, and $L'(x_1) = L(x_1)$ if $x_1 \notin X_1$. Suppose that G' with the list assignment L' satisfies assumptions of Theorem 28. Then there exists an L'-coloring φ of G', which together with φ_1 and φ_2 gives an L-coloring of G: this is obvious if $x_1 \in X_1$. If $x_1 \notin X_1$, then $|L(x_1)| = 2$, and $L(p_4) \subseteq L(x_1)$ by the minimality of G (otherwise, we could remove the edge $p_4 x_1$). By the choice of φ_1, it holds that $\varphi_1(x_2) \neq \varphi(x_1)$. Since no other vertex of $X_1 \cup X_2$ may be adjacent to x_1 by Lemmas 32 and 35, φ together with φ_1 and φ_2 is a proper coloring of G. As G is a counterexample to Theorem 28, it follows that L' violates assumptions of Theorem 28, i.e.,

(a) a vertex $v \in V(G')$ with $|L'(v)| = 2$ is adjacent to two vertices of P; or

(b) $|L'(v)| \leq 1$ for some $v \in V(G') \setminus V(P)$; or

(c) two vertices $u, v \in V(G')$ with $|L'(u)| = |L'(v)| = 2$ are adjacent.

Let us now consider each of these possibilities separately.

(a) *A vertex $v \in V(G')$ with $|L'(v)| = 2$ is adjacent to two vertices of P.* By Lemmas 32 and 35(a), this is not possible.

(b) *$|L'(v)| \leq 1$ for some $v \in V(G') \setminus V(P)$.* If $|L(x_{m+1})| = 2$, then x_{m+1} does not have any neighbor in X_2 by Lemma 35 and hence $|L'(x_{m+1})| = 2$ by the choice of φ_1. If $|L(x_{m+1})| = 3$, then the choice of φ_1 and φ_2 according to Lemma 40 ensures $|L'(x_{m+1})| \geq 2$. Therefore, $v \neq x_{m+1}$.

Since C has neither chords nor 2-chords starting in X_1 and ending at a vertex with list of size two by Lemma 38, it holds that $|L(v)| = 3$. Therefore, v has at least two neighbors $u_1, u_2 \in X_1 \cup X_2$. If at least one of u_1 and u_2 belonged to X_1, then v would be included in X_2, hence we may assume that $u_1, u_2 \in X_2$.

Consider the path $x_i u_1 v u_2 x_j$, where $x_i = r(u_1)$ and $x_j = r(u_2)$. We may assume that $i \leq j$. The cycle $x_i \dots x_j u_2 v u_1$ has length at most six, thus it bounds a face F. Note that $i = j$, as each of $R(u_1)$ and $R(u_2)$ shares at least one edge with the path induced by X_1 and $F \neq R(u_1) \neq R(u_2) \neq F$. Therefore, F is a 4-face sharing an edge with 4-face $R(u_1)$ (and also with $R(u_2)$), which is a contradiction. Therefore, $|L'(v)| \geq 2$ for every $v \in V(G') \setminus V(P)$.

(c) *Two vertices $u, v \in V(G')$ with $|L'(u)| = |L'(v)| = 2$ are adjacent.* As the vertices with lists of size two form an independent set in G, we may assume that $|L(u)| = 3$. Let y_1 be a neighbor of u in $X_1 \cup X_2$.

Consider first the case that $|L(v)| = 2$. If $u \notin V(C)$, then by Lemma 35, $y_1 \notin V(C)$, and thus $y_1 \in X_2$ and $vuy_1r(y_1)$ is a 3-chord. By Lemma 38, this 3-chord splits off a 4-face F. Note that $F \neq R(y_1)$, as $u \notin X_2$. This is impossible, as the 4-face F would share an edge with $R(y_1)$. Therefore, $u \in V(C)$, and hence $v \neq x_1$. If $y_1 \in X_2$, then $uy_1r(y_1)$ is a 2-chord, and by Lemma 35, it splits off a 4-face adjacent to $R(y_1)$, which is again a contradiction. Assume now that $y_1 \in X_1$. As C does not have chords, it follows that $y_1 = x_m$ and $u = x_{m+1}$. However, in that case $v = x_{m+2}$ and $|L(x_{m+2})| = 2$, which contradicts the choice of X_1.

Consider now the case that $|L(v)| = 3$. Let y_2 be a neighbor of v in $X_1 \cup X_2$. As $u, v \notin X_2$, at least one of y_1 and y_2, say y_1, belongs to X_2. Let us consider the possibilities $y_2 \in X_1$ and $y_2 \in X_2$ separately:

- $y_2 \in X_1$: The cycle formed by $r(y_1)y_1uvy_2$ and a part of the path $x_1x_2x_3x_4$ between $r(y_1)$ and y_2 has length at most six, thus it bounds a face F. Note that $R(y_1)$ shares an edge with F. Let k_1 and k_2 be the number of edges that $R(y_1)$ and F, respectively, share with the path induced by X_1, $k_1 \geq \ell(R(y_1)) - 3 \geq 1$ and $k_2 = \ell(F) - 4 \geq 0$. Since $|X_1| \leq 3$, it holds that $k_1 + k_2 \leq 2$. If $k_1 = 1$, then $R(y_1)$ is a 4-face.

Since 4- and 5-faces cannot be adjacent to $R(y_1)$, we obtain $\ell(F) \geq 6$. It follows that $k_2 \geq 2$, which is a contradiction. Similarly, if $k_1 = 2$, then F cannot be a 4-face, hence $\ell(F) \geq 5$ and thus $k_2 \geq 1$. This is again a contradiction.

- $y_2 \in X_2$: Let F be the cycle bounded by $r(y_1)y_1uvy_2r(y_2)$ and the part of the path $x_1x_2x_3x_4$ between $r(y_1)$ and $r(y_2)$. As $\ell(F) \leq 7$, F bounds a face. Note that $R(y_1) \neq R(y_2)$ and $\ell(R(y_1)) = \ell(R(y_2)) = 4$, as each of $R(y_1)$ and $R(y_2)$ shares an edge with the path induced by X_1. Since F shares edges with both $R(y_1)$ and $R(y_2)$, $\ell(F) \geq 6$. It follows that F shares at least one edge with the path induced by X_1 as well. However, this is impossible, since $|X_1| \leq 3$.

Therefore, the assumptions of Theorem 28 are satisfied by G' and L'. We conclude that we can find a proper coloring of G, which contradicts the choice of G as a counterexample to Theorem 28. $\square$

Chapter 6

Separating (≤ 4)-cycles in embedded 4-critical graphs

The results of this chapter are part of Dvořák et al. [27], where we give bound on the size of 4-critical graphs of girth at least five embedded in a fixed surface. As outlined in Section 2.3, one issue that we need to deal with is the presence of short non-contractible cycles, and in particular the case that we have many such short cycles drawn around a cylinder. To simplify the arguments, we allow presence of triangles and 4-cycles, as long as they are non-contractible.

In this chapter, we give an auxiliary result regarding this particular case; i.e., we have a graph G embedded in a cylinder whose boundary is formed by two cycles C_1 and C_2 of length at most four, all contractible cycles have length at least five and G is $(C_1 \cup C_2)$-critical. We will call the cycles C_1 and C_2 the *rings* of G. For technical reasons, we also allow the case that C_1 or C_2 is only a single vertex, in which case we call it a *vertex ring.*

We will need the following result on graphs embedded in the disk with the ring of length at most twelve, which follows from the results of Thomassen [71].

Theorem 41. *Let G be a graph of girth* 5 *embedded in the disk with a ring R such that $|R| \leq 12$. If G is R-critical and R is an induced cycle, then*

(a) $|R| \geq 9$ and $G - V(R)$ is a tree with at most $|R| - 8$ vertices, or

(b) $|R| \geq 10$ and $G - V(R)$ is a connected graph with at most $|R| - 5$ vertices containing exactly one cycle, and the length of this cycle is 5*, or*

(c) $|R| = 12$ and every second vertex of R has degree two and is contained in a facial 5*-cycle.*

A graph H embedded in the cylinder with (vertex-disjoint) rings C_1 and C_2 of length 4 is *basic* if every contractible cycle in H has length at least five, H is $(C_1 \cup C_2)$-critical, and one of the following holds:

- H contains a triangle, or
- H is not 2-connected, or
- the distance between C_1 and C_2 is one and $|V(H) \setminus V(C_1 \cup C_2)| \leq 2$.

Using Theorem 41, observe that all 2-connected triangle-free basic graphs are subgraphs of the graphs drawn in Figure 6.1. Note that these graphs have the following properties.

(22) *Let C_1 and C_2 be the rings of a triangle-free 2-connected basic graph H. There exists a 3-coloring ψ of C_1, vertices $v_1, v_2 \in V(C_2)$ and colors $c_1 \neq c_2$ such that if ϕ is a 3-coloring of $C_1 \cup C_2$ matching ψ on C_1 and satisfying $\phi(v_i) \neq c_i$ for $i \in \{1,2\}$, then ϕ extends to a 3-coloring of H.*

(23) *Let C_1 and C_2 be the rings of a triangle-free 2-connected basic graph H, v_1 and v_2 two distinct vertices of C_1 and $c_1 \neq c_2$ two colors. There exists a vertex $v \in V(C_2)$ and a color c such that every 3-coloring ψ of C_2 such that $\psi(v) \neq c$ extends to a 3-coloring ϕ of H satisfying $\phi(v_1) \neq c_1$ and $\phi(v_2) \neq c_2$.*

For a 4-cycle $C = x_1x_2x_3x_4$, the *type* of its 3-coloring λ is the set of the vertices x_i of C such that $\lambda(x_i) \neq \lambda(x_{i+2})$. Note that the type of λ is $\emptyset$, $\{x_1, x_3\}$ or $\{x_2, x_4\}$. In (22), any coloring of the same type as ψ has the same property (with different colors c_1 and c_2).

Let G and H be graphs with common rings $\{C_1, C_2\}$. We say that H *subsumes* G if every precoloring of $C_1 \cup C_2$ that extends to a 3-coloring of H also extends to a 3-coloring of G.

Lemma 42. *Let G be a graph embedded in the cylinder with rings $\{R_1, R_2\}$ of length at most 4. If every cycle of length at most 4 in G is non-contractible, then there exists a basic graph H with rings $\{R_1, R_2\}$ that subsumes G.*

Proof. Suppose for a contradiction that G is a counterexample such that $|V(G)| + |E(G)|$ is minimal. It follows that G is $(R_1 \cup R_2)$-critical, 2-connected and triangle-free, and in particular $|R_1| = |R_2| = 4$. Let $R_1 = a_1a_2a_3a_4$ and $R_2 = b_1b_2b_3b_4$, where the labels are assigned in the clockwise order. Since G is triangle-free and all 4-cycles are non-contractible, it follows that every internal vertex has at most one neighbor in each of the rings.

Suppose that G contains a 5-face $C = v_1v_2v_3v_4v_5$ such that all its vertices are internal and have degree three. For $1 \leq i \leq 5$, let x_i be the neighbor of v_i different from v_{i-1} and v_{i+1} (where $v_0 = v_5$ and $v_6 = v_1$). Observe that if $x_1 = x_3$, then $x_2 \neq x_4$, thus by symmetry assume that $x_1 \neq x_3$. Let $G' = (G - V(C)) + x_1x_3$. Suppose that K' is a cycle of length at most 4 in G' that contains the edge x_1x_3. Then G contains a cycle K of length at most 7 obtained from K' by replacing x_1x_3 by $x_1v_1v_2v_3x_3$. Since v_1 and v_2 have neighbors on the opposite sides of this path, K does not bound a face. By Theorem 41, we conclude that K and

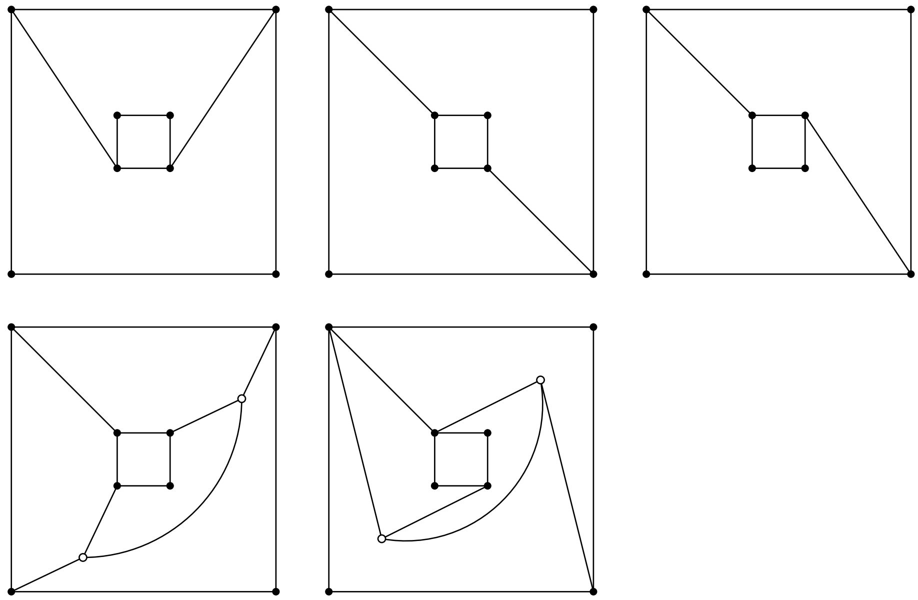


Figure 6.1: Maximal basic graphs.

K' are non-contractible. Therefore, all (≤ 4)-cycles in G' are non-contractible. Furthermore, every precoloring of R_1 and R_2 that extends to a 3-coloring of G' also extends to a 3-coloring of G (the 3-coloring of G' assigns different colors to x_1 and x_3, thus it can be extended to C). Thus, G' subsumes G, and consequently it contradicts the minimality of G. We conclude that

(24) *every* 5*-face in* G *is incident with a ring vertex or a vertex of degree at least* 4*.*

It follows that the distance between R_1 and R_2 is at least two: otherwise, if say a_1 is adjacent to b_1, then apply Theorem 41 to the subgraph bounded by the walk $a_1a_2 \ldots a_1b_1b_2 \ldots b_1$. The outcome (b) is excluded by (24), thus $G - V(R_1 \cup R_2)$ would have at most two vertices and G would be basic.

Suppose that G contains a face $C = v_1v_2 \ldots v_k$ of length $k \geq 7$. We may assume that v_1 is an internal vertex. Let G' be the graph obtained from G by identifying v_1 with v_3 to a vertex v. Consider a cycle $K' \subseteq G'$ of length at most 4 that does not appear in G. Such a cycle corresponds to a cycle K in G of length at most 6, obtained by replacing v by $v_1v_2v_3$. Note that v_2 cannot have degree two, thus K does not bound a face and it is non-contractible by Theorem 41. Therefore, all (≤ 4)-cycles in G' are non-contractible. Furthermore, every 3-coloring of G' extends to a 3-coloring of G, and we obtain a contradiction with the minimality of G. Therefore, each face of G has length at most 6.

Suppose that G contains a face $C = v_1v_2 \ldots v_6$ of length 6. We can assume that v_1 is an internal vertex. If v_3 or v_5 is an internal vertex, then let G' be the graph obtained from G by identifying v_1, v_3 and v_5 to a single vertex. As in the previous paragraph, we obtain a contradiction. It follows that v_3 and v_5 are ring vertices, and by a symmetrical argument, two of v_2, v_4 and v_6 are ring vertices. If v_2 is internal, then since the distance between R_1 and R_2 is at least two, we can assume that $V(R_1) = \{v_3, v_4, v_5, v_6\}$, and thus v_3 and v_6 are adjacent. In this situation, we consider the graph obtained from G by identifying v_1 with v_5 and v_2 with v_4 (which is isomorphic to $G - \{v_4, v_5\}$, and thus contains no non-contractible (≤ 4)-cycles), and again obtain a contradiction with the minimality of G. By symmetry, v_6 is not internal either. Therefore, v_4 is internal and v_2 and v_6 are ring vertices. Since the distance between R_1 and R_2 is at least two, we may assume that $v_2 = a_2$, $v_3 = a_3$, $v_5 = b_4$ and $v_6 = b_1$. We apply Theorem 41 to the 10-cycle $B = a_1a_2v_1b_1b_2b_3b_4v_4a_3a_4$. The case (b) is excluded by (24), thus either B is not induced or (a) holds. If B is not induced, then its chord joins v_1 with v_4. However, then the precolorings ψ of the rings that do not extend to 3-colorings of G satisfy $\psi(a_2) = \psi(b_4)$, and we can set H to be the graph consisting of R_1, R_2 and the edge between a_2 and b_4. Therefore, B is an induced cycle and $G - V(B)$ is a tree F with at most two vertices. If F has only one vertex w, then w cannot be adjacent to both v_1 and v_4, hence one of these vertices has degree two, which is a contradiction. If $V(F) = \{x, y\}$, then since v_1 and v_4 have degree at least three, we can assume that x is adjacent to v_1 and a_4 and y is adjacent to b_2 and v_4. However, then G is subsumed by the graph consisting of R_1, R_2, the edge a_2b_4, two vertices z_1 and z_2 and edges z_1z_2, z_1a_4, z_1b_4, z_2a_2 and z_2b_2, which is isomorphic to the last graph in Figure 6.1. Therefore,

(25) *all internal faces of G have length* 5.

Suppose that G contains a 4-cycle $C = v_1v_2v_3v_4$ different from R_1 and R_2. By the assumptions, C is non-contractible; let G_i be the subgraph of G drawn between R_i and C. By the minimality of G, we can assume that the distance d_i between C and R_i is at most one, and if it is exactly one, then G_i is basic, for $i \in \{1, 2\}$. Let us choose the labels of R_1 and R_2 and the cycle C so that d_1 is as small as possible. In particular, $d_1 \leq d_2$. Let us discuss the possible cases:

- $d_1 = d_2 = 0$: Since the distance between R_1 and R_2 is at least two, we conclude that $|V(R_1) \cap V(C)| = |V(R_1) \cap V(C)| = 1$. We can assume that $v_1 = a_1$ and $v_3 = b_3$. By Theorem 41, the open disks bounded by closed walks $a_1v_2v_3v_4a_1a_4a_3a_2$ and $b_3b_4b_1b_2b_3v_4a_1v_2$ contain no vertices, and since v_2 and v_4 have degree at least three, we may assume that v_2 is adjacent to a_4 and v_4 to b_2. However, then G contains a triangle $a_1v_2a_4$, which is a contradiction.

- $d_1 = 0$, $d_2 = 1$: We may assume that $a_1 = v_1$. Since all internal faces of G have length 5 and G is triangle-free, we have $|V(C) \cap V(R_1)| = 1$ and $a_3v_3 \in E(G)$. Since $d_2 = 1$, G_2 is a basic graph with all internal faces of length 5, and thus it has two adjacent vertices not belonging to $V(C \cup R)$.

Let $w_1w_2 = G_2 - V(C \cup R_2)$. Up to symmetry, there are two cases to consider:

- b_3 is adjacent to v_3. Since v_2 and v_4 have degree at least three, we can assume that w_1 is adjacent to v_4 and b_2 and w_2 is adjacent to v_2 and b_4. In this case, we let H be the graph consisting of R_1, R_2 and a vertex z, with a_1 adjacent to b_2 and z to b_2, b_3 and a_3.
- b_3 is adjacent to v_4. Since v_4 has degree at least three, we can assume that w_1 is adjacent to b_1 and v_4 and w_2 is adjacent to b_3 and v_2. We let H be the graph consisting of R_1, R_2, adjacent vertices z_1 and z_2, and edges a_1b_3, a_1z_1, b_1z_1, b_3z_2 and a_3z_2.

• $d_1 = 1$, $d_2 = 1$: By the choice of C, G does not contain a 4-cycle distinct from R_1 and R_2 that intersects one of them. Additionally, all internal faces of G have length 5 and G_1 and G_2 are basic graphs, hence we can assume that a_1 is adjacent to v_1 and $G_1 - V(R_1 \cup C) = w_1w_2$ with w_1 adjacent to a_4 and v_4 and w_2 adjacent to a_2 and v_2, and G_2 is isomorphic to G_1. Since v_3 has degree at least three, v_1 cannot have a neighbor in R_2, thus there are up to symmetry two possible cases:

- b_1 is adjacent to v_2, $G_2 - V(R_2 \cup C) = w_3w_4$, and $w_3v_1, w_3b_2, w_4v_3, w_4b_4 \in E(G)$. Then, let H be the graph consisting of R_1, R_2 and the edge a_1b_2.
- b_1 is adjacent to b_3, $G_2 - V(R_2 \cup C) = w_3w_4$, and $w_3v_2, w_3b_2, w_4v_4, w_4b_4 \in E(G)$. But then every precoloring of R_1 and R_2 extends to a 3-coloring of G, contrary to the assumption that G is $(R_1 \cup R_2)$-critical.

Therefore,

(26) *R_1 and R_2 are the only 4-cycles in G.*

Suppose that G has a face $C = v_1v_2v_3v_4v_5$ such that $v_2, \ldots, v_5$ are internal vertices of degree three. For $2 \le i \le 5$, let x_i be the neighbor of v_i that is not incident with C. By (26), the vertices x_i are distinct. If at least one of x_3 and x_4 is internal, then let G' be the graph obtained from $G - \{v_2, \ldots, v_5\} + x_2x_5$ by identifying x_3 with x_4 to a new vertex x. Observe that every 3-coloring of G' extends to a 3-coloring of G. Furthermore, suppose that K' is a cycle of length at most 4 in G' that does not appear in G, and let K be the corresponding cycle in G obtained by replacing x_2x_5 by $x_2v_2v_1v_5x_5$ or x by $x_3v_3v_4x_4$ or both. If $|K| \le 7$, then since K cannot bound a face, Theorem 41 implies that K and K' are non-contractible. If $|K| \ge 8$, then K contains both $x_2v_2v_1v_5x_5$ and $x_3v_3v_4x_4$, and since $|K'| \le 4$ and G is embedded in the cylinder, it follows that x_4 is adjacent to x_5 or x_3 is adjacent to x_2. This is excluded by (26). Therefore, G' is a smaller counterexample than G, which is a contradiction. Let us now consider the case that both x_3 and x_4 are ring vertices. Here, we exclude the possibility that x_3

and x_4 belong to different rings: If that were the case, then we can assume that $x_3 = a_1$ and $x_4 = b_1$. Since all internal faces of G have length 5, it follows that x_4 and x_5 have a common neighbor v. We apply Theorem 41 to the disk bounded by the closed walk $a_1a_2a_3a_4a_1vb_1b_2b_3b_4b_1v$ of length 12. By (24), the case (b) is excluded. Since v has degree at least three, a_1vb_1 cannot be incident with two 5-faces and the case (c) is excluded as well. Therefore, $G - V(R_1 \cup R_2) - \{v\}$ is a tree with four vertices v_2, v_3, v_4 and v_5. By (26), v is not equal to x_2, x_5 or v_2, hence two of these vertices belong to the same ring. Since G is triangle-free, (26) implies that no internal vertex has two neighbors in the same ring, thus we can assume that $v_2 \in V(R_1)$ and $x_2, x_5 \in V(R_2)$. However, the path $x_2v_2v_3v_4v_5x_5$ together with a subpath of R_2 forms a cycle that separates v_2 from R_1, which contradicts the assumption that G is embedded in the cylinder. Therefore,

(27) *if $C = v_1v_2v_3v_4v_5$ is a face such that $v_2, \ldots, v_5$ are internal vertices of degree three, then for some $i \in \{1, 2\}$, both v_3 and v_4 have a neighbor in R_i.*

Let us now assign charge to vertices and faces of G as follows: each face f gets charge $|f| - 4$ and each vertex v gets charge $\deg(v) - 4$. The sum of the charges is -8. Let us redistribute the charge: each 5-face sends $1/3$ to each incident vertex v such that v is internal and has degree three. Furthermore, for each ring vertex w of degree two, if there exists a face $f = v_1v_2v_3v_4v_5$ such all vertices incident with f except for v_1 are internal of degree three and if v_3v_4 is incident with the same face as w, then w sends $1/3$ to f. Note that all vertices and faces have non-negative charge, with the following exceptions: the ring vertices of degree two have charge at most $-7/3$ and the ring vertices of degree three have charge -1. For $i \in \{1, 2\}$, let c_i be the sum of the charges of the vertices of R_i, together with the charges of the faces that share an edge with R_i (such a face cannot share an edge with R_{3-i}, since the distance between R_1 and R_2 is at least two and all internal faces have length 5). Note that $c_1 + c_2 \leq -8$, and we may assume that $c_1 \leq -4$.

For $i \in \{1, 2, 3, 4\}$, let f_i denote the face sharing the edge a_ia_{i+1} with R_1. If a vertex a_i has degree three, then let x_i denote its internal neighbor. Since G is 2-connected, at most two vertices of R_1 have degree two. Let us discuss several cases;

- R_1 contains two vertices of degree two: Since all faces have length 5 and G is triangle-free, these two vertices are non-adjacent, say a_2 and a_4. Similarly, since G does not contain a 4-cycle different from R_1 and R_2, both a_1 and a_3 have degree at least four, and since the sum of the charges of the vertices of R_1 is at most -4, we conclude that $\deg(a_1) = \deg(a_3) = 4$. Let $f_2 = a_1a_2a_3x_3'x_1'$ and $f_4 = a_1a_4a_3x_3''x_1''$. Note that both f_2 and f_4 send charge to at most two vertices, hence their final charge is $1/3$, and since $c_1 \leq -4$, it follows that the charge of a_2 and a_4 is $-7/3$. Therefore, the vertices x_1', x_1'', x_3', x_3'' and their neighbors distinct from a_1 and a_3 are internal vertices of degree three. However, these vertices form an 8-cycle, contradicting the criticality of G.

- R_1 contains one vertex of degree two, say a_2, and a_1, a_3 and a_4 have degree three: by (25), x_1 is adjacent to x_3, x_1 and x_4 have a common neighbor x_{41} and x_3 and x_4 have a common neighbor x_{43}. Suppose that x_1 and x_3 have degree three. The path $x_{41}x_1x_3x_{43}$ is a part of a boundary of a 5-face f; let y be the fifth vertex of f. Then $x_{41}x_4x_{43}y$ is a 4-cycle, contradicting (26). Therefore, we may assume that x_1 has degree greater than three. This implies that a_2 does not send any charge and its final charge is -2. Furthermore, f_2 has charge at least $2/3$ and f_4 has charge at least $1/3$, and thus $c_1 = -4$. Furthermore, x_3, x_4, x_{41} and x_{43} are internal and have degree three.

- R_1 contains one vertex of degree two, say a_2, and at least one vertex of R_1 has degree at least four: note that the sum of the charges of a_2 and f_2 is at least -2. It follows that exactly one vertex of R_1 has degree four, two vertices have degree three, and $c_1 = -4$.

- R_1 contains no vertices of degree two. Since $c_1 \leq -4$, it follows that all vertices of R_1 have degree three and all internal vertices of the faces sharing an edge with R_1 have degree three. But then G contains an 8-cycle of internal vertices of degree 3, contradicting the criticality of G.

We conclude that $c_1 = -4$, and by symmetry, $c_2 = -4$. It follows that all charges that are not counted in c_1 and c_2 are equal to zero. Let us now go over the possible cases for the neighborhood of R_1 again, keeping the notation established in the previous paragraph:

- R_1 contains one vertex of degree two, say a_2, and a_1, a_3 and a_4 have degree three: Since all internal vertices have zero charge, x_1 has degree exactly four. Let y_1, y_{41} and y_{43} be the neighbors of x_1, x_{41} and x_{43}, respectively, not incident with f_2, f_3 and f_4. By (25), y_{43} is adjacent to y_1 and to y_{41}, and the vertices y_1 and y_{41} have a common neighbor z distinct from y_{43}. By (26), we have $R_2 = y_1y_{43}y_{41}z$. However, then we can set H to be the graph consisting of R_1, R_2 and a vertex w, with edges a_4y_{41}, wy_1, wa_1 and wa_4.

- R_1 contains one vertex of degree two, say a_2, one vertex of degree four and two of degree three. Let a_i be the vertex of degree four and x'_i and x''_i its internal neighbors. Since $c_1 = -4$, all internal vertices incident with the faces f_2, f_3 and f_4 have degree three, and by (25) they form a path P with ends x'_i and x''_i. Furthermore, x'_i and x''_i have adjacent neighbors y'_i and $y''i$. We let G' consist of $G - V(P)$ and a new vertex w adjacent to y'_i, y''_i and a_i, and observe that every 3-coloring of G' extends to a 3-coloring of G. This contradicts the minimality of G.

□

A finite graph G is an (e_1, e_2)-*chain* if either G is the complete graph on four vertices and e_1 and e_2 form a matching in G, there exists a (e_1, u_1u_2)-chain H, $e_2 = y_1y_2$ and G consists of $H - u_1u_2$, vertices y_1, y_2 and u_2' and edges y_1y_2, u_2u_2', u_1y_1, u_1y_2, $u_2'y_1$ and $u_2'y_2$. Let us note that each (e_1, e_2)-chain is a planar graph with chromatic number 4 containing exactly four triangles (two incident with each of e_1 and e_2), and all other faces of G have length 5. The graph G can be embedded in the Klein bottle by putting crosscaps on the edges e_1 and e_2; we call such an embedding *canonical*. Note that no cycle of length less than 5 is contractible in a canonical embedding of G. Thomas and Walls [63] proved the following:

Theorem 43. *If G is a 4-critical graph embedded in the Klein bottle so that no cycle of length at most 4 is contractible, then G is a canonical embedding of an (e_1, e_2)-chain, for some edges $e_1, e_2 \in E(G)$.*

For the torus, Thomassen [68] showed that the situation is even simpler.

Theorem 44. *If G is embedded in the torus so that no cycle of length at most 4 is contractible, then G is 3-colorable.*

Also, the results of Aksionov [2] imply the following:

Theorem 45. *There exists no R-critical graph embedded in the cylinder with one ring R such that R is a cycle of length at most four and every cycle of length at most four in G is non-contractible.*

Let us now give a description of $(R_1 \cup R_2)$-critical graphs on cylinder, where each of R_1 and R_2 is either a vertex ring or a triangle:

Lemma 46. *Let G be a $(R_1 \cup R_2)$-critical graph embedded in the cylinder, where each of rings R_1 and R_2 is either a vertex ring or a triangle. If every cycle of length at most 4 in G is non-contractible, then one of the following holds:*

- *G consists of R_1, R_2 and an edge between them, or*
- *R_1 and R_2 are triangles and G consists of R_1, R_2 and two edges between them, or*
- *R_1 and R_2 are triangles and G consists of R_1, R_2 and two adjacent vertices of degree three, each having a neighbor in R_1 and in R_2.*

Proof. By Theorem 45, we have that G is connected. We may assume that at least one of R_1 and R_2 is a triangle, since if both R_1 and R_2 are vertex rings, we can add a triangle containing R_2 to G. If the distance between R_1 and R_2 is at most two, then we apply Theorem 8 to the closed walk tracing the rings and a

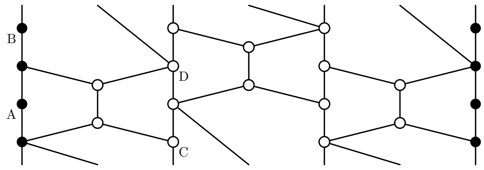


Figure 6.2: Arbitrarily large critical graph with rings of length four.

shortest path between them. The resulting graph G' is embedded in a disk and critical with respect to the cycle contained in its boundary, which has length at most 10, and thus it is described by Theorem 41. In the case that G' satisfies (b) of Theorem 41, observe that G is not $(R_1 \cup R_2)$-critical, and if G' satisfies (a), then G satisfies one of the outcomes of Lemma 46. Therefore, assume that the distance between R_1 and R_2 is at least three.

Since G is $(R_1 \cup R_2)$-critical, there exists a precoloring ψ of $R_1 \cup R_2$ that does not extend to a 3-coloring of G. We identify the vertices of R_1 and R_2 that ψ assigns the same color to and we obtain a graph G' embedded in the torus or in the Klein bottle (in the latter case, we can assume that both R_1 and R_2 are triangles). Note that G' has no loops, since R_1 and R_2 are not adjacent. Observe also that G' contains no contractible (≤ 4)-cycle. Since G' is not 3-colorable, Theorems 43 and 44 imply that G' is embedded in the Klein bottle and contains a canonical embedding of an (e_1, e_2)-chain as a subgraph. Therefore, G' contains a separating non-contractible 4-cycle C. The subgraph of G corresponding to C contains at least two paths joining R_1 and R_2. However, this implies that the distance between R_1 and R_2 is at most two, which is a contradiction. □

The main result of this chapter is a similar characterization for $(R_1 \cup R_2)$-critical graphs, where each of R_1 and R_2 has length at most four. A *broken chain* is a graph obtained from an (e_1, e_2)-chain by removing the edges e_1 and e_2, see Figure 6.2 for an illustration. Note that if A and B have different colors, then the colors of C and D must differ as well, hence there exist $(R_1 \cup R_2)$-critical graphs embedded in the cylinder, where R_1 and R_2 are arbitrarily distant 4-cycles. Dvořák and Lidický [29] gave a complete list of $(R_1 \cup R_2)$-critical graphs that are not broken chains; in particular, they showed that the there are only finitely many such graphs. However, their proof is computer assisted. In this paper, we give a much weaker bound on the size of the graphs, which however suffices for our purposes.

Theorem 47. *Let G be an $(R_1 \cup R_2)$-critical graph embedded in the cylinder Σ, where each of R_1 and R_2 is either a vertex ring or a facial ring of length at most four. Suppose that every cycle of length at most 4 in G is non-contractible. If G contains at least 34 cycles of length at most 4, then G is a broken chain.*

Proof. The graph G is connected by Theorem 45. Let C_1 and C_2 be distinct cycles of length at most 4 in G. We claim that C_1 bounds a closed disk in $\hat{\Sigma}$ that contains C_2. Indeed, otherwise each of the open disks in $\hat{\Sigma}$ bounded by C_1 contains a vertex of C_2, and we conclude that the set $X = V(C_1) \cap V(C_2)$ has size two. But then there exist three disjoint paths of length at most two between the vertices of X, and one of the (≤ 4)-cycles in this θ-graph is contractible in Σ, contradicting the assumptions.

We write $C_1 < C_2$ if the closed disk bounded by C_1 in $\Sigma + \hat{R}_2$ contains C_2. Note that $<$ is a linear ordering of the cycles of length at most four in G. Let $K_1, K_2, \ldots, K_m$ be the list of all cycles of length at most four in G sorted according to this ordering. Suppose that K_i and K_j are triangles for some $i < j$. By Theorem 41, if $V(K_i) \cap V(K_j) \neq \emptyset$, then $j = i + 1$. If K_i and K_j are vertex-disjoint, then Lemma 46 implies that $j \leq i + 3$.

For any $i < j$, if $V(K_i) \cap V(K_j) \neq \emptyset$, then by Theorem 41 the area between K_i and K_j consists either of one face or of two 5-faces, thus either $j = i + 1$, or $j = i + 2$ and K_{i+1} is a triangle. In particular, K_i and K_{i+3} are vertex-disjoint.

For $i < j$, let G_{ij} be the subgraph of G drawn between K_i and K_j. Note that if K_i and K_j are vertex-disjoint, then G_{ij} is $(K_i \cup K_j)$-critical, and by Lemma 42 it is subsumed by a (K_i, K_j)-critical basic graph H_{ij}. If K_i and K_j are not vertex-disjoint, then we define $H_{ij} = G_{ij}$. Consider indices $i < j < k$ and a graph $B \in \{G_{ij}, H_{ij}\}$, and suppose that $B \cup H_{jk}$ contains a non-contractible cycle C of length at most 4. By the definition of a basic graph, $C \not\subseteq B$ and $C \not\subseteq H_{jk}$, thus C has length 4 and $C = v_1v_2v_3v_4$, where $v_2, v_4 \in V(K_j)$, $v_1 \in V(B) \setminus V(K_j)$ and $v_3 \in V(H_{jk}) \setminus V(K_j)$. Furthermore, v_2 and v_4 must be consecutive vertices of K_j, thus both B and H_{jk} contain a triangle incident with an edge of K_j.

Let U be the set of indices i such that either there exists $t \geq \max(1, i-1)$ such that K_t is a triangle or there exists j such that $i < j \leq m$ and H_{ij} contains a cutvertex or a triangle. Let L be the set of indices j such that either there exists $t \leq \min(m, j+1)$ such that K_t is a triangle or there exists i such that $1 \leq i < j$ and H_{ij} contains a cutvertex or a triangle. Suppose that $a \in L$ and $b \in U$ satisfy $b \geq a + 6$. If there exists $t \leq \min(m, a+1)$ such that K_t is a triangle, then let $G_1 = G_{1a}$. Otherwise, there exists $k < a$ such that H_{ka} contains a cutvertex or a triangle, and we set $G_1 = G_{1k} \cup H_{ka}$. Similarly, if there exists $t \geq \max(1, b-1)$ such that K_t is a triangle, then let $G_2 = G_{bm}$, otherwise let $G_2 = H_{bl} \cup G_{lm}$ for $l > b$ such that H_{bl} contains a cutvertex or a triangle. For $i \in \{1, 2\}$, let T_i be a triangle or a cutvertex in G_i. Let $G' = G_1 \cup G_{ab} \cup G_2$. By the choice of G_1 and G_2, every (≤ 4)-cycle in G' is non-contractible. By Lemma 46, every precoloring of $T_1 \cup T_2$ extends to a 3-coloring of the subgraph of G' between T_1 and T_2. Note that G' contains no non-contractible (≤ 4)-cycles, and thus by Theorem 45, every precoloring of R_i extends to a 3-coloring of the subgraph of G' between R_i and T_i, for $i \in \{1, 2\}$. Therefore, every precoloring of $R_1 \cup R_2$ extends to a 3-coloring of G'. However, G' subsumes G, hence every precoloring of $R_1 \cup R_2$ also extends to a 3-coloring of G. This contradicts the criticality of G. We conclude that if

$a \in L$ and $b \in U$, then $b \leq a+5$. Let X be the set of indices i such that $i \in U$ and $i+2 \in L$. Observe that if X is nonempty, then $\max X - \min X \leq 7$.

Since $m \geq 34$, there exist indices $1 \leq i_1 < i_2 < \ldots < i_8 \leq m$ such that $i_{j+1} = i_j + 2$ for $1 \leq j \leq 4$ and for $5 \leq j \leq 8$, $i_5 \geq i_4 + 9$ and $i_1, i_2, i_3, i_5, i_6, i_7 \notin X$. Note that $G_{i_1 i_4}$ and $G_{i_5 i_8}$ are triangle-free, and K_{i_j} and $K_{i_{j+1}}$ are vertex-disjoint and $H_{i_j i_{j+1}}$ is 2-connected and triangle-free for $1 \leq j \leq 3$ and $5 \leq j \leq 7$.

Combining (22) and (23) shows that there exists a precoloring ψ of K_{i_4}, a vertex $v \in V(K_{i_2})$ and a color c such that every precoloring ϕ_2 of $K_{i_2} \cup K_{i_4}$ that matches ψ on K_{i_4} and satisfies $\phi_2(v) \neq c$ extends to a 3-coloring of $H_{i_2 i_3} \cup H_{i_3 i_4}$. Observe that every 3-coloring of K_{i_1} extends to a 3-coloring of $H_{i_1 i_2}$ that assigns v a color different from c. It follows that every precoloring ϕ of $K_{i_1} \cup K_{i_4}$ that matches ψ on K_{i_4} extends to a 3-coloring of $H_{i_1 i_4}$, and thus also to a 3-coloring of $G_{i_1 i_4}$. In fact, it is sufficient to assume that ϕ has the same type S_1 on K_{i_4} as ψ; thus, every precoloring of $K_{i_1} \cup K_{i_4}$ whose type on K_{i_4} is S_1 extends to a 3-coloring of $G_{i_1 i_4}$. Symmetrically, there exists a type S_2 such that every precoloring of $K_{i_5} \cup K_{i_8}$ whose type on K_{i_5} is S_2 extends to a 3-coloring of $G_{i_5 i_8}$.

Let $G' = G_{i_4 i_5}$ with rings $L_1 = K_{i_4} = a_1 a_2 a_3 a_4$ and $L_2 = K_{i_5} = b_1 b_2 b_3 b_4$. Since $i_5 \geq i_4 + 9$, the distance between L_1 and L_2 is at least three. Let G'' be the graph obtained from G' in the following way: If $S_1 = \{a_i, a_{i+2}\}$ for some $i \in \{1, 2\}$, then add the edge $a_i a_{i+2}$ to the face bounded by L_1 and add a crosscap to the middle of this edge. If $S_1 = \emptyset$, then identify a_1 with a_3 to a vertex a_{13} and a_2 with a_4 to a vertex a_{24}. Observe that at most two vertices of L_1 are incident with a (≤ 4)-cycle distinct from L_1 in G', and if there are two such vertices, then they are adjacent. By symmetry, we can assume that L_1 is the only (≤ 4)-cycle incident with a_2 and a_3. We add a crosscap on the edge $a_{13} a_{24}$ and draw the edges from a_{13} to the neighbors of a_3 and the edges from a_{24} to the neighbors of a_2 through the crosscap. Transform L_2 in the same way according to S_2. Note that G'' is embedded in the Klein bottle and it has no loops.

Consider a cycle C of length at most 4 in G''. Since the distance between L_0 and L_1 is at least three, we may assume that C does not contain any of the vertices $b_1, \ldots, b_4, b_{13}$ or b_{24}. Let us first consider the case that $S_1 = \{a_i, a_{i+2}\}$ for some $i \in \{1, 2\}$. If C does not contain the edge $a_i a_{i+2}$, then C is non-contractible in G, and thus it separates the crosscaps in G''. If C contains the edge $a_i a_{i+1}$, then C is one-sided. Suppose now that $S_1 = \emptyset$; as in the construction of G', we assume that L_1 is the only (≤ 4)-cycle incident with a_2 and a_3 in G'. If C contains the edge $a_{13} a_{24}$, then C corresponds to a (≤ 4)-cycle in G' containing one of the edges of L_1, which necessarily must be $a_1 a_4$, and C separates the crosscaps. If C contains neither a_{13} nor a_{24}, then C is non-contractible in G and separates the crosscaps in G''. If C contained both a_{13} and a_{24}, but not the edge $a_{13} a_{24}$, then since a_2 and a_3 are not incident with (≤ 4)-cycles in G', we conclude that $a_1 a_4$ is incident with two triangles in G', contradicting Theorem 41. It remains to consider the case that C contains exactly one of a_{13} and a_{24}. By symmetry, assume that C contains a_{13}. Let e'_1 and e'_2 be the edges incident with a_{13} in C, and let e_1 and e_2 be the corresponding edges in G. Since no (≤ 4)-cycle different from L_1 is incident with a_3, we may assume that e_1 is incident with a_1. If e_2 is incident with a_3, then C is one-sided. If e_2 is incident with a_1, then C separates the crosscaps. We conclude that every (≤ 4)-cycle in G'' is non-contractible.

If G'' is 3-colorable, then the corresponding 3-coloring of G' has type S_1 on L_1 and type S_2 on L_2. It follows that every precoloring of $K_{i_1} \cup K_{i_8}$ extends to a 3-coloring of the subgraph $G_{i_1 i_8}$, contradicting the criticality of G.

Therefore, G'' is not 3-colorable and it contains a 4-critical subgraph F. By Theorem 43, F is a (x_1x_2, y_1y_2)-chain, for some vertices $x_1, x_2, y_1, y_2 \in V(G'')$, and its embedding derived from the embedding of G'' is canonical. Suppose that $S_1 = \emptyset$ and that L_1 is the only (≤ 4)-cycle in G' incident with a_2 and a_3. By symmetry, we can assume that $x_1 = a_{13}$ and x_1x_2 corresponds to an edge a_3v in G'. Since x_1x_2 is incident with two triangles $x_1x_2v_1$ and $x_1x_2v_2$ in F, but a_3 is not incident with a triangle, we have $a_1v_1, a_1v_2 \in E(G')$. Note that the 4-cycle $a_1v_1vv_2$ is non-contractible in G', thus by Theorem 41, say $a_1a_2a_3vv_1$ and $a_1a_4a_3vv_2$ are faces of G' and a_2 and a_4 have degree two. On the other hand, if $S_1 = \{a_i, a_{i+2}\}$ for some $i \in \{1, 2\}$, then one of x_1x_2, y_1y_2 is equal to a_ia_{i+2} and L_1 is a subgraph of F. A symmetrical claim holds at L_2. Using Theorem 41, observe that every non-triangular face of F is also a face of G'. We conclude that G' is a broken chain.

Let us consider the subgraph G_1 of G drawn between R_1 and K_{i_4}. We choose the labelling of L_1 and L_2 so that a_1 and b_1 are vertices of degree four in G'. Observe that a precoloring ψ of $L_1 \cup L_2$ extends to a 3-coloring of G' if and only if $\psi(a_1) \neq \psi(a_3)$ or $\psi(b_1) \neq \psi(b_3)$. Let ϕ be a precoloring of $R_1 \cup K_{i_5}$ that does not extend to a 3-coloring of the subgraph G_2 of G drawn between R_1 and K_{i_5}. Let G'_1 be the graph obtained from G_1 in the following way: first, we add the edge a_1a_3 and put a crosscap on it. If R_1 is a vertex ring or a triangle, then we paste a crosscap over the cuff incident with R_1. If R_1 is a 4-cycle, then we either add an edge between two of its vertices or identify its opposite vertices according to the type of ϕ on R_1 and put a crosscap in the appropriate place, using the same rules as in the construction of G''. Note that G'_1 is embedded in the Klein bottle so that all contractible cycles have length at least five. If G'_1 is 3-colorable, then its 3-coloring corresponds to a 3-coloring of G_1 that matches ϕ on R_1 and assigns a_1 and a_3 different colors. Hence, this coloring extends to a 3-coloring of G_2 that matches ϕ on $R_1 \cup K_{i_5}$, which is a contradiction.

Therefore, G'_1 is not 3-colorable, and by Theorem 43, G'_1 contains a canonical embedding of a (e_1, e_2)-chain F_1, for some vertices $e_1, e_2 \in E(G'_1)$. Since F_1 contains four one-sided triangles, it follows that $|R_1| = 4$. However, as in the analysis of G', we conclude that then G_1 is a broken chain. By symmetry, the subgraph of G drawn between K_{i_5} and R_2 is a broken chain as well. This implies that G is a broken chain. □

Part II

Weight technique

The ideas described in Section 2.4 are usually sufficient for determining the exact list of critical graphs. However, occasionally this either is not practical (when the list is too large and complicated to describe) or not possible (when the considered class of graphs is parameterized, say by a surface in that the graphs are embedded, and thus the set of critical graphs depends on the parameter in a nontrivial way). In such a case, we might want to at least prove that the list of critical graphs is finite, i.e., that the critical graphs in the given class have a bounded size.

In such a case, an attempt to prove the claim by a straightforward application of the method of reducible configurations fails: suppose that we want to prove that all critical graphs in the class have at most B vertices, and for a contradiction, consider a counterexample G with the smallest number of vertices (or edges). Using the fact that $|V(G)| > B$, we may be able to find a reducible configuration in G, and by reducing it (and possibly restricting to a critical subgraph of the result), we obtain a smaller critical graph G'. By the minimality of G, we have $|V(G')| \le B$. However, reducing the configuration typically decreases the number of vertices by some constant k, and thus the best that we can argue is that $|V(G)| \le B + k$.

The *weight technique* sidesteps this problem: instead of bounding the number of vertices of a critical graph, we bound a different quantity, which we call weight. The exact definition of the weight depends on the considered class of graphs. The most typical setting is as follows: suppose that we consider k-colorability of graphs in a subclass $\mathcal{G}$ of planar graphs with a special facial cycle C, and we would like to bound the size of C-critical graphs from $\mathcal{G}$. We pick a function $s : N \to R^+$ and to each face $f \ne C$ of a graph $G \in \mathcal{G}$, we assign the weight $s(\ell(f))$. The weight $w(G)$ of G is defined as the sum of these weights. The choice of s depends on the particular situation, and it determines the dependence of the bound on the number of vertices compared to the length of C (e.g., if s is asymptotically quadratic, the obtained bound is $|V(G)| \le O(\ell^2(C))$. For the method to have a chance to succeed, the bound of form $w(G) \le s(\ell(C))$ has to hold. However, often a stronger bound of form $w(G) \le s(\ell(C) - k)$ (for some constant k) is necessary to give a better basis for induction (this may require excluding some special cases, e.g. the case when G is C with a single chord). Also, it is typically convenient for s to be increasing and (non-strictly) convex.

Suppose that we aim to prove that every C-critical graph G satisfies $w(G) \le s(\ell(C) - k) \le s(\ell(C))$ by the method of reducible configurations. By induction, assume that this claim holds for all graphs with fewer than $|V(G)|$ vertices. Suppose first that G is not strongly C-critical, and let $G' \supset T$ be a C-critical subgraph of G. We have $w(G') \le s(\ell(C) - k)$. By Theorem 8, if $f \ne C$ is a face of G' and G_f is the subgraph of G drawn in G_f, then $G_f = f$ or G_f is f-critical, and thus $w(G_f) \leq s(\ell(f))$. It follows that

$$w(G) = \sum_{f \in F(G')} w(G_f) \leq \sum_{f \in F(G')} s(\ell(f)) = w(G') \leq s(\ell(C) - k)$$

as required.

If G is strongly C-critical, then let φ be a coloring of C that extends to every proper subgraph of G including C, but not to G. By discharging technique (comparing the charges with the weights of faces), we argue that if $w(G') > s(\ell(C) - k)$, then G contains a reducible configuration. Let G' be the graph obtained from G by reducing the configuration, and let $G'' \supset C$ be the smallest subgraph of G' to that φ does not extend. Note that G'' is (strongly) C-critical. Let G''' be a subgraph of G whose faces correspond to the faces of G'', where G''' is obtained from G'' by adding some paths through the reduced configuration. The constant k needs to be chosen so that if f is a face of G'' and K is the corresponding cycle in G''', then $\ell(K) \leq \ell(f) + k$, and if $\ell(K) > \ell(f)$ (which may only happen for cycles passing through the reduced configuration), then K is not a face of G. Hence, if $\ell(K) > \ell(f)$, then G_K is K-critical and $w(G_K) \leq s(\ell(K) - k) \leq s(\ell(f))$ by induction. It follows that we again have

$$w(G) = \sum_{K \in F(G''')} w(G_K) \leq \sum_{f \in F(G'')} s(\ell(f)) = w(G'') \leq s(\ell(C) - k),$$

finishing the proof.

Once we have the proof of the bound on the weight, it is easy to use it to bound the number of vertices using Euler formula. Let us give an example of an application of the technique for plane graphs critical for 6-coloring. Let $s(3) = 1/8$ and $s(d) = d - 3$ for $d > 3$.

Lemma 48. *Let G be a plane graph and T its facial cycle of length t. Let $w(G) = \sum_{f \in F(G), f \neq T} s(\ell(f))$. If G is T-critical for 6-colorability and G does not consist of T with at most two chords, then $t \geq 6$ and $w(G) \leq s(t-3) + 5s(3)$.*

Proof. Let us note that if G is T with one chord, then $w(G) \leq s(t-1) + s(3)$ and if G is T with two chords, then $w(G) \leq s(t-2) + 2s(3)$. Suppose for a contradiction that G is a counterexample with the smallest number of edges not belonging to T, and subject to that with the smallest total number of vertices. Note that G is 2-connected. Furthermore, T is an induced cycle: otherwise, consider a chord e of T and let T_1 and T_2 be the cycles in $T + e$ distinct from e. Let G_1 and G_2 be the subgraphs of G drawn inside T_1 and T_2, respectively, and note that either $G_i = T_i$ or G_i is T_i-critical for $i \in \{1, 2\}$. Let t_i denote the length of T_i. If say $G_1 = T_1$, then G_2 is not T_2 with at most one chord. It follows that $t_2 \geq 5$, and thus $t \geq 6$. Furthermore, $w(G) = w(G_1) + w(G_2) \leq s(t_1) + s(t_2 - 2) + 2s(3) \leq (t_1 - 3 + s(3))) + (t_2 - 5 + s(3)) + 2s(3) = t_1 + t_2 - 8 + 4s(3) < s(t-3) + 5s(3)$.

This is excluded, since G is a counterexample; hence $G_1 \neq T_1$ and by symmetry, $G_2 \neq T_2$. It follows that $t_1, t_2 \geq 4$, $t \geq 6$ and $w(G) = w(G_1) + w(G_2) \leq (s(t_1 - 1) + s(3)) + (s(t_2 - 1) + s(3)) \leq (t_1 - 4 + 2s(3)) + (t_2 - 4 + 2s(3)) < s(t-3) + 5s(3)$, a contradiction.

As outlined in the introduction to the weight technique, we can assume that G is strongly T-critical, and thus there exists a precoloring φ of T that extends to a 6-coloring of every proper subgraph of G including T, but not to a 6-coloring of G. Furthermore, G has no separating cycles of length at most five. Let us note that since every plane graph is 4-colorable and thus every precoloring of a triangle extends to a 4-coloring of the whole planar graph, T has length at least four.

No vertex of T has degree two: otherwise, consider a subpath $t_1t_2t_3$ of T such that t_2 has degree two. Note that since T is an induced cycle and G is 2-connected, t_2 is not incident with a triangle and $t \geq 5$. If $\varphi(t_1) \neq \varphi(t_3)$, then let $G' = G - t_2 + t_1t_3$ and $T' = T - T_2 + t_1t_3$. If $\varphi(t_1) = \varphi(t_3)$, let G' be the graph obtained from $G - t_2$ by identifying t_1 with t_3 and let T' be obtained from T in the same way; note that only one parallel edge may arise (if t_2 is incident with a 4-face) and we suppress it. Observe that G' is strongly T'-critical and T' is an induced cycle; hence, $\ell(T') \geq 6$ and $w(G') \leq s(\ell(T') - 3) + 5s(3)$. Let m be the length of the face of G incident with t_2 distinct from T. If $\varphi(t_1) \neq \varphi(t_3)$, then $t = \ell(T') + 1 \geq 7$. Furthermore note that G' has a face of length $m - 1$, and thus $m - 1 \leq \ell(T') - 3$; hence, $s(m) - s(m-1) \leq s(t-3) - s(t-4)$. We have $w(G) = w(G') + s(m) - s(m-1) \leq s(t-4) + 5s(3) + s(m) - s(m-1) \leq s(t-3) + 5s(3)$. If $\varphi(t_1) = \varphi(t_3)$, then $t = \ell(T') + 2 \geq 8$ and we have $w(G) = w(G') + s(m) - s(m-2)$ if $m > 4$ and $w(G) = w(G') + s(4)$ if $m = 4$. Again, we have $s(m) - s(m-2) \leq s(t-3) - s(t-5)$ (where $s(2) = 0$) and we conclude that $w(G) \leq s(t-3) + 5s(3)$.

Observe that T does not contain two consecutive vertices of degree three such that all incident faces different from T are triangles. Otherwise, T would contain a subpath $P = t_1t_2t_3t_4$ and there would exist a vertex $x \in V(G) \setminus V(T)$ adjacent to all vertices of P. Let G' be the subgraph of G drawn inside the cycle K consisting of a subpath of T different from P and of t_1xt_4. Since x has degree at least 6, G' is not K with at most one chord, and thus $w(G') \leq s(\ell(K) - 2) + 2s(3)$. Since $\ell(K) = t - 1$ and $w(G) = w(G') + 3s(3)$, it follows that $w(G) \leq s(t-3) + 5s(3)$, which is a contradiction.

We claim that G contains a vertex z of degree 6 incident only with triangular faces. We prove this by discharging. Suppose that every vertex $z \notin V(T)$ is incident with a non-triangular face. Let the initial charge of each vertex $v \in V(G)$ be $c(v) = \deg(v) - 6$ and the charge of each face f be $c(f) = 2\ell(f) - 6$. The sum of the charges is -12. We now redistribute the charge according to the following rules:

- each vertex incident to a triangular face f of G sends $1/20$ to f.

- each face distinct from T of length at least four sends $1/4$ to each incident vertex.

Let us now discuss the final charge of vertices and faces. By the criticality of G, each vertex $v \notin V(T)$ has degree at least six. If v has degree at least 7, then it sends at most $\deg(v)/20$ to the neighboring triangular faces, and its final charge is at least $19\deg(v)/20 - 6 > 0$. If v has degree six, then it is incident with a face of length at least four, v sends at most $5/20$ to the incident triangular faces and receives at least $1/4$ from the non-triangular faces, and the final charge of v is non-negative. Each vertex $v \in V(T)$ has degree at least three and sends charge to at most $\deg(v) - 1$ triangular faces, and thus its final charge is at least $\deg(v) - 6 - (\deg(v) - 1)/20 = 19\deg(v)/20 - 119/20 \geq -31/10$. Furthermore, if $\deg(v) > 3$ or v is incident with a (≥ 4)-face other than T, then the final charge of v is at least $-31/10 + 1/4 = -57/20$. Hence, the sum of charges of two consecutive vertices of T is at least $-119/20$, and the sum of charges of all vertices of T is at least $-119t/40$.

Consider a face $f \neq T$. If f is a triangle, then it receives $3/20$ from the incident vertices. Otherwise, it sends $\ell(f)/4$, and its final charge is $7\ell(f)/4 - 6$. In both cases, the final charge of f is at least $s(\ell(f))$. The final charge of T is equal to its initial charge. Combining these inequalities, we have

$$-12 \geq -119t/40 + (2t - 6) + \sum_{f \neq T} s(f),$$

and thus $w(G) \leq 39t/40 - 6 < s(t-3) + 5s(3)$. This is a contradiction.

Therefore, there exists $z \in V(G) \setminus V(T)$ of degree 6 such that all incident faces are triangles. Let Q be the set of neighbors of z that belong to T and let $q = |Q|$. Consider the case that $q \geq 2$. Let $C_1, \ldots, C_q$ and T be the faces of the subgraph of G induced by $V(T) \cup \{z\}$. For $1 \leq i \leq q$, let $t_i = \ell(C_i)$. We apply induction to the subgraphs of G drawn inside the induced cycles $C_1, \ldots, C_q$. If each of the cycles bounds a face, then since z has degree six, we have $q = 6$. If $t = 6$, then $w(G) = 6s(3) = s(t-3) + 5s(3)$. If $t > 6$, then we can assume that C_1 is not a triangle, and thus

$$\begin{aligned}
w(G) &\leq \sum_{i=1}^{6} s(t_i) \\
&\leq -s(3) + \sum_{i=1}^{6} (t_i - 3 + s(3)) \\
&= t - 6 + 5s(3) \\
&= s(t-3) + 5s(3).
\end{aligned}$$

Otherwise, say C_1 does not bound a face, $t_1 \geq 6$ and

$$w(G) \leq s(t_1 - 3) + 5s(3) + \sum_{i=2}^{q} s(t_i)$$

$$
\begin{aligned}
&\leq && t_1 - 6 + 5s(3) + \sum_{i=2}^{q}(t_i - 3 + s(3)) \\
&= && t + 2q - 6 + (s(3) - 3)(q - 1) + 5s(3) \\
&= && t - 3 - q + (q - 1)s(3) + 5s(3).
\end{aligned}
$$

Since $w(G) > s(t-3) + 5s(3)$, we have $q \leq 3$. Furthermore, if $q = 3$, then a similar calculation shows that C_2 and C_3 bound a face.

Let $z_1, \ldots, z_6$ be the neighbors of z in the cyclic order according to their drawing around z. By the previous paragraph, we can assume that $z_1, z_3 \notin V(T)$. Note that z_1 and z_3 are not adjacent and have no common neighbor other than z_2, as otherwise G would contain a separating cycle of length at most four. Let G' be the graph obtained from $G - z$ by identifying z_1 with z_3 to a new vertex x and suppressing the resulting parallel edge between x and z_2. Note that T is an induced cycle in G'. Furthermore, φ does not extend to a 6-coloring of G', as otherwise it would give a 6-coloring of $G - z$ assigning z_1 and z_3 the same color, which would extend to a 6-coloring of G. Let G'' be a strongly T-critical subgraph of G'. Since φ extends to every proper subgraph of G including T, we have that $x \in V(G'')$. Let G''' be the subgraph of G obtained from G'' by splitting x to z_1 and z_3 and by adding the path $z_1 z z_3$; if $xz_2 \in E(G')$, we put the edge z_1z_2 to $E(G''')$, but we do not include edges z_2z and z_2z_3. Note that z_3 has degree at least two in G''', as otherwise G'' would be a subgraph of G, contrary to the fact that G is strongly T-critical. Let f_1 and f_2 be the faces of G''' incident with z, where the edge zz_2 is drawn inside f_1 in G. Since G'' is T-critical, it is 2-connected, and thus $f_1 \neq f_2$. Let us note that every face f of G''' corresponds to a face $\overline{f}$ of G'' in a natural way, $\ell(f_i) = \ell(\overline{f_i}) + 2$ for $1 \leq i \leq 2$ and $\ell(f) = \ell(\overline{f})$ for any other face f of G'''. Let G_i denote the subgraph of G drawn inside f_i, for $1 \leq i \leq 2$. Observe that G_2 is not f_2 with at most two chords, as the degree of z in G_2 is at least 5. Similarly, G_1 is not f_1 with at most one chord, since $zz_2, z_2z_3 \notin E(G_1)$. By induction, we have

$$
\begin{aligned}
w(G) &\leq s(\ell(f_1) - 2) + 2s(3) + s(\ell(f_2) - 3) + 5s(3) + \sum_{f \in F(G''') \setminus \{f_1, f_2, T\}} s(\ell(f)) \\
&= w(G'') + 7s(3) + s(\ell(f_2) - 3) - s(\ell(f_2) - 2) \\
&\leq w(G'') + 8s(3) - 1 \\
&\leq w(G'') \\
&\leq s(t - 3) + 5s(3).
\end{aligned}
$$

This contradicts the assumption that G is a counterexample to the lemma, and finishes the proof. □

Let us remark that Lemma 48 is tight, with the equality achieved for graphs consisting of T and a vertex with 6 neighbors in T, with five triangular faces and one face of length $t-3$. It is possible (and sufficient) to prove a weaker bound of form $w(G) \le s(t-2)$. However, the analysis of the last case is a bit more involved, as we cannot compensate for the extra $2s(3)$ from f_1 by using the stronger bound for f_2 (so, if G_1 were f_1 with two chords, we would need to consider the graph drawn inside the cycle bounded by $z_1z_2z_3$ and a subpath of f_2, instead of G_2).

On the other hand, the numeric value of $s(3)$ is certainly not optimal and could be improved by a more detailed analysis.

Theorem 49. *Let G be a plane graph and T its facial cycle of length t. If G is T-critical for 6-colorability, then G has at most $5t-21$ vertices.*

Proof. The claim clearly holds if $V(G) = V(T)$. If $V(G) \neq V(T)$, then by Lemma 48 we have $t \ge 6$ and $w(G) \le t - 6 + 6s(3)$. Let us give a charge $s(\ell(f))$ to each face $f \neq T$ of G, so that the sum of the charges is $w(G)$. Then, each face other than T sends $s(3)/3$ to each incident vertex. The final charge of each vertex $v \notin V(T)$ is at least $2s(3)$, and the charge of faces is nonnegative, thus $2s(3)(|V(G)|-t) \le t-6+6s(3)$. It follows that $|V(G)| \le \left(1 + \frac{1}{2s(3)}\right)t - \frac{3-3s(3)}{s(3)} = 5t-21$. $\square$

So far, we have presented the technique for plane graphs with a precolored facial cycle. This is the simplest setting and it is a prerequisite for dealing with more complicated cases (we need it to apply Theorem 8 to switch between the critical graph and its subgraphs). However, what we really would like to obtain using the weight technique is a bound on the weight (and consequently, the size) of critical graphs embedded in a fixed surface. Such an extension is fairly straightforward, although numerous technical details complicate the exposition. For concreteness, suppose that we want to extend Lemma 48 to show that the weight of a graph embedded in a surface of genus at most g is at most linear in g:

- When identifying z_1 with z_3, loops could arise if z_1 and z_3 are adjacent, i.e., if z_1zz_3 form a non-contractible cycle. In this case, we apply Lemma 9 and cut the surface along z_1zz_3 instead, decreasing its genus (and possibly splitting it to two pieces) and introduce two new precolored triangles (or one precolored 6-cycle, if z_1zz_3 is one-sided). Thus, we will need to consider a T-critical graph G embedded in a surface Σ with boundary, where T is the intersection of G with the boundary of Σ and each component of the boundary of Σ is a cycle in G. Let g denote the genus of Σ and k the number of components of its boundary. We will prove a bound of form

$$w(G) \le |E(T)| + C_1g + C_2k - C_3,$$

where C_1, C_2 and C_3 are appropriately chosen constants. For the ideas of this paragraph to work, we need $C_1 > C_2$: cutting along a non-separating cycle decreases the genus by a and increases the number of components of the boundary by a, where $a = 1$ if the cycle is one-sided and $a = 2$ if it is two-sided. Furthermore, cutting along a separating cycle does not change the sum of genera of the resulting surfaces and increases the number of components of the boundary by 2, thus we need $C_3 > 2C_2$. Both inequalities must be strict, since cutting along triangle also increases $|E(T)|$ by 6.

- We run into problems in the cases that the term $C_1 g + C_2 k - C_3$ is negative. As we will see later, the constants C_1, C_2 and C_3 need to be rather large; hence, at least for the small values of $|E(T)|$ the resulting inequality would be violated for some critical graphs. By the inequalities of the previous paragraph, the term $C_1 g + C_2 k - C_3$ is negative if $g = 0$ and $k \leq 2$. There are no $\emptyset$-critical plane graphs for 6-colorability, hence the case $g = k = 0$ is fine. The case $g = 0$ and $k = 1$ was dealt with in Lemma 48 (and in the argument of the preceding paragraph, it only arises when we cut the projective plane along a one-sided cycle; however, there are no $\emptyset$-critical projective planar graphs for 6-colorability, thus this does not happen). That leaves us with the case $g = 0$ and $k = 2$, i.e., when Σ is the cylinder. In this case, we will need to prove an inequality $w(G) \leq |E(T)| + C_4$ for some constant C_4, instead.

- Consequently, the argument that we used to deal with non-contractible triangles will fail if the triangle K surrounds one of the components of the boundary, i.e., if it becomes contractible after we cap the component with a disk (this is natural, since cutting along K splits Σ to a cylinder and a surface homeomorphic to Σ, thus this does not simplify the problem). However, by Lemma 3, there are at most $18k$ such triangles in G, and if $C_2 \gg 18$, we can increase the charge of the vertices incident with them in the discharging phase of the argument and thus ensure that no such triangle passes through the vertex z forming the considered reducible configuration.

- Another way how the reduction could fail is if all the neighbors of z belong to T. The arguments in Lemma 48 exclude this case unless either two of the neighbors z_1 and z_2 of z belong to different components of T, or if they belong to the same component of T, but the path $z_1 z z_2$ is not homotopic to any of the two paths joining z_1 with z_2 in T. Both of the cases are dealt with by cutting along the path $z_1 z z_2$. In the latter case, the analysis is the same as in the case of non-contractible triangles. In the former case, we are decreasing the number of components of the boundary by 1, which compensates the increase in $|E(T)|$.

- The analysis of the relationship between the weights of G'' and G is somewhat more complicated, since G'' can have faces that are not homeomorphic to open disks. However, it turns out that the inductive argument still goes through and the non-disk faces actually result in a stronger bound on the weight of G. Thus, the most difficult case is the one where all faces of G'' are open disks. Then, the analysis from the plane cases goes through, with a single exception: it is possible that $f_1 = f_2$. In that case, the difference between the length of the face in G'' and the length of the corresponding cycle in G''' would be 4, and if we applied the bound of Lemma 48 to it, we would not be able to show that the weight of the subgraph of G''' drawn in it is bounded by $s(\ell(f_1))$.

In this case, note that there exists a closed simple curve intersecting G'' only in x. We aim to improve the bound on the weight of G'' by first cutting along this curve and applying the induction hypothesis to the resulting graph(s) instead of G''. To enable this, we need to allow the graph intersect a component of the boundary of Σ also in a single vertex, and actually to get a better bound in this case. That is, if k' is the number of the components of the boundary that intersect G only in one vertex, then we will be proving a bound of form $w(G) \leq |E(T)| + C_1 g + C_2 k - C_5 k' - C_3$ if $g > 0$ or $k > 2$ and $w(G) \leq |E(T)| + C_4 - C_5 k'$ if $g = 0$ and $k = 2$. Again, there is a problematic case when we cut along the closed curve intersecting G in a single vertex, one of the resulting surfaces is a cylinder and the other one is homeomorphic to Σ. Still, because of the extra term $-C_5$, we obtain an improved bound, assuming that $2C_5 > C_4$. In the case that $g = 0$, $k = 2$ and $k' = 2$, this implies a negative bound on $w(G)$. Hence, this case (planar graph with T consisting of two isolated vertices) needs to be dealt with separately. It turns out that the only T-critical graph in this case is an edge joining the two vertices of T. We can avoid creating this case in the inductive argument by ensuring that the identified neighbors of z are not adjacent to an isolated vertex of T, which is possible by modifying the discharging phase of the proof.

In Chapter 7, we give an example of the application of the technique to 4-critical graphs of girth at least 5 embedded in surfaces, in which the issues and approaches described in the preceding paragraphs are worked out in details.

The weights technique is not limited to the method of reducible configurations. In Chapter 8, we show its usage it in combination with the precoloring extension method. The chapter also demonstrates the use of the weight technique with a non-linear weight function.

Chapter 7

4-critical graphs of girth 5 on surfaces[1]

Thomassen [71] proved the following.

Theorem 50. *For every surface Σ there are only finitely many 4-critical graphs of girth at least five that can be embedded in Σ.*

This important result shows that it is possible to decide whether a graph of girth at least five embedded in a fixed surface is 3-colorable in polynomial (actually, linear) time by checking for the presence of one of the finitely many 4-critical subgraphs. To prove Theorem 50, Thomassen gives a bound on the size of a 4-critical graph embedded in Σ. However, this bound is double-exponential in the genus of Σ. In this chapter, we give a different (somewhat simpler) proof of the result, which additionally improves the bound to linear.

Theorem 51. *There exists a constant C with the following property. If G is a 4-critical graph of Euler genus g and girth at least 5, then $|V(G)| \leq Cg$.*

In the exposition of the proof of Theorem 51, we essentially follow the outline presented in the introduction to Part II using the method of reducible configurations. In the following section, we give some definitions. In Section 7.2, we describe more precisely what we mean by a reducible configuration, its appearance in the considered graph and its reduction. In Section 7.3, we show that the reductions preserve 3-colorings. In Section 7.4, we give the discharging argument used to show the existence of a reducible configuration. In Section 7.5, we argue that the reductions preserve the assumptions of the theorem. In Section 7.6, we analyze the change of the weights during the reduction. The Section 7.7 is devoted to the case of a plane graph with one precolored facial cycle. In Section 7.9, we similarly consider the special case of a plane graph with two precolored facial cycles. We finish the proof for a general surface in Section 7.10.

[1]The results of this chapter are based on Dvořák et al. [28, 26, 27].

7.1 Definitions

All graphs that in this paper are simple, with no loops or parallel edges.

A *surface* is a compact 2-manifold with (possibly null) boundary. Each component of the boundary is homeomorphic to the circle, and we call it a *cuff*. For non-negative integers a, b and c, let $\Sigma(a, b, c)$ denote the surface obtained from the sphere by adding a handles, b crosscaps and removing interiors of c pairwise disjoint closed discs. A standard result in topology shows that every connected surface is homeomorphic to $\Sigma(a, b, c)$ for some choice of a, b and c. Note that $\Sigma(0, 0, 0)$ is a sphere, $\Sigma(0, 0, 1)$ is a closed disk, $\Sigma(0, 0, 2)$ is a cylinder, $\Sigma(1, 0, 0)$ is a torus, $\Sigma(0, 1, 0)$ is a projective plane and $\Sigma(0, 2, 0)$ is a Klein bottle. The *Euler genus* $g(\Sigma)$ of the surface $\Sigma = \Sigma(a, b, c)$ is defined as $2a + b$. For a cuff C of Σ, let $\hat{C}$ denote an open disk with boundary C disjoint from Σ, and let $\Sigma + \hat{C}$ be the surface obtained by gluing Σ and $\hat{C}$ together, that is, by closing C with a patch. Let $\hat{\Sigma} = \Sigma + \hat{C}_1 + \ldots + \hat{C}_c$, where $C_1, \ldots, C_c$ are the cuffs of Σ, be the surface without boundary obtained by patching all the cuffs.

Consider a graph G drawn in the surface Σ; when useful, we identify G with the topological space consisting of the points corresponding to the vertices of G and the simple curves corresponding to the edges of G. A *face* f of G is a maximal connected subset of $\hat{\Sigma} - G$. The boundary of a face is equal to a union of closed walks of G, which we call the *boundary walks* of f.

An embedding of G in Σ is *normal* if each cuff C that intersects G either does so in exactly one vertex v or is equal to a cycle B in G. In the former case, we call v a *vertex ring* and the face of G that contains C the *cuff face* of v. In the latter case, note that B is the boundary walk of the face $\hat{C}$ of G; we say that B is a *facial ring*. A face of G is a *ring face* if it is equal $\hat{C}$ for some ring C, and *internal* otherwise. We write $F(G)$ for the set of internal faces of G. A vertex v of G is a *ring vertex* if v is incident with a ring (i.e., it is drawn in the boundary of Σ), and *internal* otherwise. A cycle K in G is *separating the surface* if $\hat{\Sigma} - K$ has two components, and K is *non-separating* otherwise. A cycle K is *contractible* if there exists a closed disk $\Delta \subseteq \Sigma$ with boundary equal to K. A cycle K *surrounds the cuff* C if K is not contractible in Σ, but it is contractible in $\Sigma + \hat{C}$. We say that K *surrounds a ring* R if K surrounds the cuff incident with R.

An *embedding of G in Σ with rings $\mathcal{R}$* is a triple $(\mathbf{G}, \mathcal{R}, W)$, where $\mathbf{G}$ is a normal embedding of G in Σ, $\mathcal{R}$ is the set of rings of G and W is a subset of vertex rings. The elements of W are called *weak vertex rings*. At this point, let us remark that weak vertex rings (which behave slightly differently with respect to coloring, see the next paragraph) are just a technicality devised to deal with cutvertices in the case of general surfaces; thus, the reader may ignore them for the moment. Unless explicitly specified otherwise, we assume that every cuff of Σ is incident with a ring in $\mathcal{R}$.

The length $|R|$ of a facial ring is the length of the corresponding face. For a vertex ring R, we define $|R| = 0$ if R is weak and $|R| = 1$ otherwise. For an internal face f, by $|f|$ we mean the sum of the lengths of the boundary walks of f (in particular, if an edge appears twice in the boundary walks, it contributes 2 to $|f|$); if a boundary walk consist just of a vertex ring R, it contributes $|R|$ to $|f|$.

Let G be a graph with rings $\mathcal{R}$. A *precoloring* of $\mathcal{R}$ is a proper 3-coloring of the graph $H = \bigcup \mathcal{R}$. Note that H is a (not necessarily induced) subgraph of G. A precoloring of $\mathcal{R}$ *extends to a 3-coloring of* G if there exists a proper 3-coloring ϕ of G such that $\phi(v) \neq \psi(v)$ for every weak vertex ring v and $\phi(v) = \psi(v)$ for every other vertex v incident with one of the rings. The graph G is $\mathcal{R}$*-critical* if $G \neq H$ and for every proper subgraph G' of G that contains $\mathcal{R}$, there exists a precoloring of $\mathcal{R}$ that extends to a 3-coloring of G', but not to a 3-coloring of G. For a precoloring ϕ of the rings, the graph G is ϕ*-critical* if ϕ does not extend to a 3-coloring of G, but it extends to a 3-coloring of every proper subgraph of G that contains $\mathcal{R}$. Let us remark that if G is ϕ-critical for some ϕ, then it is $\mathcal{R}$-critical, but the converse is not true (for example, consider a graph consisting of a single facial ring with two chords). On the other hand, if ϕ is a precoloring of the rings of G that does not extend to a 3-coloring of G, then G contains a (not necessarily unique) ϕ-critical subgraph.

7.2 Reducible configurations

By a plane graph we mean a graph G drawn in the plane with no crossings. Thus G has exactly one unbounded face, called the *infinite face*; all the other faces are called *finite*. An *isomorphism* of plane graphs maps finite faces to finite faces and the infinite face to the infinite face.

A *configuration* is a quintuple $\gamma = (G, \mathcal{F}, d, \mathcal{I}, \mathcal{A})$, where

- G is a plane graph,
- $\mathcal{F}$ is a set of finite faces of G,
- d is a function that maps a set $\mathrm{dom}(d) \subseteq V(G)$ to $\{3, 4, \ldots\}$,
- $\mathcal{A}$ is a subset of $V(G)$ of size zero or two, and
- $\mathcal{I}$ is a subset of $V(G)$.

If γ is a configuration, then we define $G_\gamma := G$, $\mathcal{F}_\gamma := \mathcal{F}$, $d_\gamma := d$, $\mathcal{I}_\gamma := \mathcal{I}$ and $\mathcal{A}_\gamma := \mathcal{A}$.

Two configurations γ and γ' are *isomorphic* if there exists an isomorphism ϕ of the plane graphs G_γ and $G_{\gamma'}$ that maps $\mathcal{F}_\gamma$ to $\mathcal{F}_{\gamma'}$, $\mathcal{I}_\gamma$ to $\mathcal{I}_{\gamma'}$, $\mathcal{A}_\gamma$ to $\mathcal{A}_{\gamma'}$, $\mathrm{dom}(d_\gamma)$ to $\mathrm{dom}(d_{\gamma'})$ and $d_\gamma(v) = d_{\gamma'}(\phi(v))$ for every $v \in \mathrm{dom}(d_\gamma)$. Figure 7.1 contains the depictions of several configurations, using the following conventions.

The graph G_γ is drawn in the figure (ignoring the "half-edges" and dashed edges for a moment), $\mathcal{F}_\gamma$ consists of all the finite faces of G_γ that do not include any half-edges in their interior, the elements of $\mathcal{I}_\gamma$ are indicated by $\mathcal{I}$ next to them, if $u, v \in \mathcal{A}_\gamma$ are distinct, then they are joined by a dashed edge, the set $\mathrm{dom}(d_\gamma)$ consists of vertices drawn by empty circles, and the value $d_\gamma(v)$ is equal to the number of edges and half-edges incident with v in the figure. A configuration is *good* if it is isomorphic to one of the configurations depicted in Figure 7.1.

Let γ be a good configuration and either let $H = G_\gamma$, or let H be a plane graph obtained from G_γ by identifying two vertices of $V(G_\gamma) \setminus \mathrm{dom}(d_\gamma)$ that are at distance at least five in G_γ. (The latter is only possible when γ is R7 or R7$''$.) In those circumstances we say that H is an *imprint* of γ. It follows that every face in $\mathcal{F}_\gamma$ may be regarded as a face of H, and that $\mathrm{dom}(d_\gamma) \subseteq V(H)$.

Let G be a graph in a surface Σ with rings $\mathcal{R}$. We say that a configuration γ *faintly appears* in the pair $(G, \mathcal{R})$ if

- some imprint H of γ is a subgraph of G,
- every face in $\mathcal{F}_\gamma$ is an internal face of G,
- $\mathrm{dom}(d_\gamma) \cap V(\mathcal{R}) = \emptyset$,
- if $v \in \mathrm{dom}(d_\gamma)$, then $\deg_G(v) = d_\gamma(v)$,
- at most one vertex of $\mathcal{I}_\gamma$ belongs to $V(\mathcal{R})$, and
- no cuff face of a vertex ring belongs to $\mathcal{F}_\gamma$.

If a configuration γ faintly appears in $(G, \mathcal{R})$, then we say that a subgraph J of G *touches* γ if an edge of J is incident with a face in $\mathcal{F}_\gamma$. We say that γ *weakly appears* in the pair $(G, \mathcal{R})$ if it faintly appears and

- no cycle of length at most four distinct from facial rings touches γ and if γ is R7, then $x_3 \neq x_7$ or $x_1 \neq x_6$,
- if $u, v \in \mathrm{dom}(d_\gamma)$ are adjacent in G, then u, v are adjacent in G_γ,
- if γ is isomorphic to R4 and the vertices corresponding to x_4 and x_5 both belong to $\mathcal{R}$, then the vertex corresponding to v_2 does not belong to $\mathcal{R}$.

Let a good configuration γ weakly appear in $(G, \mathcal{R})$. We wish to define a new graph G' in Σ with rings $\mathcal{R}$. For the definition we need to distinguish several cases. Assume first that γ is not isomorphic to R4. Let the graph G' be obtained from $G \backslash \mathrm{dom}(d_\gamma)$ by adding an edge joining the vertices in $\mathcal{A}_\gamma$ if $\mathcal{A}_\gamma \neq \emptyset$ and by identifying the vertices in $\mathcal{I}_\gamma$. If parallel edges are created, remove all edges but one from each bunch of parallel edges, so that each edge of G' corresponds to a unique edge of G. Since no cycle of length at most four touches γ and if γ is R7,

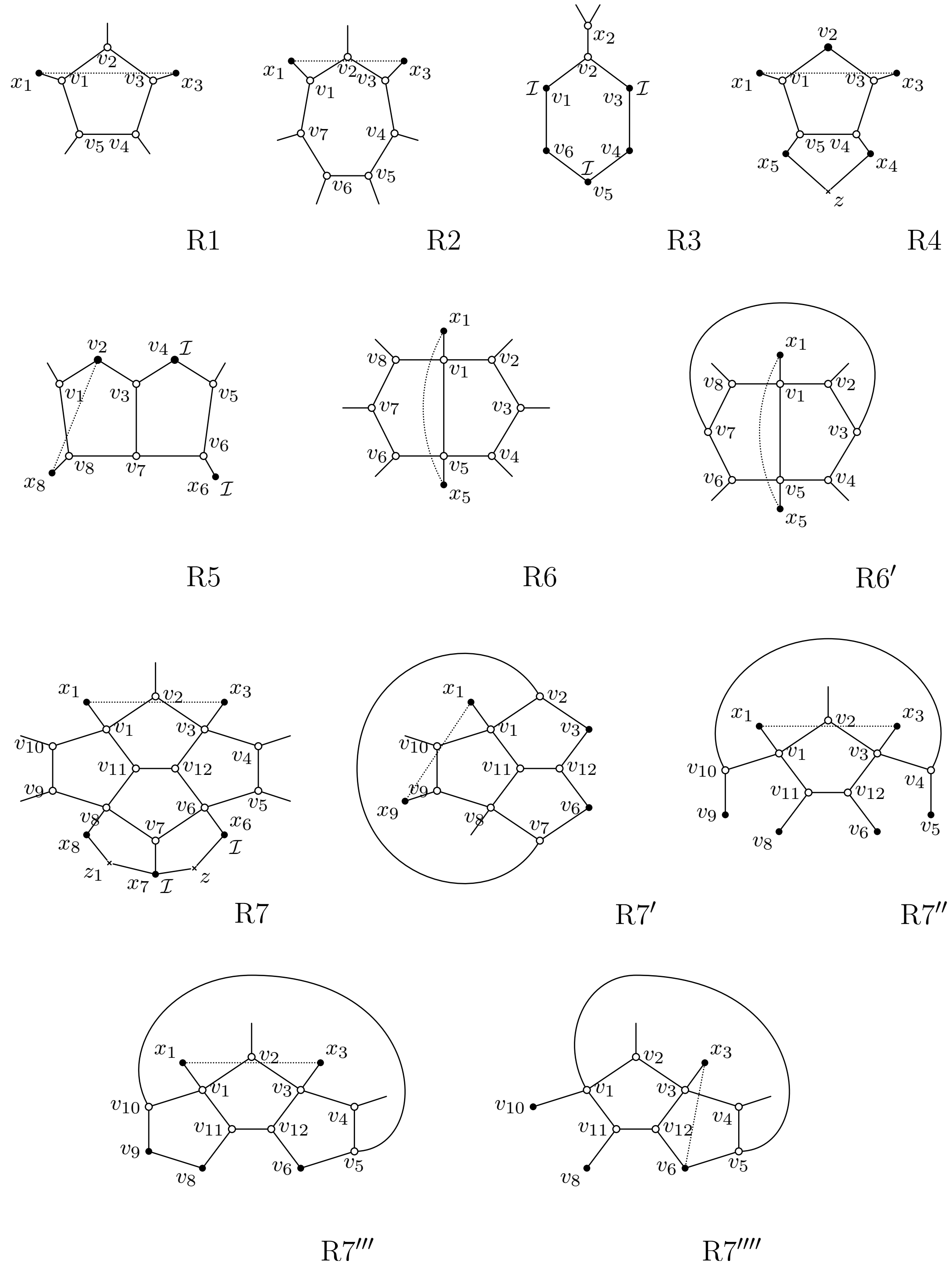


Figure 7.1: Reducible configurations.

then $x_3 \neq x_7$ or $x_1 \neq x_6$, it follows that G' has no loops. It also follows that $\mathcal{R}$ is a set of rings for G'. We will refer to the added edge as the *new edge* and to the vertex that resulted from the identification of vertices as the *new vertex*. If two vertices $u, v \in \mathcal{I}_\gamma$ have a common neighbor $x \in V(G_\gamma) \setminus \mathrm{dom}(d_\gamma)$ and w is the new vertex arising by identification of u and v, then we call the edge wx *squashed*.

We also need to specify an embedding of G' in Σ. There is a unique natural way to make the edge additions and vertex identifications inside the faces of $\mathcal{F}_\gamma$, and that is how the embedding of G' will be defined. Formally, for every pair $u, v \in \mathcal{A}_\gamma \cup \mathcal{I}_\gamma$ of distinct vertices we define the *replacement* u, v*-path* as the shortest path from u to v in G_γ. It follows by inspecting all the good configurations that the replacement path is unique. Now we identify u and v or join them by an edge along the replacement u, v-path P, with the proviso that if P includes a vertex $v \in V(G_\gamma) \setminus \mathrm{dom}(d_\gamma)$ (specifically, vertex v_4 or v_6 of R3 or vertex z of R7), then prior to making the edge addition or vertex identification we shift P slightly into the unique face f of $\mathcal{F}_\gamma$ incident with v. By the conditions of weak appearance, f is not a cuff face of v, hence P stays in Σ and its homotopy does not change by such a shift. This completes the definition of G' when γ is not R4.

Now let γ be R4. If not both x_4 and x_5 belong to $\mathcal{R}$, then we proceed as above, treating the configuration as if $\{x_4, x_5\}$ belonged to $\mathcal{I}_\gamma$; that is, identifying those vertices. We may therefore assume that both x_4, x_5 belong to $\mathcal{R}$. Let ϕ be a 3-coloring of $\mathcal{R}$; the definition of G' will now depend on ϕ. If $\phi(x_4) = \phi(x_5)$, then we define G' exactly as in the previous two paragraphs; in particular, we do not identify x_4 and x_5. If $\phi(x_4) \neq \phi(x_5)$, then we let G' be obtained from $G \setminus \{v_1, v_3, v_4, v_5\}$ by identifying v_2 and x_5 along the "replacement path" $v_2 v_1 v_5 x_5$. Let us remark that the last condition in the definition of weak appearance guarantees that in this case v_2 does not belong to $\mathcal{R}$. Then G' is a graph in Σ with rings $\mathcal{R}$, and we say that it is the *γ-reduction of G*. When we wish to emphasize the dependence on ϕ we will say that G' is the *γ-reduction of G with respect to ϕ.*

7.3 Colorings

In this section, we show that each 3-coloring of the γ-reduction of a graph G extends to a 3-coloring of G. Most of the reductions were used earlier [43, 68], but R5, R7 and their variants seem to be new. For the sake of completeness we include proofs of extendability for all good configurations.

Lemma 52. *Let G be a graph in a surface Σ with rings $\mathcal{R}$, let γ be a good configuration that weakly appears in $(G, \mathcal{R})$, let ϕ_0 be a 3-coloring of $\mathcal{R}$, and let G_1 be the γ-reduction of G with respect to ϕ_0. If ϕ_0 extends to a 3-coloring of G_1, then it extends to a 3-coloring of G.*

Proof. Let γ be as stated, and let the vertices of G_γ be labeled as in Figure 7.1. Let ϕ be a 3-coloring of G_1 that extends the coloring ϕ_0. Then ϕ can be regarded as a 3-coloring of $G \setminus \mathrm{dom}(d_\gamma)$, and our objective is to extend it to a 3-coloring of G. For each vertex $v_i \in \mathrm{dom}(d_\gamma)$ that has a unique neighbor outside of the configuration, let this neighbor be denoted by x_i. We will use the following easy observations:

(28) *Suppose that $u_1, u_2 \in V(G)$ are adjacent vertices of degree three, w_1 and w_2 are the neighbors of u_1 distinct from u_2 and w_3 and w_4 are the neighbors of u_2 distinct from u_1. A 3-coloring ψ of $w_1, \ldots, w_4$ extends to u_1 and u_2, unless $\psi(w_1) = \psi(w_3) \neq \psi(w_2) = \psi(w_4)$ or $\psi(w_1) = \psi(w_4) \neq \psi(w_2) = \psi(w_3)$.*

(29) *Let $P = u_1 u_2 \ldots u_k$ be a path in G and $L_1, \ldots, L_k$ lists of colors of size two, such that $L_i \neq L_j$ for some $1 \le i < j \le k$. Then there exist colorings ψ_1, ψ_2 and ψ_3 of P such that $\psi_i(v_j) \in L_j$ for $1 \le i \le 3$ and $1 \le j \le k$, and for each $1 \le i < j \le 3$ either $\psi_i(u_1) \neq \psi_j(u_1)$ or $\psi_i(u_k) \neq \psi_j(u_k)$.*

Let us now consider each configuration separately.

Configurations R1 **and** R2**.** Each of the vertices of the cycle $v_1 v_2 \ldots v_k$ (where $k = 5$ for the configuration R1 and $k = 7$ for R2) has a list of two available colors, and the lists of v_1 and v_3 are not the same. By ((29)), there exists a coloring of the path $v_1 \ldots v_k$ from these lists such that the colors of v_1 and v_k are not the same, giving a coloring of G, as desired.

Configuration R3**.** The vertices v_1, v_3 and v_5 inherit the color of the new vertex. Then we can color the vertices x_2 and v_2 in order, because at the time each of those vertices is colored it is adjacent to vertices of at most two different colors.

Configuration R4**.** Suppose first that at least one of x_4 and x_5 is internal, or that both belong to $\mathcal{R}$ and $\phi_0(x_4) = \phi_0(x_5)$. If $\phi(x_1) = \phi(v_2)$, then color the vertices in the order v_3, v_4, v_5 and v_1 (each of them has neighbors of at most two different colors when it is being colored). The case that $\phi(x_3) = \phi(v_2)$ is symmetric. Therefore, we may assume that $\phi(x_1) = 1$, $\phi(v_2) = 2$ and $\phi(x_3) = 3$. Set $\phi(v_1) = 3$ and $\phi(v_3) = 1$ and extend the coloring to v_4 and v_5 by ((28)). Then ϕ is a desired 3-coloring of G.

We may therefore assume that both x_4 and x_5 belong to $\mathcal{R}$ and $\phi_0(x_4) \neq \phi_0(x_5)$. In this case, the definition of γ-reduction ensures that $\phi(v_2) = \phi(x_5)$. We may assume that $\phi(v_2) = \phi(x_5) = 1$ and $\phi(x_4) = 2$. Let us set $\phi(v_4) = 1$ and color v_3, v_1 and v_5 in this order.

Configuration R5**.** The reduction ensures that $\phi(v_2) \neq \phi(x_8)$ and $\phi(v_4) = \phi(x_6)$. If $\phi(v_2) = \phi(v_4)$, then ϕ extends—color the vertices in the order v_1, v_8, v_5, v_6, v_7 and v_3, and observe that for each of these vertices, at most two different colors appear on already colored neighbors. Thus we may assume that $\phi(v_2) = 1$ and $\phi(v_4) = \phi(x_6) = 2$. We set $\phi(v_3) = 3$ and $\phi(v_7) = 2$, and color the vertices

v_5 and v_6. As $\phi(x_8) \neq \phi(v_2) \neq \phi(v_7)$, ((28)) implies that the coloring extends to v_1 and v_8.

Configurations R6 **and** R6′**.** In both cases, the reduction ensures that $\phi(x_1) \neq \phi(x_5)$, say $\phi(x_1) = 1$ and $\phi(x_5) = 2$. If $\phi(x_6) = 1$, then set $\phi(v_5) = 1$, and color the vertices in order v_4, v_3, v_2, v_1, v_8, v_7 and v_6. Therefore, we may assume that this is not the case. By symmetry, we may also assume that $\phi(x_4) \neq 1$ and $\phi(x_2), \phi(x_8) \neq 2$. If $\phi(x_2) = \phi(x_8) = 3$, then set $\phi(v_1) = 3$, $\phi(v_5) = 1$ and color v_6, v_7, v_8, v_4, v_3 and v_2 in this order. Otherwise, by symmetry we may assume that $\phi(x_2) = 1$. If v_3 and v_7 are adjacent, or if $\phi(x_3) \neq 1$, then set $\phi(v_3) = \phi(v_5) = 1$ and color v_4, v_6, v_7, v_8, v_1 and v_2 in this order. Therefore, assume that v_3 and v_7 are not adjacent and $\phi(x_3) = 1$.

If $\phi(x_6) = 3$, then set $\phi(v_4) = 1$, $\phi(v_1) = \phi(v_3) = 2$ and $\phi(v_2) = \phi(v_5) = 3$ and color v_8, v_7 and v_6 in this order. Thus, assume that $\phi(x_6) = 2$. By the argument symmetrical to the one used for x_3, we conclude that ϕ extends unless $\phi(x_7) = 2$. If $\phi(x_8) = 3$, then set $\phi(v_4) = \phi(v_6) = \phi(v_8) = 1$, $\phi(v_1) = \phi(v_3) = 2$ and $\phi(v_2) = \phi(v_5) = \phi(v_7) = 3$. Thus assume that $\phi(x_8) = 1$ and by symmetry, $\phi(x_4) = 2$. In this case, set $\phi(v_5) = \phi(v_7) = 1$, $\phi(v_1) = \phi(v_3) = 2$ and $\phi(v_2) = \phi(v_4) = \phi(v_6) = \phi(v_8) = 3$.

Configuration R7**.** The reduction ensures that $\phi(x_1) \neq \phi(x_3)$, say $\phi(x_1) = 1$ and $\phi(x_3) = 2$. To preserve the symmetry of the configuration, let us for a while ignore the identification of x_6 and x_7.

Suppose first that $\phi(x_8) = 1$. By ((29)), there exists a coloring ψ of the path $v_1 v_2 \ldots v_8$ such that $\psi(v_1) = \psi(v_8) \in \{2, 3\}$. We can extend ψ to v_{12} and v_{11}. By ((28)), if $\phi(x_9) \neq \phi(x_{10})$, then ψ extends to v_9 and v_{10} as well. Consider the case that $\phi(x_9) = \phi(x_{10}) = c$. Set $\phi(v_{11}) = 1$. If $\phi(x_2) = 1$, then color v_3 by 1, and color the vertices v_4, v_5, …, v_{10}, v_1, v_2, v_{12} in this order. If $\phi(x_6) = 2$, then color v_{12} by 2 and extend the coloring to the 10-cycle $v_1 \ldots v_{10}$. Therefore, assume that $\phi(x_2) \neq 1$ and $d = \phi(x_6) \neq 2$. Let us distinguish several cases:

- $d = 3$, $\phi(x_4) = 1$ and $\phi(x_5) = 3$: In this case, set $\phi(v_{12}) = 3$, $\phi(v_3) = 1$ and color v_2, v_1, v_{10}, v_9, …, v_4 in order.
- $d = 1$ and $\phi(x_4) = \phi(x_5)$: Set $\phi(v_3) = 1$ and color the vertices v_2, v_1, v_{10}, v_9, …, v_6, v_{12} in order. Note that $\phi(v_3) = 1 \neq \phi(v_6)$, thus ϕ extends the coloring to v_4 and v_5 by ((28)).
- Otherwise, set $\phi(v_2) = 1$, $\phi(v_3) = 3$, $\phi(v_{12}) = 2$, $\phi(v_6) = 4 - d$, and color vertices v_7, …, v_{10}, v_1 in order. By ((28)), this coloring extends to v_4 and v_5.

We conclude that if ϕ does not extend to the empty-circle vertices, then $\phi(x_8) = c_1 \neq 1$, and by the symmetry, $\phi(x_6) = c_2 \neq 2$.

There are four possible colorings of v_1 and v_8 (two choices of colors for each of these vertices, so that the color of v_1 is not 1 and the color of v_8 is not c_1). By

((28)), out of these four colorings, all but at most one extend to v_9 and v_{10}; if such a coloring of v_1 and v_8 exists, let it be denoted by ω_1; otherwise, set $\omega_1(v_1) = 1$ and $\omega_1(v_8) = c_1$. Symmetrically, let ω_2 be the unique coloring of v_3 and v_6 such that $\omega_2(v_3) \neq 2$, $\omega_2(v_6) \neq c_2$ and ω_2 does not extend to v_4 and v_5, if such a coloring exists, and $\omega_2(v_3) = 2$ and $\omega_2(v_6) = c_2$ otherwise.

If $\phi(x_2) = 2$, then let $a = 2$, otherwise let $a = 3$. Note that any color $c \neq 2 = \phi(x_3)$ satisfies $|\{a, c, \phi(x_2)\}| = 2$. In the following cases, we can extend ϕ to a coloring ψ of the path $v_1v_{10}v_9v_8v_7v_6$ such that $\psi(v_1) = a$ and $b = \psi(v_6) \neq \omega_2(v_6)$:

- $\omega_1(v_1) \neq a$: choose $b \notin \{\phi(x_6), \omega_2(v_6)\}$, color v_7 and v_8, and note that we can extend this coloring to v_9 and v_{10} by the definition of ω_1.
- $\omega_2(v_6) = c_2$: color the vertices v_{10}, v_9, …, v_6 in this order.
- $\phi(x_7) \notin \{c_1, \omega_1(v_8)\} \cap \{c_2, \omega_2(v_6)\}$ or $\{c_1, \omega_1(v_8)\} = \{c_2, \omega_2(v_6)\}$: excluding the previous two cases, we may assume that $c_1 \neq \omega_1(v_8)$ and $c_2 \neq \omega_2(v_6)$. Color v_8 by the color $d \notin \{c_1, \omega_1(v_8)\}$ and v_6 by the color $b \notin \{c_2, \omega_2(v_6)\}$, extend the coloring to v_9 and v_{10} by the definition of ω_1, and observe that $|\{\phi(x_7), b, d\}| \leq 2$, thus v_7 can be colored as well.

If such a coloring ψ exists, then choose a color $c \neq \phi(x_3)$ such that $c = b$ or $\{b, c\} \neq \{a, \psi(v_8)\}$; this ensures that the coloring extends to v_{11} and v_{12} by ((28)). Since $b \neq \omega_2(v_6)$, this coloring extends to v_4 and v_5 as well. Finally, the choice of a ensures that $|\{a, c, \phi(x_2)\}| = 2$, hence the coloring extends to v_2. Therefore, we may assume that such the coloring ψ does not exist, i.e., $\omega_1(v_1) = a$, $\omega_2(v_6) \neq c_2$, $\{c_1, \omega_1(v_8)\} \neq \{c_2, \omega_2(v_6)\}$ and $\phi(x_7) \in \{c_1, \omega_1(v_8)\} \cap \{c_2, \omega_2(v_6)\}$.

Let us now distinguish two cases:

- $\phi(x_9) \neq \phi(x_{10})$: By ((28)), $a = \omega_1(v_1) = \phi(x_9)$. If $c_1 \neq a$, then set $\phi(v_1) = \phi(v_8) = a$ and color v_{10}, v_9, v_7, v_6, …, v_2 in this order (v_2 can be colored by the choice of a), and color v_{12} and v_{11}; hence, assume that $c_1 = a$.

  If $\phi(x_{10}) = 5 - a$, then set $\phi(v_1) = \phi(v_8) = 5 - a$, $\phi(v_{10}) = a$, and $\phi(v_9) = 1$. Note that $\phi(x_7) \in \{c_1, \omega_1(v_8)\} = \{a, 5 - a\}$ and $\{c_2, \omega_2(v_6)\} = \{1, \phi(x_7)\}$. Set $\phi(v_7) = 1$ and choose $\phi(v_6) \notin \{c_2, \omega_2(v_6)\}$, i.e., $\phi(v_6) = 5 - \phi(x_7)$. Extend the coloring to v_2, v_3, v_{12} and v_{11} in this order. As$\phi(v_6) \neq \omega_2(v_6)$, this coloring extends to v_4 and v_5, giving a coloring of the whole configuration.

  Therefore, assume that $\phi(x_{10}) = 1$. Then $\omega_1(v_8) = 1$ and $\phi(x_7) \in \{1, a\}$. Let us set $\phi(v_1) = \phi(v_7) = \phi(v_9) = 5 - a$, $\phi(v_{10}) = a$ and $\phi(v_8) = 1$. Let us choose color $\phi(v_6) \notin \{c_2, \omega_2(v_6)\}$; note that $\phi(v_6) \neq 5 - a$, since $\{c_2, \omega_2(v_6)\} \neq \{c_1, \omega_1(v_8)\} = \{1, a\}$. Color v_2 and v_3, and extend the coloring to v_4 and v_5 (this is possible, since $\phi(v_6) \neq \omega_2(v_6)$). We may assume that this coloring does not extend to v_{11} and v_{12}, i.e., $\{\phi(v_3), \phi(v_6)\} = \{1, 5-a\}$, hence $\phi(v_3) = 5 - a$ and $\phi(v_6) = 1$. As $\phi(v_6) \notin \{c_2, \omega_2(v_6)\}$, we get

$\{c_2, \omega_2(v_6)\} = \{a, 5-a\}$ and $\phi(x_7) = a$. Since $c_2 \neq 2$, we have $c_2 = 3$ and$\omega_2(v_6) = 2$. As $\omega_2(v_3) \neq 2$, it follows that $\phi(x_4) = 2$ and $\phi(x_5) \neq 2$.

Consider the coloring ψ with $\psi(v_8) = 5-a$, $\psi(v_7) = \psi(v_9) = 1$, $\psi(v_6) = 2$, $\psi(v_3) = \psi(v_5) = 4 - \phi(x_5)$ and $\psi(v_4) = \phi(v_5)$, and assume that this coloring does not extend to the coloring of the whole configuration. On one hand, we may color v_1 by a and v_{10} by $5-a$; then ψ extends to v_2 by the definition of a, and since it does not extend to v_{11} and v_{12}, we have $\{a, 5-a\} = \{2, 4-\phi(x_5)\}$, and $\phi(x_5) = 1$. On the other hand, we may color v_1 by $5-a$, v_{12} by 1 and v_{10} and v_{11} by a. Since this coloring does not extend to v_2, we have $|\{5-a, 3, \phi(x_2)\}| = 3$, and $a = 3$ and $\phi(x_2) = 1$. In that case, we can color the configuration by setting $\phi(v_3) = \phi(v_6) = \phi(v_8) = 1$, $\phi(v_1) = \phi(v_5) = \phi(v_7) = \phi(v_9) = \phi(v_{12}) = 2$ and $\phi(v_2) = \phi(v_4) = \phi(v_{10}) = \phi(v_{11}) = 3$.

- $\phi(x_9) = \phi(x_{10})$: By symmetry, we may also assume that $\phi(x_4) = \phi(x_5)$. At this point, we use the second relation guaranteed by the reduction, $\phi(x_7) = c_2$. If $c_2 \neq 3$, then set $\phi(v_7) = 3$, $\phi(v_8) = 1$ and $\phi(v_6) = 2$, color the 5-cycle $v_1v_2v_3v_{12}v_{11}$, and extend the coloring to v_4, v_5, v_9 and v_{10} by ((28)). Thus, assume that $c_2 = 3$.

  If $\phi(x_2) \neq 1$, then set $\phi(v_2) = \phi(v_6) = \phi(v_8) = 1$, $\phi(v_1) = \phi(v_7) = \phi(v_{12}) = 2$ and $\phi(v_3) = \phi(v_{11}) = 3$, and extend the coloring to v_4, v_5, v_9 and v_{10} by ((28)).

  Finally, if $\phi(x_2) = 1$, then set $\phi(v_2) = \phi(v_8) = 5 - c_1$, $\phi(v_1) = c_1$, $\phi(v_3) = \phi(v_7) = \phi(v_{11}) = 1$, $\phi(v_6) = 2$ and $\phi(v_{12}) = 3$, and extend the coloring to v_4, v_5, v_9 and v_{10} by ((28)).

Configuration R7$'$**.** If $\phi(v_3) = \phi(v_6)$, then first color the 6-cycle $v_2v_1v_{10}v_9v_8v_7$ (this is possible, as each of the vertices has at most one colored neighbor), and then color v_{11} and v_{12}. Thus, assume that $\phi(v_3) = 1$, $\phi(v_6) = 2$ and $\phi(v_{12}) = 3$. Color the 5-cycle $v_1v_{11}v_8v_9v_{10}$ (this is possible, as $\phi(x_1) \neq \phi(x_9)$). Note that in this coloring, $\phi(v_1) \neq 2$ or $\phi(v_8) \neq 1$, as $\phi(v_{11}) \neq \phi(v_{12}) = 3$. Therefore, the coloring extends to v_2 and v_7 by ((28)).

Configuration R7$''$**.** The reduction ensures that $\phi(x_1) \neq \phi(x_3)$, say $\phi(x_1) = 1$ and $\phi(x_3) = 2$. Also, by symmetry, we may assume that $c = \phi(x_2) \neq 1$. Suppose first that $\phi(v_8) \neq 1$. Then try coloring v_{11} and v_3 by 1 and v_1 by c. By ((28)), this coloring extends unless $\phi(v_9) = 1$ and $\phi(v_5) = c$. If $\phi(v_6) \neq 2$, then set the color of v_3 to 3, instead, and observe that the coloring extends. Otherwise, $\phi(v_6) = 2$, and set $\phi(v_{12}) = \phi(v_2) = 1$, $\phi(v_3) = 3$, and color v_{11} and v_1. The coloring extends to v_{10} and v_4 by ((28)).

Therefore, we may assume that $\phi(v_8) = 1$. Suppose that $\phi(v_6) \neq c$. Then try coloringv_1 and v_{12} by c, v_{11} and v_2 by $5-c$ and v_3 by 1. By ((28)), this coloring extends to v_4 and v_{10} unless $\phi(v_5) = c$ and $\phi(v_9) = 1$. In that case, set $\phi(v_2) = 1$,

$\phi(v_3) = 3$, color v_{12}, v_{11} and v_1 in this order, and extend the coloring to v_4 and v_{10} by ((28)). Thus, we may assume that $\phi(v_6) = c$.

If $c \neq 2$, then set $\phi(v_3) = c$ and color v_4, v_{10}, v_1, v_2, v_{11} and v_{12} in this order; hence, assume that $c = 2$. Consider the coloring that assigns 1 to v_2 and v_{12}, 3 to v_{11} and v_3 and 2 to v_1. If this coloring does not extend to v_4 and v_{10}, then ((28)) implies that $\phi(v_5) = 2$ and $\phi(v_9) = 3$. In that case, set $\phi(v_2) = \phi(v_4) = \phi(v_{12}) = 1$, $\phi(v_{10}) = \phi(v_{11}) = 2$ and $\phi(v_1) = \phi(v_3) = 3$.

Configuration R7‴. The reduction ensures that $\phi(x_1) \neq \phi(x_3)$, say $\phi(x_1) = 1$ and $\phi(x_3) = 2$. If $\phi(v_8) \neq 1$ and $\phi(v_6) \neq 2$, then color v_{11} by 1, v_{12} by 2 and extend the coloring to the 6-cycle $v_{10}v_1v_2v_3v_4v_5$.

Assume now that $\phi(v_8) = 1$ or $\phi(v_6) = 2$. Suppose first that $\phi(v_6) \neq 2$, and thus $\phi(v_8) = 1$. Then try setting the color of v_1, v_5 and v_{12} to 2 and coloring v_{11} and v_{10}. If $\phi(x_2) = 2$ or $\phi(x_4) = 2$ or $\phi(x_2) = \phi(x_4)$, then the coloring extends to v_2, v_3 and v_4, thus assume that $\{\phi(x_2), \phi(x_4)\} = \{1, 3\}$. If $\phi(v_9) \neq 2$ or $\phi(v_6) \neq 3$, then set $\phi(v_2) = \phi(v_4) = \phi(v_{11}) = 2$, $\phi(v_1) = 3$, color v_{12} and v_3 and extend the coloring to v_5 and v_{10} by ((28)). Otherwise, $\phi(v_9) = 2$ and $\phi(v_6) = 3$ and we set $\phi(v_5) = 1$, $\phi(v_1) = \phi(v_4) = \phi(v_{12}) = 2$, $\phi(v_{10}) = \phi(v_{11}) = 3$, $\phi(v_2) = \phi(x_4)$ and $\phi(v_3) = \phi(v_2)$.

Therefore, it suffices to consider the case that $\phi(v_6) = 2$. If $\phi(x_4) \neq 2$, then set $\phi(v_4) = 2$, color the 5-cycle $v_1v_2v_3v_{12}v_{11}$, and color v_{10} and v_5. So we have $\phi(x_4) = 2$. Suppose that $\phi(x_2) \neq 2$. Then set $\phi(v_2) = 2$ and $\phi(v_1) = 3$. If $\phi(v_8) \neq 2$, then color v_{11} by 2 and color v_{10}, v_5, v_4, v_3 and v_{12} in this order. On the other hand, if $\phi(v_8) = 2$, then note that $\phi(v_9) \neq 2$, and set $\phi(v_{10}) = 2$, $\phi(v_3) = \phi(v_5) = \phi(v_{11}) = 1$ and $\phi(v_4) = \phi(v_{12}) = 3$. Thus, assume that $\phi(x_2) = 2$.

Try setting $\phi(v_2) = \phi(v_4) = \phi(v_{12}) = 1$ and $\phi(v_3) = \phi(v_5) = 3$. If $\phi(v_9) \neq 1$, then set $\phi(v_{10}) = 1$ and color v_{11} and v_1; thus assume that$\phi(v_9) = 1$. If $\phi(v_8) \neq 2$, then set $\phi(v_{10}) = \phi(v_{11}) = 2$ and $\phi(v_1) = 3$.

Finally, consider the case that $\phi(v_9) = 1$ and $\phi(v_8) = 2$. Then, we set $\phi(v_3) = \phi(v_5) = \phi(v_{11}) = 1$, $\phi(v_1) = 2$ and $\phi(v_2) = \phi(v_4) = \phi(v_{10}) = \phi(v_{12}) = 3$.

Configuration R7⁗. The reduction ensures that $\phi(x_3) \neq \phi(v_6)$, say $\phi(v_6) = 1$ and $\phi(x_3) = 2$. Suppose first that $\phi(v_8) \neq \phi(v_{10})$. If $\phi(v_{10}) \neq 2$, then let $\phi(v_{12}) = 2$, $\phi(v_{11}) = \phi(v_{10})$ and extend the coloring to the 5-cycle $v_1v_2v_3v_4v_5$; thus assume that $\phi(v_{10}) = 2$. If $\phi(x_2) \neq 2$, then set $\phi(v_2) = 2$, $\phi(v_3) = 1$, and color v_4, v_5, v_1, v_{11} and v_{12} in this order. If $\phi(x_2) = 2$, then set $\phi(v_1) = \phi(v_3) = 1$, $\phi(v_2) = 3$, and color v_{11}, v_{12}, v_4 and v_5, in this order.

Therefore, assume that $\phi(v_8) = \phi(v_{10}) = c$. If $c = 2$, then color v_{12} by 2, extend the coloring to the 5-cycle $v_1 \dots v_5$, and color v_{11}. If $c = 3$, then set $\phi(v_1) = \phi(v_3) = 1$, $\phi(v_{11}) = 2$, $\phi(v_{12}) = 3$, and color v_2, v_4 and v_5 in this order. Thus, assume that $c = 1$. Try setting $\phi(v_1) = \phi(v_{12}) = 2$ and $\phi(v_{11}) = \phi(v_5) = 3$. If $\phi(x_4) \neq 2$, then set $\phi(v_4) = 2$ and color v_2 and v_3. If $\phi(x_4) = 2$ and $\phi(x_2) \neq 1$, then set $\phi(v_2) = \phi(v_4) = 1$ and $\phi(v_3) = 3$.

Finally, consider the case that $\phi(x_2) = 1$ and $\phi(x_4) = 2$. Then, set $\phi(v_3) = 1$, $\phi(v_2) = \phi(v_5) = \phi(v_{11}) = 2$ and $\phi(v_1) = \phi(v_4) = \phi(v_{12}) = 3$. $\square$

7.4 Discharging

Let G be a graph in a surface Σ with rings $\mathcal{R}$. A face is *open* 2*-cell* if it is homeomorphic to an open disk. A face is *closed* 2*-cell* if it is open 2-cell and bounded by a cycle. A face f is *omnipresent* if it is not open 2-cell and each of its boundary walks is either a vertex ring or a cycle bounding a closed disk $\Delta \subseteq \hat{\Sigma} \setminus f$ containing exactly one ring. We say that G has an *internal* 2*-cut* if there exist sets $A, B \subseteq V(G)$ such that $A \cup B = V(G)$, $|A \cap B| = 2$, $A - B \neq \emptyset \neq B - A$, A includes all vertices of $\mathcal{R}$, and no edge of G has one end in $A - B$ and the other in $B - A$.

We wish to consider the following conditions that the triple $G, \Sigma, \mathcal{R}$ may or may not satisfy:

- (I0) every internal vertex of G has degree at least three,
- (I1) G has no even cycle consisting of internal vertices of degree three,
- (I2) G has no cycle C consisting of internal vertices of degree three, and two distinct adjacent vertices $u, v \in V(G) - V(C)$ such that both u and v have a neighbor in C,
- (I3) every internal face of G is closed 2-cell and has length at least 5,
- (I4) if a path of length at most two has both ends in $\mathcal{R}$, then it is a subgraph of $\mathcal{R}$,
- (I5) no two vertices of degree two in G are adjacent.
- (I6) if Σ is the sphere and $|\mathcal{R}| = 1$, or if G has an omnipresent face, then G does not contain an internal 2-cut,
- (I7) the distance between every two distinct members of $\mathcal{R}$ is at least four,
- (I8) every cycle in G that does not separate the surface has length at least seven,
- (I9) if a cycle C of length at most 9 in G bounds an open disk Δ in $\hat{\Sigma}$ such that Δ is disjoint with all rings, then Δ is a face, a union of a 5-face and a $(|C| - 5)$-face, or C is a 9-cycle and Δ consists of three 5-faces intersecting in a vertex of degree three.

Let G be a graph in a surface Σ with rings $\mathcal{R}$ satisfying (I3). We say that a good configuration γ *appears* in $(G, \mathcal{R})$ if it faintly appears and the following conditions hold:

- no vertex ring belongs to $\mathcal{I}_\gamma$,
- if γ is isomorphic to R3, then either $\mathcal{I}$ contains a vertex of $\mathcal{R}$ or there exists a vertex $v \in \mathcal{I}$ such that v and all its neighbors are internal,
- if γ is isomorphic to R4, then the vertex that corresponds to v_2 is internal and has degree at least 4, and neither x_4 nor x_5 is a vertex ring,
- if γ is isomorphic to R5, then v_4 is an internal vertex and the face whose boundary contains the path corresponding to $v_6v_7v_8$ has length at least seven,
- if γ is isomorphic to R6 or R6$'$, then both vertices of $\mathcal{A}_\gamma$ are internal, and all neighbors of at least one of them are internal,
- if γ is isomorphic to one of $\mathrm{R7}, \mathrm{R7}', \mathrm{R7}'', \mathrm{R7}''', \mathrm{R7}''''$, then all vertices in $\mathcal{A}_\gamma \cup \mathcal{I}_\gamma$ and all their neighbors are internal, and
- if γ is isomorphic to R7, then the vertex corresponding to x_8 and all its neighbors are internal.

Let G be a graph in a surface Σ with rings $\mathcal{R}$, and let M be a subgraph of G. We define the *initial charge* of the triple $(G, \Sigma, \mathcal{R})$ as follows. The faces bounded by facial rings get charge 0. Every internal face f gets charge $|f| - 4$. A vertex of degree two belonging to a facial ring gets charge $-1/3$, a vertex of degree d forming a vertex ring gets charge d, any other vertex of $\mathcal{R}$ of degree d gets charge $d - 3$, and all internal vertices get charge $d - 4$. Finally, we increase the charge of each face incident with an edge of M by $5/3$.

Lemma 53. *Let G be a graph in a surface Σ with rings $\mathcal{R}$, let g be the Euler genus of Σ, let n_2 be the number of vertices of degree two in facial rings, and let M be a subgraph of G. Then the sum of initial charges of all vertices and faces of G is at most $4g + 4|\mathcal{R}| + 2n_2/3 + 10|E(M)|/3 - 8$.*

Proof. By Euler's formula, the sum of the initial charges of all vertices and faces is at most

$$\textstyle\sum_{f \in F(G)}(|f| - 4) + \sum_{v \in V(G)}(\deg(v) - 4) + 4|\mathcal{R}| + 2n_2/3 + 10|E(M)|/3$$
$$\leq 4g + 4|\mathcal{R}| + 2n_2/3 + 10|E(M)|/3 - 8,$$

as desired. □

A 5-face f is *k-dangerous* if f is not incident with an edge of M and f is incident with exactly k internal vertices of degree three. Let $f_1 = uvawb$ be a 4-dangerous face, where w is the unique incident vertex that is not internal of degree three. Let f_2 be the face incident with uv distinct from f_1. We say that f_2 is *linked to* f_1 *(through the edge* uv*)*. Let xy be an edge such that y has degree three, and let g_1, g_2, g_3 be the faces incident with y such that xy is incident with q_1 and g_2. Then the face g_3 is *opposite* to x. A 4-dangerous face f is *extremely 4-dangerous* if it is neither incident with a vertex of $\mathcal{R}$ nor opposite to any vertex ring.

Let us apply the following *primary discharging rules*, resulting in the *primary charge*:

Rule 1: Every internal face f sends $1/3$ to each incident vertex v such that $\deg(v) = 2$ and v belongs to a facial ring, or $\deg(v) = 3$ and v is internal.

Rule 2: Every vertex v belonging to a facial ring sends $1/3$ to each 4-dangerous face incident with v. Every vertex ring sends $1/3$ to each non-cuff incident face, $8/9$ to its cuff face and $1/3$ to each opposite face.

Rule 3: Let f be a face linked to an extremely 4-dangerous face f' through an edge uv. If f has length at least 6, or f is incident with an edge of M, or f is the cuff face of a vertex ring, then f sends $1/3$ to f' across the edge uv.

Rule 4: Let $v_1v_2v_3v_4$ be a subwalk of the boundary walk of an internal face f' of length at least seven, such that f' is linked to extremely 4-dangerous faces through both v_1v_2 and v_3v_4. Let f be the other face incident with the edge v_2v_3. If f has length at least six, then f sends $1/9$ to f' across the edge v_2v_3.

Lemma 54. *Let G be a graph in a surface Σ with rings $\mathcal{R}$ satisfying* (I0) *and* (I3) *and let M be a subgraph of G. Then the primary charge of each vertex is non-negative, the primary charge of a vertex of degree d forming a vertex ring is at least $d/18$, and the primary charge of a vertex of degree $d \geq 3$ incident with a facial ring is at least $2(d-3)/3$. Moreover, the primary charge of a vertex of degree $d \geq 4$ that does not belong to $\mathcal{R}$ is exactly $d-4$.*

Proof. This is ensured by Rule 1 for internal vertices of degree three. The charge of internal vertices of degree $d \geq 4$ is unchanged, i.e., $d-4 \geq 0$. A vertex forming a vertex ring has initial charge d and sends at most $(6d+5)/9$ by Rule 2. By (I3), we have $d \geq 2$; thus its primary charge is at least $(3d-5)/9 \geq d/18$. Consider now a vertex v of degree d that belongs to a facial ring. If $d = 2$, then the initial charge of v is $-1/3$ and v receives $1/3$ by Rule 1. Observe that v sends nothing by Rule 2, since both neighbors of v belong to $\mathcal{R}$; thus the primary charge of v is 0. If $d \geq 3$, then v sends charge by Rule 2 to at most $d-3$ faces, and hence its primary charge is at least $d-3-(d-3)/3 = 2(d-3)/3$, as desired. $\square$

The primary charge of a face corresponding to a facial ring R is zero, as it is equal to its initial charge. Let us now estimate the primary charge of internal faces. A subgraph $M \subseteq G$ *captures* (≤ 4)*-cycles* if M contains all cycles of G of length at most 4 and furthermore, M is either null or has minimum degree at least two.

Lemma 55. *Let G be a graph in a surface Σ with rings $\mathcal{R}$ satisfying* (I0), (I1), (I3), (I4), (I5) *and* (I7)*, let M be a subgraph of G that captures* (≤ 4)*-cycles and assume that if a configuration isomorphic to one of* R1, R2, …, R5 *appears in G, then it touches M. If f is an internal face of G, then the primary charge of f is non-negative. Furthermore, if the primary charge of f is zero, then f has length exactly five and*

- *(a) f is* 3*-dangerous, or*
- *(b) f is incident with a vertex of $\mathcal{R}$, or*
- *(c) f is* 4*-dangerous and a face of length at least* 6 *is linked to f, or*
- *(d) f is* 4*-dangerous, the a face h linked to f has length five and either h is incident with an edge of M or h is the cuff face of a vertex ring, or*
- *(e) f is* 4*-dangerous and is opposite to a vertex ring.*

Otherwise, the primary charge of f is least $2/9$*, and if $|f| \geq 8$, then the primary charge of f is at least* $5|f|/9 - 4$*. Also, if f is a* 6*-face incident with a vertex of degree two belonging to a facial ring, then f has primary charge at least* $2/3$.

Proof. Suppose first that f has length five. Let us consider the case that f is incident with an edge of M. Then f sends charge by Rules 1 and 3. If f sends charge across an edge uv by Rule 3 to a face f', then both u and v have degree three and no edge of f' belongs to M. Since M has minimum degree at least two, it follows that no edge incident with u or v belongs to M; hence f sends charge by Rule 3 to at most two faces. The primary charge of f is at least $1 + 5/3 - 5/3 - 2/3 = 1/3 > 2/9$.

Therefore, we may assume that f is not incident with any edge of M, and in particular, f does not share an edge with any cycle of length at most 4. Suppose that f is the cuff face of a vertex ring v. Then f sends charge by Rules 1 and 3; however, f is linked to at most one extremely 4-dangerous face and f receives $8/9$ from v. The primary charge of f is at least $1 + 8/9 - 4/3 - 1/3 = 2/9$, as desired.

If f is not the cuff face of a vertex ring, then f sends charge only by Rule 1. Let us distinguish several cases according to the number of internal vertices of degree three incident with f.

• *All vertices incident with f are internal and have degree three.* Then f and its incident vertices form a configuration isomorphic to R1 that appears in G, which is a contradiction.

• *The face f is incident with exactly four internal vertices of degree three.* Let $f = v_1v_2v_3v_4v_5$ and suppose that all these vertices except for v_2 are internal and have degree three. If v_2 is not internal, then either v_2 is a vertex ring or it has degree at least four, since v_1 and v_3 are internal. The charge of f after applying Rule 1 is $-1/3$.

The face f is incident with no edge of M, hence f is 4-dangerous. If v_2 belongs to a ring, then f receives 1/3 by Rule 2, making its charge zero, and hence f satisfies (b). Thus we may assume that v_2 is internal and of degree at least 4. Similarly, if f is opposite to a vertex ring, then f receives 1/3 by Rule 2 and f satisfies (e), hence it suffices to consider the case that f is extremely 4-dangerous.

If the face h with that f shares the edge v_4v_5 has length five, then the faces f and h form an imprint of R4 (v_2 is distinct from the vertices incident with h, since f does not share an edge with a cycle of length at most 4). If h is the cuff face of a vertex ring, then f receives 1/3 from h by Rule 3, the primary charge of f is zero and f satisfies (d). Otherwise, a configuration isomorphic to R4 appears in G. By hypothesis the face h is incident with an edge of M.

We conclude that h either has length at least 6 or is incident with an edge of M. In both cases, h sends 1/3 to f by Rule 3. Thus the primary charge of f is zero, and f satisfies (c) or (d).

• *The face f is incident with exactly three internal vertices of degree three.* In this case f sends 1/3 to each of the three incident internal vertices of degree three by Rule 1, making its charge zero. (The face f is not incident with a vertex of $\mathcal{R}$ of degree two belonging to a facial ring, since both neighbors of such a vertex belong to $\mathcal{R}$). Since f does not share an edge with M, f is 3-dangerous and satisfies (a).

• *The face f is incident with exactly two internal vertices of degree three.* Then f sends 1/3 to each of them, and at most 1/3 to a vertex of $\mathcal{R}$ of degree two by Rule 1, making its charge non-negative. Furthermore, if the charge is zero, then f satisfies (b); otherwise the charge is at least 1/3, as desired.

• *The face f is incident with at most one internal vertex of degree three.* Then f sends at most 2/3 by Rule 1 and (I5), and its primary charge is at least 1/3, as desired.

Thus we have proved the lemma when f has length five. Let us now consider the case that f has length six, and let $f = v_1v_2v_3v_4v_5v_6$. By (I1) not all vertices incident with f are internal and of degree three. Thus f sends at most 5/3 by Rule 1 and at most 4/3 by Rules 3 and 4. If f is incident with an edge of M, then its primary charge is at least $2 + 5/3 - 5/3 - 4/3 = 2/3$, as desired. If f does not send charge using Rules 3 or 4, then its primary charge is at least $2 - 5/3 = 1/3$. Furthermore, if say v_2 is a vertex of degree two belonging to a facial ring R, then by (I5), v_1 and v_3 belong to R and have degree at least three, thus the primary charge of f is at least $2 - 4/3 = 2/3$.

If say v_1 is a vertex ring, then the faces incident with the edges v_1v_2, v_1v_6, v_2v_3 and v_5v_6 are not extremely 4-dangerous; hence, f sends at most 2/3 by Rule

3 and nothing by Rule 4. Furthermore, f receives at least 1/3 from v_1 by Rule 2, thus the primary charge of f is at least $2 - 5/3 - 2/3 + 1/3 = 0$. If f sends less than 5/3 by Rule 1 or less than 2/3 by Rule 3, or if f is the cuff face of v_1, then the primary charge is at least 1/3, as desired. Otherwise, f forms an appearance of $\gamma = \mathrm{R3}$, with $\mathcal{I}_\gamma = \{v_2, v_4, v_6\}$, contradicting the hypothesis of the lemma.

We show that the situation that f is incident neither with a vertex ring nor with an edge of M and sends charge by Rule 3 or 4 cannot occur. Suppose that f sends charge across v_2v_3 by Rule 3 or 4. It follows that v_2 and v_3 are internal and of degree three. Let x_2 be the neighbor of v_2 other than v_1 and v_3, and let x_3 be defined analogously. Then both x_2 and x_3 are internal vertices of degree three. If v_1 and v_5 both belong to $\mathcal{R}$, then by (I4) and (I7), v_6 is a vertex of degree two, and by (I5) v_4 is an internal vertex, implying that $\gamma = \mathrm{R3}$ appears in G (with $\mathcal{I}_\gamma = \{v_2, v_4, v_6\}$). This contradicts the hypothesis; hence, assume that at least one of v_1 and v_5 is internal, and symmetrically, at least one of v_4 and v_6 is internal.

If both v_1 and v_5 are internal, then by (I4) and (I7) at least one of v_4 and v_6 is internal as well. Since neither v_4 nor v_6 is a vertex ring, $\gamma = \mathrm{R3}$ appears in G with $\mathcal{I}_\gamma = \{v_2, v_4, v_6\}$. This is a contradiction, hence exactly one of v_1 and v_5 belongs to $\mathcal{R}$. Therefore, $\gamma = \mathrm{R3}$ appears in G with $\mathcal{I}_\gamma = \{v_1, v_3, v_5\}$. This is a contradiction, finishing the case that $|f| = 6$.

Finally, we consider the case that $|f| \geq 7$. Let us estimate the amount of charge sent from f and received by f using Rules 3 a 4. If $v_1v_2v_3v_4$ is a subwalk of the boundary walk of f and f sends 1/3 across v_2v_3 by Rule 3, then assign 1/9 of this charge to each of v_1v_2, v_2v_3 and v_3v_4. If f sends 1/9 across v_2v_3 by Rule 4, then add 1/9 to the charge assigned to v_2v_3; if f receives 1/9 across v_2v_3, then remove 1/9 from the charge assigned to v_2v_3. We claim that each edge has at most 1/9 assigned to it, and hence that f sends at most $|f|/9$ by Rules 3 and 4.

Suppose for a contradiction that more than 1/9 is assigned to the edge v_2v_3. By symmetry, we can assume that f sends charge by Rule 3 to the face f_{12} across v_1v_2. Let $f_{23} \neq f$ be the face incident with the edge v_2v_3. If f sends charge across v_2v_3 by Rule 3, then the faces f_{12} and f_{23} form an appearance of a configuration isomorphic to R5. It follows that f_{12} or f_{23} is incident with an edge of M. This is a contradiction, because Rule 3 sends charge to 4-dangerous faces only. Furthermore, f does not send charge across v_2v_3 by Rule 4, because f is linked to f_{12} through v_1v_2.

Since more than 1/9 is assigned to v_2v_3, it follows that f sends charge across v_3v_4 by Rule 3 and does not receive charge by Rule 4 across v_2v_3. Therefore, f_{23} has length five and f_{12} and f_{23} form an appearance of a configuration isomorphic to R5 as before. Since f_{12} is 4-dangerous, some edge of M is incident with f_{23} but not with f_{12}. Since all neighbors of v_2 and v_3 have degree three and M has minimum degree at least two, it follows that some edge of M is incident with the face $f_{34} \neq f$ that is incident with v_3v_4. This is a contradiction, because f sends charge to f_{34} by Rule 3.

We can now bound the primary charge of f. If f has length at least eight, then f sends at most $|f|/3$ by Rule 1 and at most $|f|/9$ by Rules 3 and 4; thus its primary charge is at least $|f| - 4 - |f|/3 - |f|/9 = 5|f|/9 - 4 > 2/9$, as desired.

Finally, assume that f has length exactly seven. If f is incident with an edge of M, then f sends at most $7/3$ by Rule 1, making the primary charge of f at least $3+5/3-7/3-7/9 = 14/9$. If f is incident with no edge of M, then f and its incident vertices do not form an appearance of a configuration isomorphic to R2, and that in turn implies that f is incident with no more than six internal vertices of degree three. Thus f sends at most 2 by Rule 1, and hence the primary charge of f is at least $3 - 2 - 7/9 = 2/9$, as desired. □

We now modify the primary charges using three additional rules into what we will call "final charges". A vertex is *safe* if its degree is at least five, or if it belongs to $\mathcal{R}$, or if it is incident with a face with strictly positive primary charge. A face f is *k-reachable* from a vertex v if there exists a path P of length at most k (P may have length zero), joining v to a vertex incident with f, such that no vertex of $P \backslash v$ is safe. In particular, every vertex of $P \backslash v$ is internal and has degree at most four, and all faces incident with them have length 5, which implies that the number of faces that are 3-reachable from a vertex of degree d is bounded by $20d$. Furthermore, if v is incident to a face f with strictly positive primary charge, then two of the neighbors of v are safe, and we conclude that at most $20(d-3)+26$ faces distinct from f are 3-reachable from v.

Let $\epsilon > 0$ be a real number, to be specified later. Starting from the primary charges we now apply the following three rules, resulting in the *final charge*:

Rule 5: The charge of each vertex of degree three that belongs to a facial ring is increased by 26ϵ,

Rule 6: each face of strictly positive primary charge sends 46ϵ units of charge to each incident vertex,

Rule 7: if v is either a vertex ring or a safe vertex of degree at least three, then v sends a charge of ϵ to each internal face of zero primary charge that is 3-reachable from v.

Lemma 56. *Let G be a graph in a surface Σ with rings $\mathcal{R}$, let g be the Euler genus of Σ, let n_2 and n_3 be the number of vertices of degree two and three, respectively, incident with facial rings, let $\epsilon > 0$, and let M be a subgraph of G. Then the sum of final charges of all vertices and faces of G is at most $4g+4|\mathcal{R}| + 26\epsilon n_3 + 2n_2/3 + 10|E(M)|/3 - 8$.*

Proof. This follows from Lemma 53 and the description of the discharging rules. □

Lemma 57. *Let $G, \Sigma, \mathcal{R}, M$ be as in Lemma 55, and let $\epsilon \leq 1/360$. Then the final charge of every vertex is non-negative and the final charge of every vertex of degree $d \geq 4$ belonging to a facial ring is at least $(2/3 - 20\epsilon)(d-3) - 26\epsilon$.*

Proof. Let v be a vertex of G of degree d. Lemma 54 tells us that the primary charge of v is non-negative. If v is safe, then it sends at most $20\epsilon d$ units of charge by Rule 7; otherwise it sends nothing using Rules 5–7. Assume first that v is not in $\mathcal{R}$. If $d \geq 5$, then the primary charge of v is $d-4$, and its final charge is at least $d - 4 - 20\epsilon d$, which is non-negative by the choice of ϵ. If $d \leq 4$ and v is not incident with a face of positive primary charge, then its final charge is the same as its primary charge, and so the conclusion follows from Lemma 54. If $d \leq 4$ and v is incident with a face of positive primary charge, then it receives at least 46ϵ units of charge using Rule 6 and sends at most 46ϵ units using Rule 7. Thus v has non-negative final charge.

Let us now assume that v belongs to $\mathcal{R}$. If v is a vertex ring, then v has primary charge at least $d/18$, making its final charge at least $(1/18 - 20\epsilon)d \geq 0$. Suppose that v is incident with a facial ring. If $d = 2$, then v sends no charge and its final charge is zero. If $d = 3$, then v receives 26ϵ units using Rule 5, and sends at most 26ϵ units using Rule 7. Finally, if $d \geq 4$, then v has primary charge at least $2(d-3)/3$ by Lemma 54, and it sends at most $20(d-3)\epsilon + 26\epsilon$ units of charge, and hence its final charge is at least $(2/3 - 20\epsilon)(d-3) - 26\epsilon$, which is non-negative by the choice of ϵ. □

Lemma 58. *Let $G, \Sigma, \mathcal{R}, M$ be as in Lemma 55, and let $\epsilon > 0$ be arbitrary. Then the final charge of every internal face of length six or seven is at least $2/9 - 322\epsilon$, and the final charge of every internal face of length $l \geq 8$ is at least $(5/9 - 46\epsilon)l - 4$.*

Proof. Lemma 55 gives a lower bound on the primary charge of a face f, and f sends at most $46\epsilon|f|$ units of charge using Rule 6. □

Lemma 59. *Let $G, \Sigma, \mathcal{R}, M$ be as in Lemma 55, satisfying additionally* (I8), *and assume that if a configuration isomorphic to one of* R1, R2, …, R6 *or* R7 *appears in G, then it touches M. Then every internal face of zero primary charge is 3-reachable from some safe vertex.*

Proof. Let f be an internal face of zero primary charge. Lemma 55 implies that f is a 5-face, and unless f is 1-reachable from a safe vertex, we have that f is 3-dangerous and all vertices incident with f are internal and have degree at most four. Let $f = w_1w_2w_3w_4w_5$, and suppose first that w_1 and w_5 have degree four. In this case, we prove the following stronger claim: both w_1 and w_5 are in distance at most two from a safe vertex.

Let f' be the other face incident with the edge w_1w_5. Then f' has primary charge zero, because no vertex incident with f is safe. By Lemma 55 we may assume that f' is 3-dangerous, for otherwise some vertex incident with f' is safe and in distance at most two from w_1 and w_5. Since f and f' have zero primary charge, they do not share an edge with M, and in particular, they do not share an edge with any cycle of length at most four. We deduce that the faces f and f' and their incident vertices form a faint appearance of a configuration isomorphic to R6. Since f and f' are incident with no edge of M, this is not an appearance, hence either w_1 or w_5 has a neighbor in $\mathcal{R}$, or the distance from both w_1 and w_5 to a vertex of $\mathcal{R}$ is most two. In both cases, w_1 and w_5 are in distance at most two from a safe vertex, as desired. This concludes the case when w_1 and w_5 have degree four.

We may therefore assume that w_1 and w_3 have degree four. Let f_1, f_2, f_3, f_4 and f_5 be the other faces incident with the edges w_1w_2, w_2w_3, w_3w_4, w_4w_5 and w_5w_1, respectively. Similarly as before we may assume that f_1, f_2, f_3, f_4 and f_5 are all 3-dangerous, for otherwise f is 3-reachable from a safe vertex. If any of those faces contained two consecutive vertices x and y of degree four, then by the previous paragraph, both x and y would be in distance at most two from a safe vertex, and hence f would be 3-reachable from such a safe vertex. We may therefore assume that this is not the case. Since no cycle of length at most 4 shares an edge with f or f_i for $1 \leq i \leq 5$, we deduce that the faces $f, f_1, f_2, f_3, f_4, f_5$ and their incident vertices and edges form a faint appearance of a configuration γ isomorphic to R7, unless f_3 and f_5 are incident with a common vertex, i.e., unless v_4 is identified with v_9, or v_5 is identified with v_{10} in the depiction of R7 in Figure 7.1. Suppose that say $v_4 = v_9$. Since this vertex has degree three, we conclude that $\{v_3, v_5\} \cap \{v_8, v_{10}\} \neq \emptyset$. As f does not share an edge with M, we have $v_3 \neq v_8$, $v_3 \neq v_{10}$ and $v_5 \neq v_8$. However, if $v_5 = v_{10}$, then the cycles $v_5v_6v_{12}v_{11}v_1$ and $v_4v_3v_{12}v_{11}v_8$ are non-separating, contradicting (I8).

It follows that R7 faintly appears, but not appears, in G. Thus, using the labeling of the vertices as in Figure 7.1, one of x_1, x_3, x_6, x_7, x_8 or one of their neighbors belongs to $\mathcal{R}$. Therefore, f is 3-reachable from a safe vertex, as desired. $\square$

Lemma 60. *Let $G, \Sigma, \mathcal{R}, M$ be as in Lemma 59, let $\epsilon \leq 2/2079$, and assume that if a good configuration appears in G, then it touches M. Then the final charge of every internal face of length five is at least ϵ.*

Let us remark that $2079 = 9(5 \cdot 46 + 1)$.

Proof. Let f be an internal face of length five. If f has positive primary charge, then by Lemma 58, f has final charge at least $2/9 - 5 \cdot 46\epsilon \geq \epsilon$.

We may therefore assume that f has primary charge zero. By Lemma 59, f is 3-reachable from some safe vertex, and hence has final charge at least ϵ because of Rule 7, as desired. $\square$

Let $s : \{5, 6, \ldots\} \to \mathbb{R}$ be an increasing convex function (that we specify later) satisfying

- (S1) $s(5) = 2\epsilon$,

- (S2) $s(7) \leq 4/9 - 644\epsilon$,
- (S3) $s(l) \leq (10/9 - 92\epsilon)l - 8$ for every integer $l \geq 8$.

Suppose that we are given such a function and a graph G in Σ with rings $\mathcal{R}$. For an internal face f of G, we define $w(f) = s(|f|)$ if f is open 2-cell and $|f| \geq 5$, and $w(f) = |f|$ otherwise. We define $w(G, \mathcal{R})$ as the sum of $w(f)$ over all internal faces f of G.

Lemma 61. *Let G be a graph in a surface Σ with rings $\mathcal{R}$ satisfying* (I0)–(I8)*, let M be a subgraph of G that captures (≤ 4)-cycles and assume that if a configuration isomorphic to one of* R1, R2, …, R7 *appears in G, then it touches M. Let ϵ be a real number satisfying $0 < \epsilon < 2/2079$, and let $s : \{5, 6, \ldots\} \to \mathbb{R}$ be a function satisfying* (S1)–(S3)*. Then the final charge of every vertex is non-negative, the final charge of every face bounded by a ring is zero, and the final charge of every internal face f is at least $s(|f|)/2$.*

Proof. The final charge of every face bounded by a ring is clearly zero. The remaining assertions follow from Lemmas 57, 58 and 60 using conditions (S1)–(S3). □

Lemma 62. *Let $G, \Sigma, \mathcal{R}, M, \epsilon, s$ be as in Lemma* 61*, Then $w(G, \mathcal{R}) \leq 8g + 8|\mathcal{R}| + 52\epsilon n_3 + 4n_2/3 + 20|E(M)|/3 - 16$.*

Proof. By Lemma 61 the quantity $w(G, \mathcal{R})$ is at most twice the sum of the final charges of all vertices and faces of G, and hence the lemma follows from Lemma 56. □

We need the following refinement of the previous lemma.

Lemma 63. *Let $G, \Sigma, \mathcal{R}, M, \epsilon, s$ be as in Lemma* 61*. Then $w(G, \mathcal{R}) \leq 8g + 8|\mathcal{R}| + 52\epsilon n_3 + 4n_2/3 + 20|E(M)|/3 - 8b/9 - 16$, where b is the number of internal 6-faces of G incident with a vertex of degree two contained in a facial ring, plus the number of vertices of degree at least four incident with a facial ring.*

Proof. This follows similarly as Lemma 62, since according to Lemma 55, each 6-face incident with a vertex of degree two contained in a facial ring has charge by at least 4/9 higher than the bound used to derive Lemma 62, and since the final charge of a vertex of degree at least four incident with a facial ring is at least $2/3 - 46\epsilon > 4/9$. □

7.5 Reductions

In this section, we argue that subject to a few assumptions, reducing a good configuration does not create cycles of length at most four.

Let G be a graph in a surface Σ with rings $\mathcal{R}$, and let P be a path of length at least one and at most four with ends $u, v \in V(\mathcal{R})$ and otherwise disjoint from $\mathcal{R}$. We say that P is *allowable* if

- u, v belong to the same ring of $\mathcal{R}$, say R,
- P has length at least three,
- there exists a subpath Q of R with ends u, v such that $P \cup Q$ is a cycle of length at most eight that bounds an open disk $\Delta \subset \Sigma$,
- if P has length three, then $P \cup Q$ has length five and Δ is a face of G, and
- if P has length four, then Δ includes at most one edge of G, and if it includes one, then that edge joins the middle vertex of P to the middle vertex of Q.

We say that G is *well-behaved* if every path P of length at least one and at most four with ends $u, v \in V(\mathcal{R})$ and otherwise disjoint from $\mathcal{R}$ is allowable.

We say that a configuration γ *strongly appears in* G if it both appears and weakly appears in G and

- if $u, v \in \mathcal{A}_\gamma$ are distinct, then at least one of u, v is internal,
- if $u, v \in \mathcal{I}_\gamma$ are distinct, $u \in V(\mathcal{R})$, and $w \in V(\mathcal{R})$ is a neighbor of v, then u and w are adjacent and $uw, wv \in E(G_\gamma)$, and
- if γ is isomorphic to R7, then the vertices corresponding to v_2 and z are distinct, non-adjacent and have no common neighbor distinct from v_1, v_3, x_6 and x_7.

Lemma 64. *Let G be a graph in a surface Σ with rings $\mathcal{R}$ satisfying* (I0)*,* (I2) *and* (I8)*, and assume that G is well-behaved. If a configuration isomorphic to one of* R1*,* R2*, ...,* R7 *appears in G and no cycle in G of length four or less touches it, then either a good configuration strongly appears in G, or G is a planar graph with one ring of length $2s$ for some $s \in \{5, 7\}$, such that $V(G) = V(R) \cup V(C)$ for a cycle C of length s, each vertex of C is internal of degree three and has one neighbor in R.*

Proof. Let γ be a good configuration appearing in G, such that no cycle in G of length four or less touches γ. If possible, we choose γ so that it is equal to one of R1, R6′, R7′, R7″, R7‴ or R7⁗. We claim that, possibly after relabeling the vertices of G_γ, γ strongly appears in G. To prove that we first notice that the first condition of weak appearance holds by hypothesis and (I8)—if $x_3 = x_7$, then $x_3v_3v_{12}v_6v_7$ is a 5-cycle separating x_1 from x_6. The third condition is implied by appearance. The second condition of weak appearance our choice of γ and the fact that no cycle of length at most four touches γ. For example, if γ is R7, then v_2 and v_7 are not adjacent, because R7$'$ does not appear in G by the choice of γ. Additionally, when γ is R5, we use (I2) to show that v_1 is not adjacent to v_5.

It remains to prove that γ satisfies the conditions of strong appearance. Let us discuss the configurations separately. If γ is R1 or R2, it suffices to show that we can choose the labels of the vertices of γ so that x_1 is internal. If that is not possible, then each vertex of γ is adjacent to a vertex belonging to $\mathcal{R}$. Since G is well-behaved it follows that there exists a ring $R \in \mathcal{R}$ that satisfies the conclusion of the lemma for $s = 5$ if γ is R1 and for $s = 7$ when γ is R2.

If γ is R3, we only need to prove the second condition of strong appearance. Suppose that say $v_3 \in V(\mathcal{R})$ and v_5 has a neighbor x_5 in $\mathcal{R}$ other than v_4. Since G is well-behaved, v_4 is an internal vertex and $v_3v_4v_5x_5$ together with a path in $\mathcal{R}$ bound a 5-face, implying that v_4 has degree two. This contradicts (I0).

If γ is R4, then note that the path $x_1v_1v_2v_3x_3$ is not allowable, since by the definition of appearance, v_2 has degree four. Therefore, at least one of x_1 and x_3 is internal, and γ strongly appears.

If γ is R5, we need to prove the first and the second condition of strong appearance. For the first one, observe that the path $v_2v_1v_8x_8$ is not allowable, since v_1 has degree at least three. For the second condition, since γ appears in G, we have that v_4 is internal, thus it suffices to consider the case that x_6 and a neighbor x_4 of v_4 belongs to $\mathcal{R}$. Since $v_3v_4v_5v_6v_7$ is not an appearance of R1 in G, v_4 has degree at least four, and thus the paths $v_2v_3v_4x_4$ and $x_4v_4v_5v_6x_6$ cannot both be allowable. It follows that v_2 is internal, and similarly all neighbors of v_2 are internal. However, then we can relabel the vertices of γ, switching v_2 with v_4, v_6 with v_8, etc., and obtain a strong appearance of R5 in G.

For the configurations R6, …, R7$''''$, the first two conditions follow from the definition of appearance. Therefore, suppose that γ is R7 and let us now consider the last condition in the definition of strong appearance. Again, we we use symmetry: if the condition does not hold for γ we swap v_1 and v_3, v_6 and v_8, and so on. The vertex v_2 cannot be equal to or adjacent to both z and z_1, since $v_2 \neq x_7$ (otherwise, R7$'$ would appear in G), x_7 has degree at least three and no cycle of length at most four touches γ. Unless the condition holds, we can assume that $z_1 \neq v_2$, z_1 is not adjacent to v_2 and that z_1 and v_2 have a common neighbor x_2 distinct from v_1, v_3, x_7 and x_8. Since no cycle of length at most four touches γ, we have $z \notin \{v_2, v_3, x_2\}$. If $z = v_1$, then the cycle $K = v_1v_{11}v_86v_7x_7$ separates z_1 from v_2, thus $x_2 \in V(K)$. This is a contradiction, since then a cycle of length at most four touches γ. Therefore, z is distinct from and non-adjacent to v_2. Furthermore, z is not adjacent to x_2, as otherwise x_2zx_7z' touches γ. $\square$

Lemma 65. *Let G be a graph in a surface Σ with rings $\mathcal{R}$ satisfying* (I0)*,* (I3)*,* (I8) *and* (I9)*, let γ be a good configuration that strongly appears in G, and let G' be the γ-reduction of G with respect to a 3-coloring ϕ of $\mathcal{R}$. If G is $\mathcal{R}$-critical and C' is a cycle in G' of length at most four, then either C' is a cycle in G, or*

C' is noncontractible and there exists a noncontractible cycle C in G such that C touches γ and $|C| - |C'| \leq 3$. Furthermore, all ring vertices of C' belong to C, and if r_1, r_2 and r_3 are ring vertices adjacent to mutually distinct vertices of C', then r_1, r_2 and r_3 also have mutually distinct neighbors in C.

Proof. Suppose that $C' \subseteq G'$ is a cycle of length 3 or 4 such that C' is not a cycle in G. Let us discuss the possible configurations γ:

- γ is isomorphic to one of R1, R2, R6, R6′, R7′, R7″, R7‴ or R7⁗, or to R4 and both x_4 and x_5 belong to $\mathcal{R}$ and $\phi(x_4) = \phi(x_5)$. We are adding an edge e between vertices $x, x' \in \mathcal{A}_\gamma$ along the replacement path $P \subset G$ of length at most 4. Note that $e \in E(C')$. Let $C \subseteq G$ be the cycle obtained from C' by replacing e with P. Clearly, $|C| \leq |C'| + 3 \leq 7$. Let us remark that C is indeed a cycle (i.e., if γ is R4, then $v_2 \notin V(C')$), since no non-ring cycle of length at most four touches γ. Note that P is not a part of a boundary of a face in any of the configurations, thus C does not bound a face in G. By (I9), C is not contractible, hence C' is not contractible, either.

- γ is R3: Let w be the vertex of G' obtained by identifying v_1 with v_3 and v_5. Note that $w \in V(C')$ and consider the edges $e_1, e_2 \in E(C')$ incident with w. Unless C' corresponds to a cycle of length $|C'|$ in G, e_1 and e_2 are incident with distinct vertices $a, b \in \mathcal{I}_\gamma$, and the cycle C obtained from C' by adding the replacement path avb between a and b has length at most $|C'| + 2 \leq 6$. Note that C' and C have the same homotopy. Suppose that they are contractible. By (I9), we conclude that implies that C' bounds a face h and v has degree two. By (I0), v belongs to $\mathcal{R}$, and since at least one of a and b is internal, v is a vertex ring. However, then h is the cuff face of v and C' is not contractible. This is a contradiction.

- γ is R4 and at least one of x_4 and x_5 is internal: Let w be the vertex obtained by identifying x_4 and x_5. If x_1x_3 is not an edge of C', then the cycle C obtained from C' by replacing w by the path $x_4v_4v_5x_5$ such that $6 \leq |C| \leq 7$ does not bound a face, thus neither C nor C' is contractible. Let us assume that $x_1x_3 \in E(C')$. Similarly, we deal with the case that $w \notin V(C')$ or that both edges incident with w in C' correspond to edges incident to one of x_4 and x_5.

  Suppose now that the neighbors of w in C' are adjacent to x_4 and x_5. Since no cycle of length at most four touches γ, we have $x_1x_5, x_3x_4 \notin E(G)$, thus by symmetry we may assume that $x_1x_4 \in E(C')$ and x_3 and x_5 are joined by a path P of length at most two in C'. By (I8), the 5-cycle $K = x_1v_1v_5v_4x_4$ separates x_3 from x_5, thus P is not disjoint with K. However, then a cycle of length at most four touches γ.

- γ is R4, neither x_4 nor x_5 is internal and $\phi(x_4) \neq \phi(x_5)$: Let w be the vertex created by identifying v_2 and x_5. The claim of the lemma follows by considering the non-facial cycle C obtained from C' by replacing w with $v_2v_1v_5x_5$.

- γ is R5: Let w be the vertex obtained by identifying v_4 and x_6. Let C be the cycle obtained from C' by replacing v_2x_8 by $v_2v_1v_8x_8$ or w by $v_4v_5v_6x_6$ or both. If we performed only one replacement, then $|C| \leq |C'|+3$ and the claim follows from (I9).

  Otherwise, $v_2x_8 \in E(C')$ and $w \in V(C')$, and since no cycle of length at most four touches γ, there exist paths P_1 between v_2 and x_6 and P_2 between v_4 and x_8 of total length at most three. Let K_1 be the cycle consisting of $v_2v_3v_7v_6x_6$ and P_1 and K_2 the cycle consisting of $v_4v_3v_7v_8x_8$ and P_2, and by symmetry assume that $|K_1| = 5$ and $|K_2| \leq 6$. By (I8) the cycle K_1 separates v_4 from v_8, thus P_2 intersects K_1. However, that contradicts the fact that no cycle of length at most four touches γ.

- γ is R7: Let w be the vertex obtained by identifying x_6 and x_7. Let C_1 be the cycle obtained from C' by replacing x_1x_3 by $x_1v_1v_2v_3x_3$ or w by $x_6v_6v_7x_7$ or both. If we performed only one replacement, then $|C_1| = |C'| + 3$ and the claim of the lemma follows from (I9), with $C = C_1$.

  Otherwise, let C_2 be the closed walk obtained from C_1 by replacing $x_6v_6v_7x_7$ by x_6zx_7; we have $|C_2| = |C'| + 5 \leq 9$. Since γ appears, observe that all vertices of C' are internal and at most one of them has a neighbor in a ring. Note that C_2 is a cycle, since otherwise a cycle of length at most four touching γ is a subgraph of C_2. Suppose now that C_2 consists of $x_1v_1v_2v_3x_3$, a path P_1 from x_3 to x_7, the path x_7zx_6 and a path P_2 from x_6 to x_1, where the total length of P_1 and P_2 is at most three. Let K_1 be the cycle consisting of P_1 and $x_3v_3v_{12}v_6v_7x_7$ and K_2 the cycle consisting of P_2 and $x_1v_1v_{11}v_{12}v_6x_6$. Note that $\min(|K_1|, |K_2|) \leq 6$, and by (I8), the shorter of the two cycles is separating. It follows that K_1 and K_2 intersect in a vertex distinct from v_{12} and v_6, This is a contradiction, since the vertices of C_2 are mutually distinct and none of them is equal to $v_7, v_{11} \notin V(G')$.

  Therefore, C_2 consists of $x_1v_1v_2v_3x_3$, a path Q_1 of length $l_1 \geq 1$ from x_3 to x_6, the path x_6zx_7 and a path Q_2 of length l_2 from x_7 to x_1, where $l_1 + l_2 \leq 3$. Let L_1 be the cycle consisting of Q_1 and $x_3v_3v_{12}v_6x_6$ and L_2 the cycle consisting of Q_2 and $x_1v_1v_{11}v_8v_7x_7$. Note that neither L_1 nor L_2 bounds a face, $|L_1| = 4 + l_1 \leq 7$ and $|L_2| = 5 + l_2 \leq 7$, thus by (I9) neither L_1 nor L_2 is contractible. Furthermore, $|L_1| + |L_2| \leq 9 + l_1 + l_2 \leq 12$, thus there exists a cycle $C \in \{L_1, L_2\}$ of length at most $6 \leq |C'| + 3$ touching γ.

  Let us now show that the cycle C' is not contractible. Assume for a contradiction that C', and hence also C_2, is contractible. Let $\Delta \subseteq \Sigma$ be an open disk bounded by C_2. Note that Δ does not consist of a single face, since at least one edge incident with v_1 or v_2 lies inside Δ. By (I9), Δ consists of two or three faces, and in the latter case, $|C_2| = 9$ and three vertices of C_2 have a common neighbor.

It follows that $v_{11}, v_{12} \notin \Delta$, and thus the edge joining v_2 with its neighbor $x_2 \notin \{v_1, v_3\}$ lies in Δ. Since γ appears strongly in G, we have that $x_2 \neq z$ and that z is an internal vertex. We conclude that $\deg(z) = 3$ and z has a neighbor inside Δ distinct from x_6 and x_7. By (I3) and (I9), this neighbor is equal to x_2. However, this contradicts the assumption that γ appears strongly in G.

□

7.6 Contributions of faces

Let G be a graph in a surface Σ with rings $\mathcal{R}$ satisfying (I3). Let γ be a good configuration that strongly appears in G, let G' be the γ-reduction of G, and let G'' be a subgraph of G' that includes all the rings and satisfies (I0).

Let f'' be an internal face of G'', and let H be the subgraph of G'' that forms the boundary of f''. We wish to define a subgraph $J_{f''}$ of G that will correspond to H, and a union of faces of $J_{f''}$ that will correspond to f''.

Let us recall that during the construction of the graph G', parallel edges could have been removed (e.g., if γ is R5 and v_4 and x_6 have a common neighbor), but we have retained the correspondence of each non-squashed edge e of G' to a unique edge of G (which also determined the placement of e in the embedding of G'). We now define the edge-set of $J_{f''}$, by replacing pieces of the boundary of f'' by appropriate replacement paths. More precisely, we apply the following construction to each boundary walk C of f''. Let C be $v_1, e_1, v_2, e_2, \ldots, v_m, e_m$ and let $e_{m+1} = e_1$, $v_{m+1} = v_1$, $e_0 = e_m$ and $v_0 = v_m$. Replace each edge e_i of C by a path P_i defined as follows:

- If e_i is a new edge, then P_i is the corresponding replacement path.
- If e_i is a squashed edge, then a vertex v_j with $j \in \{i, i+1\}$ is a new vertex. Let $e = e_{2j-i-1}$ be the other edge of C incident with v_j. Note that $e \neq e_i$, since otherwise v_j would have degree one; however, by the assumption (I0), v_j would be a vertex ring, and the corresponding vertex of G would belong to $\mathcal{I}_\gamma$, contrary to the definition of appearance. The edge of G corresponding to e is incident with a vertex $v \in \mathcal{I}_\gamma$. Let P_i be the edge vv_{2i-j+1}.
- Suppose that e_i is neither new nor squashed. If v_{i+1} is a new vertex, e_{i+1} is not a squashed edge, e_i is incident in G with a vertex $u \in \mathcal{I}_\gamma$, e_{i+1} is incident in G with $v \in \mathcal{I}_\gamma$ and $u \neq v$, then let P_i consist of e_i and the replacement path between u and v.
- Otherwise, let P_i be the path with edge-set $\{e_i\}$.

The newly constructed walk has the same homotopy as C. It should be noted that even though f'' is a face of G'', it may correspond to several faces of $J_{f''}$. Let the set of these faces of $J_{f''}$ be denoted by $S_{f''}$. For example, suppose that G' was created by reducing the configuration R3, G'' does not contain any of the squashed edges and f'' is bounded by a cycle that contains the new vertex and edges that were incident with v_3 and v_5, and suppose that v_4 is incident with f'' as well. Then $J_{f''}$ contains the replacement path $v_3v_4v_5$, which can split f'' to two faces sharing the vertex v_4.

Let us also remark on another somewhat subtle issue. In the definition of "faintly appear", we require that vertex rings do not belong to $\mathcal{I}_\gamma$. The reason for this restriction is the following. Later, we use the fact that the faces in $S_{f''}$ do not contain any rings. If say $u \in \mathcal{I}_\gamma$ were a vertex ring identified with another vertex $v \in \mathcal{I}_\gamma$ to a new vertex w and all edges in f'' that are incident with w corresponded to edges of G incident with v, then this condition could fail.

The *elasticity* of f'' is $\mathrm{el}(f'') = \left(\sum_{f \in S_{f''}} |f|\right) - |f''|$. Note that f'' can have non-zero elasticity only if $J_{f''}$ contains at least one replacement path. Using this fact and the inspection of the configurations, we observe the following.

Lemma 66. *Let G, γ, G', G'' be as above. Then G'' has at most three faces with non-zero elasticity, and the sum of the elasticities of the faces of G'' is at most 10. Furthermore, if an internal face f'' of G'' is closed 2-cell or omnipresent, then $el(f'') \le 5$, and if the inequality is strict, then $el(f'') \le 3$.*

Let G be a graph in a surface Σ with rings $\mathcal{R}$, let J be a subgraph of G, and let S be a subset of faces of J such that

(30) *J is equal to the union of the boundaries of the faces in S and whenever C is a cuff intersecting a face $f \in S$, then C is incident with a vertex ring belonging to J.*

We define $G[S]$ to be the subgraph of G consisting of J and all the vertices and edges drawn inside the faces of S. Let $C_1, C_2, \ldots, C_k$ be the boundary walks of the faces in S (in case that a vertex ring $R \in \mathcal{R}$ forms a component of a boundary of a face in S, we consider R itself to be such a walk). We would like to view $G[S]$ as a graph with rings $C_1, \ldots, C_k$. However, the C_i's do not necessarily have to be disjoint, and they do not have to be cycles or isolated vertices. To overcome this difficulty, we proceed as follows: Let Z be the set of cuffs incident with the vertex rings that form a component of J by themselves, and let $\hat{Z} = \bigcup_{R \in Z} \hat{R}$. Suppose that $S = \{f_1, \ldots, f_m\}$. For $1 \le i \le m$, let Σ'_i be a surface with interior homeomorphic to f_i (i.e., we add cycles forming the boundary of Σ_1). Let $\theta_i : f_i \to \Sigma'_i$ be the homeomorphism and let $\Sigma_i = \Sigma'_i \setminus \theta_i(\hat{Z} \cap f_i)$. Let G_i be the graph with a normal embedding in Σ_i corresponding to the subgraph of G drawn in the closure of f; i.e., $\theta_i(f \cap G) \subset G_i$ and each component of the boundary of Σ'_i forms a cycle in G_i. The set $\{(G_i, \Sigma_i) : 1 \le i \le m\}$ is called the

G-expansion of S. Note that there is a one-to-one correspondence between the boundary walks of the faces of S and the rings of the graphs in the G-expansion of S; however, each vertex of J may be split to several copies.

We define the *G-expansion of f''* to be the G-expansion of $S_{f''}$. The following lemma is straightforward.

Lemma 67. *Let G, γ, G', G'' be as above, and let f be an internal face of G. Then either there exists a unique internal face f'' of G'' such that f corresponds to an internal face of the G-expansion of f'' or γ is isomorphic to* R3 *and f is the* 6*-face of $\mathcal{F}_\gamma$.*

Let us now give an informal summary of what we are trying to achieve in this section. We assign weights to the faces of embedded graphs according to the function s as described in Section 7.4, and we aim to show that the sum of the weights of the faces of G is bounded by the sum of the weights of the faces of G''. To do so, we would like to claim that the sum w of the weights of the faces in the G-expansion $G_{f''}$ of f'' is bounded by the weight w'' of f''. In Theorem 73, we will show that this claim holds, provided that the elasticity of f'' is small and $G_{f''}$ is not one of a few exceptional graphs. Here, we assign a *contribution* $c(f'')$ to each face f'' of G'' according to the criteria that we later prove to ensure that $w \leq w'' - c(f'')$. Furthermore, we argue that the sum of the contributions of all faces is non-negative.

Let us now proceed more formally. We say that a plane graph G with one ring R of length $l \geq 5$ is *exceptional* if it satisfies one of the conditions below (see Figure 7.2):

- (E0) $G = R$,
- (E1) $l \geq 8$ and $E(G) - E(R) = 1$,
- (E2) $l \geq 9$, $V(G) - V(R)$ has exactly one vertex of degree three, and the internal faces of G have lengths $5, 5, l - 4$,
- (E3) $l \geq 11$, $V(G) - V(R)$ has exactly one vertex of degree three, and the internal faces of G have lengths $5, 6, l - 5$,
- (E4) $l \geq 10$, $V(G) - V(R)$ consists of two adjacent degree three vertices, and the internal faces of G have lengths $5, 5, 5, l - 5$,
- (E5) $l \geq 10$, $V(G) - V(R)$ consists of five degree three vertices forming a facial cycle of length five, and the internal faces of G have lengths $5, 5, 5, 5, 5, l - 5$,

We say that G is *very exceptional* if it satisfies (E1), (E2) or (E3).

Let us now show the following lemma, which we use to analyze omnipresent faces.

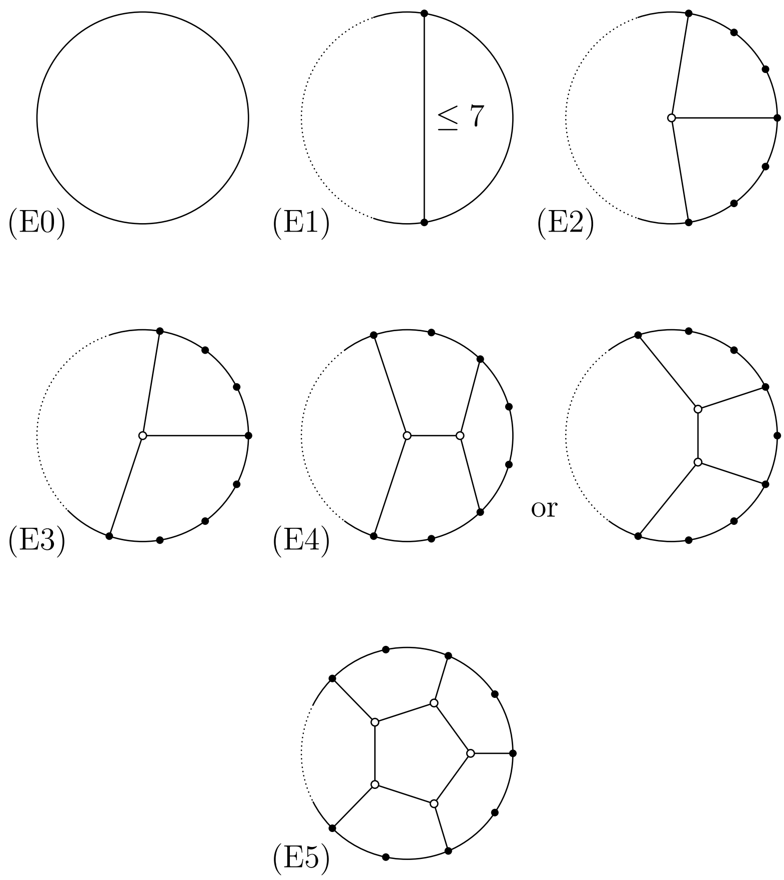


Figure 7.2: Exceptional graphs.

Lemma 68. *Suppose that γ strongly appears in a well-behaved graph G, G satisfies (I0), (I4) and (I8), and let H be a component of G'' that contains a new edge or a new vertex. Assume that Σ is a disk and $|\mathcal{R}| = 1$ and every face of G'' is closed 2-cell, or that G'' has an omnipresent face. Then H is not very exceptional. Furthermore, if γ is isomorphic to one of* R6*,* R6$'$*,* R7*,* R7$'$*,* R7$''$*,* R7$'''$ *or* R7$''''$*, then H is not exceptional.*

Proof. Let $R \in \mathcal{R}$ be the ring belonging to H. If γ is isomorphic to one of R7, R7$'$, R7$''$, R7$'''$ or R7$''''$, then all vertices in $\mathcal{A}_\gamma \cup \mathcal{I}_\gamma$ and all their neighbors are internal, and thus each new edge or new vertex is in distance at least two from R. It follows that H cannot be exceptional. Similarly, we exclude the case that γ is isomorphic to R6 or R6$'$. Thus, assume that γ is one of R1, …, R5.

Suppose that H contains a new edge xy. Note that since γ does not touch a non-ring cycle of length at most four, neither x nor y is a new vertex. Since γ appears strongly in G, we may assume that x is an internal vertex, thus H does not satisfy (E1). Suppose that H satisfies (E2) or (E3). Then, in H the vertex x has three neighbors in R. On the other hand, x has at most one neighbor in R in G, by (I4). We conclude that x is adjacent to a new vertex in G'' that belongs to R. It follows that γ is R4 or R5, and in the former case at least one of x_4 and x_5 is internal. Let $\mathcal{I} = \mathcal{I}_\gamma$ if γ is R5 and $\mathcal{I} = \{x_4, x_5\}$ if γ is R4. Note that there exists a vertex in $\mathcal{I}$ belonging to R, and another vertex of $\mathcal{I}$ is adjacent to x in G. If γ is isomorphic to R4, then by symmetry we may assume that x_1 is adjacent to x_4 and x_3 and x_5 belong to R. However, by (I8), the cycle $x_1v_1v_5v_4x_4$ consisting of internal vertices separates x_3 from x_5, which is a contradiction. If γ is isomorphic to R5, then by the conditions of appearance, v_4 is an internal vertex, hence x_6 belongs to R. Since v_2 and v_4 are not adjacent, we conclude that v_4 is adjacent to x_8 and that v_2 belongs to R. However, this again contradicts (I8).

Therefore, we may assume that H contains a new vertex, but not a new edge. Suppose first that γ is not isomorphic to R4. If H satisfied (E1), then by (I4) there would exist vertices $x \in \mathcal{I}_\gamma \cap V(R)$ and $y \in \mathcal{I}_\gamma \setminus V(R)$ and a neighbor z of y in R, where z is not adjacent to x. However, this contradicts the assumption that γ appears strongly in G. If H satisfies (E2) or (E3), then by (I4) we have $|\mathcal{I}_\gamma| = 3$ (thus γ is R3), all elements of $\mathcal{I}_\gamma$ are internal and each of them has exactly one neighbor in R. This is excluded, since γ appears in G.

Finally consider the case that γ is R4 and H does not contain a new edge. By (I4), H does not satisfy (E2) or (E3), thus suppose that H satisfies (E1). If x_4 is an internal vertex, this implies that $x_5 \in V(R)$ and x_4 has a neighbor w in R distinct from z. By (I4), z is an internal vertex. Since G is well-behaved, the path x_5zx_4w forms a part of a boundary of a 5-face, thus z has degree two, contrary to (I0). The case that x_5 is internal is symmetric, thus assume that both x_4 and x_5 belong to R. Then v_2 is an internal vertex of degree at least four and has a neighbor $w \in V(R)$. However, since G is well-behaved, the subpaths

v_3v_2w and v_1v_2w of the paths $x_4v_4v_3v_2w$ and $x_5v_5v_1v_2w$ form parts of boundaries of faces, implying on contrary that v_2 has degree three. $\square$

Suppose that all faces of G'' are either closed 2-cell of length at least 5 or omnipresent. Let us now define the *contribution* of an internal face f'' of G''. Let $s : \{5, 6, \ldots\} \to \mathbb{R}$ be an increasing convex function to be chosen later, such that

- (S0) $18s(5) \leq s(6)$, $135s(5) \leq s(7)$, $4s(6) \leq s(7)$, $3s(7) \leq s(8)$ and $s(l) = l - 8$ for $l \geq 9$.

If f'' is closed 2-cell, then its contribution is defined as follows.

- If $G_{f''}$ satisfies (E0), then $c(f'') = -\infty$ if f'' has non-zero elasticity and $c(f'') = 0$ otherwise.
- If $G_{f''}$ satisfies (E1), then $c(f'') = -\infty$ if $\mathrm{el}(f'') = 5$ and $c(f'') = s(8 - \mathrm{el}(f'')) - 2s(5)$ otherwise.
- If $G_{f''}$ satisfies (E2), then $c(f'') = -\infty$ if $\mathrm{el}(f'') = 5$ and $c(f'') = s(9 - \mathrm{el}(f'')) - 3s(5)$ otherwise.
- If $G_{f''}$ satisfies (E3), then $c(f'') = s(11 - \mathrm{el}(f'')) - 2s(6) - s(5)$.
- If $G_{f''}$ satisfies (E4) or (E5), or if $|S_{f''}| = 2$ and $G_{f''}$ consists of two cycles such that one of them has length 5, then $c(f'') = s(10 - \mathrm{el}(f'')) - 6s(5)$.
- If $|S(f'')| = 1$ and $G_{f''}$ is not exceptional, and
  - $G_{f''}$ contains a path $P = v_1v_2v_3v_4$ such that $v_1, v_4 \in V(J_{f''})$, $v_2, v_3 \notin V(J_{f''})$ and both of the open disks bounded by P and paths in $J_{f''}$ contain at least two vertices of G, then $c(f'') = s(7)$.
  - Otherwise, $c(f'') = s(11 - \mathrm{el}(f'')) - s(6) + 5s(5)$.
- If $|S_{f''}| = 2$ and $G_{f''}$ does not consist of two cycles such that one of them has length 5, or if $|S_{f''}| \geq 3$, then $c(f'') = s(12 - \mathrm{el}(f'')) - 2s(6)$.

Suppose now that f'' is an omnipresent face of G''. Let $G''_1, G''_2, \ldots, G''_k$ be the components of G'' such that G''_i contains the ring $R_i \in \mathcal{R}$. If there exists $i \neq j$ such that $G''_i \neq R_i$ and $G''_j \neq R_j$, then $c(f'') = 1$. Otherwise, suppose that $G''_i = R_i$ for $i \geq 2$. If G''_1 satisfies (E0), (E1), (E2) or (E3), then $c(f'') = -\infty$. If G''_1 satisfies (E4) or (E5), then $c(f'') = 5 - \mathrm{el}(f'') - 5s(5)$, otherwise $c(f'') = 5 - \mathrm{el}(f'') + 5s(5)$.

Let $c(G'') = -\delta + \sum_{f'' \in F(G'')} c(f'')$, where δ is $s(6)$ if γ is isomorphic to R3 and 0 otherwise.

Let us remark on the following consequences of the convexity of s which we often use:

- If $a_1 \leq a_2 \leq \ldots \leq a_k$ and $x_i \geq a_i$ for each $1 \leq i \leq k$, then

$$s(x_1)+s(x_2)+\ldots+s(x_k) \leq s(a_1)+s(a_2)+\ldots+s(a_{k-1})+s\left(\sum_{i=1}^{k} x_i - \sum_{i=1}^{k-1} a_i\right).$$

- If $x \geq a$ and $m \geq 0$, then $s(x) \leq s(x+m) - (s(a+m) - s(a))$.

Lemma 69. *Let G be a well-behaved graph in a surface Σ with rings $\mathcal{R}$ satisfying (I0)–(I4) and (I8), let γ be a good configuration strongly appearing in G, let G' be the γ-reduction of G. Suppose that G'' is a subgraph of G' that includes $\mathcal{R}$, G'' satisfies (I6), and G'' contains a new vertex or a new edge. If each face of G'' is closed 2-cell or omnipresent, then $c(G'') \geq 0$. Furthermore, if each face of G'' is closed 2-cell, then G'' has a face of length at least six, and if Σ is a disk and $|\mathcal{R}| = 1$, then $c(G'') \geq 10s(5)$.*

Proof. Let us note that G'' satisfies the assumptions of Lemma 68, and thus the contribution of the omnipresent face (if G'' has such a face) is not $-\infty$.

We may assume that there exists a face of non-zero elasticity, for otherwise all faces have non-negative contribution and the face f'' of G'' that includes all vertices of $\mathrm{dom}(d_\gamma)$ satisfies $c(f'') \geq s(7)$.

Let us argue that if a face f'' that is closed 2-cell has non-zero elasticity, then either $S_{f''}$ has at least two components or the interior of the face of $S_{f''}$ contains an edge of G. Indeed, most replacement paths are incident with edges on both of its sides, thus if such a replacement path is used in $J_{f''}$, then at least one such edge lies in $S_{f''}$. The exceptions are the replacement paths in R3, R4 and the replacement path between the vertices of R7 in $\mathcal{I}_\gamma$. In these configurations, the middle vertex v of the replacement path could also lie on the boundary walk of f'', in which case all the edges incident with v could belong to $J_{f''}$ or lie outside of $S_{f''}$. However, then $S_{f''}$ has at least two components. We conclude that if $c(f'') = -\infty$, then $\mathrm{el}(f'') = 5$ and two replacement paths are used in the construction of $J_{f''}$.

Let us now consider the case that either G'' contains no new edge, or that for every internal face f'' of G'', no replacement path between vertices of $\mathcal{I}_\gamma$ is added. Then every face bounded by a cycle has elasticity at most three, and hence the contribution of each face is greater or equal to $-s(5)$. If G'' has an omnipresent face, then $c(G'') \geq 2 - 7s(5) - s(6)$, hence assume that all faces of G'' are closed 2-cell. Observe that G has a face f'' such that at least one vertex of $\mathrm{dom}(d_\gamma)$ is contained inside a face of $S_{f''}$. For this face, we have $c(f'') \geq s(6) - 3s(5)$. Furthermore, if γ is R3, then the elasticity of f'' is at most two, thus $c(f'') \geq s(7) - 3s(5)$, and all other faces of G'' have non-negative contribution. Therefore, $c(G'') \geq \min(s(6) - 4s(5), s(7) - s(6) - 3s(5)) \geq 10s(5)$.

Thus we may assume that γ is R4, R5, or R7, and G'' contains a new edge incident with two faces of non-zero elasticity, say f_1 and f_2, where f_2 contains all vertices of $\mathrm{dom}(d_\gamma)$. Furthermore, G'' contains a new vertex w incident with f_2 and possibly another face f_3 of non-zero elasticity.

The elasticity of f_2 is 5, and by the inspection of the configurations, we conclude that $c(f_2) \geq -5s(5)$. Furthermore, if γ is isomorphic to R7, then $c(f_2) = s(7)$ if f_2 is closed 2-cell, and by Lemma 68, we have $c(f_2) = 5s(5)$ if f_2 is omnipresent.

Assume now that either f_2 is the only face of G'' with non-zero elasticity that is incident with w, or that $f_1 \neq f_3$. Consider a face $f \in \{f_1, f_3\}$ with non-zero elasticity. Since $\mathrm{el}(f) \leq 3$, we have $c(f) \geq -s(5)$. Furthermore, if f is omnipresent, then by Lemma 68, we have $c(f) \geq 2 - 5s(5)$ and $c(G'') \geq 2 - 11s(5) > 10s(5)$, thus assume that each such face f is closed 2-cell. If γ is R5, then $\mathrm{el}(f_1) = 2$ and $c(f_1) \geq s(6) - 2s(5)$. Similarly if γ is R4, then by the assumptions v_2 has degree at least 4 in G_{f_1}, hence $c(f_1) \geq s(7) - 6s(5)$. In both cases we get $c(G'') \geq 10s(5)$. If γ is R7, Σ is a disk and $|\mathcal{R}| = 1$, then f_2 is not omnipresent, and hence $c(G'') \geq s(7) - 2s(5) \geq 10s(5)$. Otherwise, $c(G'') \geq 3s(5)$.

Therefore, we may assume that $f_1 = f_3$ and f_1 has elasticity 5. If Σ were a disk and $|\mathcal{R}| = 1$, or if f_1 or f_2 were omnipresent, then w together with a vertex of the new edge would form a 2-cut in G'', contradicting (I6). We conclude that both f_1 and f_2 are closed 2-cell and that either Σ is not a disk or $|\mathcal{R}| \neq 1$; hence, it suffices to show that $c(G'')$ is non-negative.

Suppose that γ is R4. Since γ weakly appears in G, we have that no cycle of length at most 4 touches γ, and thus $z \neq v_2$. The fact that v_2 has degree at least four in G_{f_1} implies that $c(f_1) \geq 5s(5)$, unless G_{f_1} consists of a 5-cycle and a $|f_1|$-cycle. In that case z would be a vertex of degree two, and by (I0) it would form a vertex ring. However, then f_1 could not be closed 2-cell, since z would be either an isolated vertex or a vertex of degree one forming a part of the boundary of f_1. This is a contradiction, thus $c(G'') \geq 0$.

Assume next that γ is R5. By (I1) and (I2) we have that G_{f_1} is not an exceptional graph (considering the cycle formed by the path $v_1v_8v_7v_6v_5$ together with a path in G_{f_1}), thus again $c(f_1) \geq 5s(5)$ and $c(G'') \geq 0$.

Finally, let γ be R7. If $|S_{f_1}| \geq 2$, then $c(f_1) \geq -5s(5)$. Otherwise, note that z is not a vertex ring, thus by (I0), it is incident with an edge lying inside the face of S_{f_1}. Since γ appears strongly in G, we have that v_2 is not adjacent to z, and v_2 and z have no common neighbor distinct from v_1, v_3, x_6 and x_7. It follows that G_{f_1} does not satisfy (E1), (E2) or (E3), and thus $c(f_1) \geq -5s(5)$. Therefore, $c(G'') \geq s(7) - 5s(5) > 0$.

Therefore, both inequalities from the statement of the lemma hold. Furthermore, note that in all the cases, at least one face f'' of G'' has positive contribution; and if f'' is closed 2-cell, then $|f''| \geq 6$. □

7.7 Plane graphs with one ring

Before we turn our attention to plane graphs with one ring, let us show several properties of critical graphs.

Lemma 70. *Let G be a graph in a surface Σ with rings $\mathcal{R}$. If G is $\mathcal{R}$-critical, then it satisfies* (I0), (I1) *and* (I2).

Proof. If G contains an internal vertex v of degree at most two, then let $G' = G - v$. If G contains a cycle C consisting of internal vertices of degree three that has even length or two vertices of C have adjacent neighbors, then let $G' = G - V(C)$. For any precoloring ψ of $\mathcal{R}$ that extends to a 3-coloring ϕ of G', observe that ϕ can be extended to a 3-coloring of G. This contradicts the assumption that G is $\mathcal{R}$-critical. □

By Grötzsch's theorem, no component of a critical graph is a triangle-free planar graph. This observation can be strengthened as follows.

Lemma 71. *Let G be a graph in a surface Σ with rings $\mathcal{R}$. Suppose that each component of G is a planar graph containing exactly one of the rings. If G is $\mathcal{R}$-critical and contains no non-ring triangle, then each component of G is 2-connected and G satisfies (I6).*

Proof. We can consider each component of G separately, thus assume that Σ is the sphere and G has only one ring R. Firstly, observe that G is 2-connected; otherwise, it contains proper subgraphs G_1 and G_2 such that $G = G_1 \cup G_2$ and $|V(G_1) \cap V(G_2)| \leq 1$. Since R is 2-connected, we can assume that $R \subseteq G_1$. However, G_2 is 3-colorable, and since we can permute the colors arbitrarily, any precoloring of the common vertex of G_1 and G_2 extends to a 3-coloring of G_2. It follows that any 3-coloring of G_1 extends to a 3-coloring of G, contrary to the criticality of G.

Suppose now that G has an internal 2-cut, i.e., there exist proper induced subgraphs G_1 and G_2 of G such that $G = G_1 \cup G_2$, $V(G_1) \cap V(G_2) = \{u, v\}$ for some vertices $u, v \in V(G)$, and $R \subseteq G_1$. Since G is 2-connected and planar, both u and v are incident with the same face of an embedding of G_2 in the plane. If u and v are adjacent, then we argue as in the previous paragraph that every precoloring of u and v by distinct colors extends to a 3-coloring of G_2, contrary to the criticality of G. If u and v are not adjacent, then let G_2' be the graph obtained from G_2 by adding vertices z_1 and z_2 and edges of paths uz_1v and uz_2v. The resulting graph is triangle-free, and by [43], every precoloring of the cycle uz_1vz_2 by three colors extends to a 3-coloring of G_2'; hence, every precoloring of u and v extends to a 3-coloring of G_2. Again, this contradicts the criticality of G. □

Let G be a graph in a surface Σ with rings $\mathcal{R}$. If Σ is the sphere and $\mathcal{R} = \{R\}$, then we say that G is a plane graph with one ring R (we can imagine G drawn in plane, with R bounding the infinite face of G), and we abbreviate $\{R\}$-critical to *R-critical*. Such graphs are very important for the study of critical graphs on surfaces, for the following reason:

Lemma 72. *Let G be a graph in a surface Σ with rings $\mathcal{R}$, and assume that G is $\mathcal{R}$-critical. Let C be a closed walk in G bounding an open disk $\Delta \subseteq \hat{\Sigma}$ such that Δ is disjoint from the rings, and let G' be the graph consisting of the vertices and edges of G drawn in Δ, together with a cycle C' bounding Δ (corresponding to C). Then G' may be regarded as a plane graph with one ring C', and as such it is C'-critical.*

Proof. If G' is not C'-critical, then let $e \in E(G') \setminus E(C')$ be an edge such that every precoloring of C' that extends to $G' - e$ also extends to G. Observe that every precoloring of $\mathcal{R}$ that extends to $G - e$ also extends to G, contrary to the assumption that G is $\mathcal{R}$-critical. $\square$

Let us recall that critical planar graphs with ring of length at most twelve were described by Thomassen [71], see Theorem 41 In this section, we generalize this result by giving bounds on the weight of planar critical graphs with one ring.

Theorem 73. *Let $\epsilon \le 1/1278$ be a fixed real number and let $s : \{5, 6, \ldots\} \to \mathbb{R}$ be an increasing convex function satisfying conditions (S0)–(S3) formulated in Sections 7.4 and 7.6. Let G be a plane graph with one ring R of length $l \ge 5$ such that G is R-critical and has no cycle of length at most four, and let w be the weight function arising from s as described in Section 7.4. Then*

- *$w(G, \{R\}) \le s(l - 3) + s(5)$, and furthermore,*
- *if R does not satisfy (E1), then $w(G, \{R\}) \le s(l - 4) + 2s(5)$,*
- *if (G, R) is not very exceptional, then $w(G, \{R\}) \le s(l - 5) + 5s(5)$, and*
- *if (G, R) is not exceptional, then $w(G, \{R\}) \le s(l - 5) - 5s(5)$.*

Proof. If G satisfies $(E1)$, then $l \ge 8$ and G has a face of length a such that $a \le 7$. We can assume that the other face of G is at least as long as a, that is, $l + 2 - a \ge a$. Then, $w(G, \{R\}) = s(a) + s(l + 2 - a) \le s(l - 3) + s(5)$, where the inequality holds by convexity. If G satisfies (E2), then it is very exceptional and $w(G, \{R\}) = s(l-4)+2s(5)$. If G satisfies (E3), then it is very exceptional, $l \ge 11$ and $w(G, \{R\}) = s(l - 5) + s(5) + s(6) \le s(l - 4) + 2s(5)$, where the inequality follows from convexity. If G satisfies (E4) or (E5), then $l \ge 10$ and $w(G, \{R\}) \le s(l - 5) + 5s(5) \le s(l - 4) + 2s(5)$, where the second inequality follows from convexity and (S0). Finally, suppose that G is not exceptional. By Theorem 41, we have $l \geq 11$, thus $s(l-5)-5s(5) \leq s(l-5)+5s(5) \leq s(l-4)+2s(5)$ by convexity and (S0). Therefore, it suffices to prove that $w(G, \{R\}) \leq s(l-5) - 5s(5)$.

Suppose for a contradiction that (G, R) is not exceptional, and yet $w(G, \{R\}) > s(l-5) - 5s(5)$. By induction, assume that the theorem holds for all graphs with fewer edges than G. We may assume that R bounds the outer face of G.

(31) *There is no path of length at most two with both ends in R that is otherwise disjoint from R (i.e., G satisfies* (I4)*).*

To prove ((31)) let P be a path in G of length one or two with ends $u, v \in V(R)$, and otherwise disjoint from R. Let C_1, C_2 be the two cycles of $R \cup P$ other than R, and for $i = 1, 2$ let G_i be the subgraph of G drawn in the closed disk bounded by C_i and $l_i = |C_i|$. Note that $l_1 + l_2 = l + 2|P|$.

Since G does not satisfy (E1) and satisfies (I0), we can assume that $G_1 \neq C_1$. Hence G_1 is C_1-critical by Lemma 72. Assume for a moment that $G_2 = C_2$. If G_1 is not very exceptional, then using the minimality of G, we have $w(G, \{R\}) = w(G_1, \{C_1\}) + s(l_2) \leq s(l_1-5)+5s(5)+s(l_2) \leq s(l_1+l_2-10)+6s(5) \leq s(l-5) - 5s(5)$ by the convexity and (S0), a contradiction. Similarly, we exclude the case that P has length one and G_1 is very exceptional. Finally, if G_1 is very exceptional and $|P| = 2$, then $G \backslash V(R)$ consists of one or two adjacent vertices of degree three in G. Let $a_1 \leq a_2 \leq \ldots$ be the lengths of the internal faces of G. In the former case, since G does not satisfy (E2) and (E3), we have $a_1 \geq 6$ (and $l \geq 12$) or $a_1 = 5$ and $a_2 \geq 7$ (and $l \geq 13$), thus $w(G, \{R\}) = s(a_1) + s(a_2) + s(a_3) \leq \min(2s(6), s(5) + s(7)) + s(l-6) \leq s(l-5) - 5s(5)$, by convexity and (S0). In the latter case, since G does not satisfy (E4), we have $a_3 \geq 6$ and $l \geq 12$, thus $w(G, \{R\}) = s(a_1)+s(a_2)+s(a_3)+s(a_4) \leq 2s(5)+s(6)+s(l-6) \leq s(l-5)-5s(5)$. This is a contradiction.

Thus we may assume that $G_1 \neq C_1$ and $G_2 \neq C_2$. Therefore, G_1 is C_1-critical and G_2 is C_2-critical by Lemma 72. Furthermore, we may assume that P cannot be chosen so that $G_2 = C_2$. That implies that G_1 and G_2 are not very exceptional, and hence $w(G, \{R\}) \leq s(l_1-5) + 5s(5) + s(l_2-5) + 5s(5) \leq s(l-5) - 5s(5)$. a contradiction. This proves ((31)).

Let ϕ be a precoloring of R that does not extend to a 3-coloring of G.

(32) *G is ϕ-critical.*

On contrary, suppose that G is not ϕ-critical. Then G contains a proper ϕ-critical subgraph G'. By Lemma 71, G' is 2-connected, thus all its faces are bounded by cycles. Note that G' is not very exceptional by ((31)). Since G' has fewer edges than G, we have $w(G', \{R\}) \leq s(l-5) + 5s(5)$ by induction. For $f \in \mathcal{F}(G')$ let G_f be the subgraph of G drawn inside the closed disk corresponding to f. By Lemma 72, G_f is either equal to the face f or it is f-critical, thus by induction, the convexity of s and (S0), we have $w(G_f, \{f\}) \leq s(|f|)$. Furthermore, if G_f is not equal to f, then $w(G_f, \{f\}) \leq s(|f|-3) + s(5)$. Let f_0 be a face of G' such that G_{f_0} is not equal to f_0. Note that $|f_0| \geq 8$ by Theorem 41. We have

$$\begin{aligned}
w(G,\{R\}) &= \sum_{f\in\mathcal{F}(G)} s(|f|) = \sum_{f'\in\mathcal{F}(G')} w(G_{f'},\{f'\}) \\
&\leq s(|f_0|-3)+s(5)-s(|f_0|)+\sum_{f'\in\mathcal{F}(G')} s(|f'|) \\
&= s(|f_0|-3)+s(5)-s(|f_0|)+w(G',\{R\}) \\
&\leq s(|f_0|-3)-s(|f_0|)+s(l-5)+6s(5) \leq s(l-5)-5s(5),
\end{aligned}$$

where the last inequality holds by convexity and (S0).

(33) *The graph G does not have two adjacent vertices of degree two (i.e., G satisfies* (I5)*). Furthermore, every vertex of degree two is incident with a face of length at most six.*

To prove ((33)) let u and v be two adjacent vertices of degree two in R. Let G' and R' be the graphs obtained from G and R, respectively, by identifying u and v into a single vertex w. Let ϕ' be a 3-coloring of R' matching ϕ on $R'-w$. Note that G' is ϕ'-critical, and by ((31)), G' has no cycle of length at most four. Let d be the length of the internal face f of G incident with the edge uv. By ((31)), if G' is exceptional, then it satisfies (E5), hence G has four faces of length five, a 6-face and a face of length $l-6$ and $w(G,\{R\}) = s(l-6)+s(6)+4s(5) \leq s(l-5)-5s(5)$. Therefore, assume that G' is not exceptional. By the minimality of G we have $w(G',\{R'\}) \leq s(l-6)-5s(5)$, and since the face corresponding to f contributes $s(d-1)$ to $w(G',\{R'\})$, we conclude that $d-1 < l-6$. Thus $w(G,\{R\}) = w(G',\{R'\})-s(d-1)+s(d) \leq s(l-6)-5s(5)-s(d-1)+s(d) \leq s(l-5)-5s(5)$ by convexity. The case that a vertex v of degree two is incident with a face of length at least 7 is handled similarly, with G' obtained either by suppressing v or by identifying its neighbors, depending on whether the colors of these neighbors according to ϕ differ or not. This proves ((33)).

(34) *Some good configuration appears in G.*

To prove ((34)) suppose for a contradiction that no good configuration appears in G. By Lemma 70 the graph satisfies (I0), (I1) and (I2). By Lemma 71, the graph G satisfies (I3) and (I6). By ((31)) and ((33)) it satisfies (I4) and (I5). The assumptions (I7) and (I8) are trivially satisfied by planar graphs with only one ring. Let M be the null graph. We deduce from Lemma 62 that $w(G,\mathcal{R}) \leq 4n_2/3+52\epsilon n_3-8$. By (I5) we have $n_2 \leq l/2$, thus $4n_2/3+52\epsilon n_3 \leq (2/3+26\epsilon)l$. If $l \geq 16$, then

$$w(G,\mathcal{R}) \leq (2/3+26\epsilon)l-8 \leq l-13-10\epsilon = s(l-5)-5s(5)$$

because $\epsilon \leq 1/1278$, a contradiction. Thus we may assume that $l \leq 15$, and hence $n_2 \leq 7$. If $l = 15$, then $w(G,\mathcal{R}) \leq 28/3+8\cdot 52\epsilon-8 \leq l-13-10\epsilon = s(l-5)-5s(5)$, again a contradiction. If $l = 13$, then we $n_2 \le 6$ and $w(G, \mathcal{R}) \le 7 \cdot 52\epsilon \le s(8) - 5s(5)$.

Suppose that $l = 12$, $n_2 = 6$ and $n_3 = 6$. By Theorem 41(c), all internal faces sharing an edge with R have length 5, thus the internal vertices that have a neighbor in R form a 6-cycle K. By Lemma 72 and Theorem 41, we have that K bounds a face, thus all its vertices have degree three. This contradicts (I1). It follows that if $l = 12$ and $n_2 = 6$, then $n_3 \le 5$; thus $w(G, \mathcal{R}) \le 260\epsilon \le s(7) - 5s(5)$ by (S0).

Thus we may assume that $l = 14$. If $n_2 \le 6$, then we have $w(G, \mathcal{R}) \le 8 \cdot 52\epsilon \le s(9) - 5s(5)$, hence $n_2 = 7$. Furthermore, using Lemma 63, we conclude that $b = 0$, where b is as in that lemma. Then vertices of degree two and three alternate on R, and every internal face that shares an edge with R has length five. The neighbors of the vertices of R of degree three form a 7-cycle, which bounds a face by Theorem 41. Then, $w(G, \{R\}) = s(7) + 7s(5) \le s(9) - 5s(5)$. This proves ((34)).

(35) *The graph G is well-behaved.*

To prove ((35)), assume to the contrary that G is not well-behaved. Thus there exists a path P of length at most four, with ends $u, v \in V(R)$ and otherwise disjoint from R, that is not allowable. We may assume that P is such a path of the shortest possible length. By ((31)), the path P has length at least three.

Let C_1, C_2, R be the three cycles of $R \cup P$, and for $i = 1, 2$ let G_i be the subgraph of G consisting of all vertices and edges drawn in the closed disk bounded by C_i. We claim that C_1 and C_2 are induced cycles. To prove this claim suppose to the contrary that some edge has ends $x, y \in C_i$ for some $i \in \{1, 2\}$, but that the edge itself does not belong to C_i. Then one of x, y, say x, belongs to the interior of P, and y does not belong to P. By ((31)), the vertex x is not a neighbor of u or v, and hence P has length four, and x is the middle vertex of P. Let the vertices of P be u, u', x, v', v, in order. Since P was chosen minimal, the two paths $uu'xy$ and $vv'xy$ are allowable, hence G_i consists of two 5-faces and the path P is allowable, a contradiction. This proves that C_1 and C_2 are induced cycles.

It follows from ((31)) and ((33)) that G_1 and G_2 are not very exceptional and that $G_i \ne C_i$. By Lemma 72 the graph G_i is C_i-critical for $i = 1, 2$. Let $l_i = |C_i|$. By induction we have

$$\begin{aligned} w(G, \{R\}) &= w(G_1, \{C_1\}) + w(G_2, \{C_2\}) \\ &\le s(l_1 - 5) + 5s(5) + s(l_2 - 5) + 5s(5) \\ &\le s(l_1 + l_2 - 15) + 11s(5) \le s(l - 5) - 5s(5), \end{aligned}$$

by convexity and (S0). This proves ((35)).

It follows from ((34)), ((35)) and Lemma 64 that some good configuration strongly appears in G, for if the second outcome of Lemma 64 holds, then (G, R)

either is exceptional or satisfies the conclusion of the theorem. Let γ be a good configuration that strongly appears in G, and let G' be the γ-reduction of G. By Lemma 52 the 3-coloring ϕ does not extend to a 3-coloring of G'. Thus G' has a ϕ-critical subgraph G''. By Lemma 65 the graph G'' has no cycles of length at most four (G satisfies (I9) by Lemma 72 and Theorem 41). By Lemma 71, the graph G'' satisfies (I3) and (I6). Since G is R-critical, G'' is not a subgraph of G; hence G'' contains a new vertex or edge.

(36) *Let f'' be a face of G'' and let $G_{f''}$ be the G-expansion of $S_{f''}$ as defined in Section 7.6. Let $C^1_{f''}$, $C^2_{f''}$, …, $C^{k_{f''}}_{f''}$ be the rings of $G_{f''}$ (corresponding to $J_{f''}$), let $G^1_{f''}$, $G^2_{f''}$, …, $G^{k_{f''}}_{f''}$ be the components of $G_{f''}$ such that $C^i_{f''} \subseteq G^i_{f''}$ and let $c(f'')$ be the contribution of f''. Then*

$$\sum_{i=1}^{k_{f''}} w(G^i_{f''}, \{C^i_{f''}\}) \leq s(|f''|) - c(f'').$$

Note that by Lemma 72, we have that either $G^i_{f''} = C^i_{f''}$ or $G^i_{f''}$ is $C^i_{f''}$-critical for each i. To prove ((36)), let us discuss the possible cases in the definition of the contribution of a face:

- If $G_{f''}$ satisfies (E0), then by Lemma 69 we have $c(f'') \neq -\infty$, hence f'' has zero elasticity, $c(f'') = 0$ and $w(G^1_{f''}, \{C^1_{f''}\}) = s(|f''|)$.
- If $G_{f''}$ satisfies (E1), then similarly we have $\mathrm{el}(f'') < 5$ and $c(f'') = s(8 - \mathrm{el}(f'')) - 2s(5)$. Note that by Lemma 66 we have $\mathrm{el}(f'') \leq 3$. By induction, $w(G^1_{f''}, \{C^1_{f''}\}) \leq s(|C^1_{f''}| - 3) + s(5) = s(|f''| + \mathrm{el}(f'') - 3) + s(5)$, and $s(|f''| + \mathrm{el}(f'') - 3) + s(5) \leq s(|f''|) - s(8 - \mathrm{el}(f'')) + 2s(5)$ by convexity.
- If $G_{f''}$ satisfies (E2), then $\mathrm{el}(f) \leq 3$, $c(f'') = s(9 - \mathrm{el}(f'')) - 3s(5)$, and $w(G^1_{f''}, \{C^1_{f''}\}) = s(|C^1_{f''}| - 4) + 2s(5) = s(|f''| + \mathrm{el}(f'') - 4) + 2s(5) \leq s(|f''|) - c(f'')$ by convexity.
- If $G_{f''}$ satisfies (E3), then $c(f'') = s(11 - \mathrm{el}(f'')) - 2s(6) - s(5)$ and $w(G^1_{f''}, \{C^1_{f''}\}) = s(|C^1_{f''}| - 5) + s(5) + s(6) = s(|f''| + \mathrm{el}(f'') - 5) + s(5) + s(6) \leq s(|f''|) - c(f'')$.
- If $G_{f''}$ satisfies (E4) or (E5), then $c(f'') = s(10 - \mathrm{el}(f'')) - 6s(5)$ and $w(G^1_{f''}, \{C^1_{f''}\}) \leq s(|C^1_{f''}| - 5) + 5s(5) = s(|f''| + \mathrm{el}(f'') - 5) + 5s(5) \leq s(|f''|) - c(f'')$.
- Suppose that $k_{f''} = 2$, $G^1_{f''} = C^1_{f''}$ and $G^2_{f''} = C^2_{f''}$, where $|C^1_{f''}| \leq |C^2_{f''}|$. If $|C^1_{f''}| = 5$, then $c(f'') = s(10 - \mathrm{el}(f'')) - 6s(5)$ and $w(G^1_{f''}, \{C^1_{f''}\}) + w(G^2_{f''}, \{C^2_{f''}\}) = s(|C^2_{f''}|) + s(5) = s(|f''| + \mathrm{el}(f'') - 5) + s(5) < s(|f''|) - c(f'')$. Otherwise, $c(f'') = s(12 - \mathrm{el}(f'')) - 2s(6)$ and $w(G^1_{f''}, \{C^1_{f''}\}) + w(G^2_{f''}, \{C^2_{f''}\}) = s(|C^1_{f''}|) + s(|C^2_{f''}|) \leq s(6) + s(|f''| + \mathrm{el}(f'') - 6) \leq s(|f''|) - c(f'')$.

- Suppose that $k_{f''} = 1$ and $G_{f''}$ is not exceptional.
  - Let us consider the case that $G^1_{f''}$ contains a path $P = v_1v_2v_3v_4$ such that $v_1, v_4 \in V(C^1_{f''})$, $v_2, v_3 \notin V(C^1_{f''})$ and both of the open disks Δ_1 and Δ_2 bounded by P and paths in $C^1_{f''}$ contain at least two vertices of G. In this case, $c(f'') = s(7)$. Let H_i be the subgraph of $G^1_{f''}$ drawn in Δ_i and K_i the cycle bounding Δ_i, for $i \in \{1, 2\}$. Neither H_1 nor H_2 is very exceptional, thus we have $w(G^1_{f''}, \{C^1_{f''}\}) = w(H_1, K_1) + w(H_2, K_2) \leq s(|K_1| - 5) + s(|K_2| - 5) + 10s(5) \leq s(|K_1| + |K_2| - 15) + 11s(5) = s(|f''| + \mathrm{el}(f'') - 9) + 11s(5) < s(|f''|) - s(7)$, since $\mathrm{el}(f'') \leq 5$ and $|f''| + \mathrm{el}(f'') \geq |K_1| + |K_2| - 6 \geq 14$.
  - Otherwise, $c(f'') = s(11 - \mathrm{el}(f'')) - s(6) + 5s(5)$. In this case, we have $w(G^1_{f''}, \{C^1_{f''}\}) \leq s(|C^1_{f''}| - 5) - 5s(5) = s(|f''| + \mathrm{el}(f'') - 5) - 5s(5) \leq s(|f''|) - c(f'')$.
- If $k_{f''} = 2$ and $G^1_{f''} \neq C^1_{f''}$, then $c(f'') = s(12 - \mathrm{el}(f'')) - 2s(6)$ and $w(G^1_{f''}, \{C^1_{f''}\}) + w(G^2_{f''}, \{C^2_{f''}\}) \leq s(|C^1_{f''}| - 3) + s(5) + s(|C^2_{f''}|) \leq s(|f''| + \mathrm{el}(f'') - 8) + 2s(5) < s(|f''|) - c(f'')$
- If $k_{f''} \geq 3$, then $c(f'') = s(12 - \mathrm{el}(f'')) - 2s(6)$ and $\sum_{i=1}^{k_{f''}} w(G^i_{f''}, \{C^i_{f''}\}) \leq s(|f''| + \mathrm{el}(f'') - (k_{f''} - 1)5) + (k_{f''} - 1)s(5) < s(|f''|) - c(f'')$.

Therefore, in all the cases, ((36)) holds.

By Lemma 67, we have $w(G, \{R\}) \leq \delta + \sum_{f'' \in \mathcal{F}(G'')} \sum_{i=1}^{k_{f''}} w(G^i_{f''}, \{C^i_{f''}\})$, where $\delta = s(6)$ if γ is isomorphic to R3 and $\gamma = 0$ otherwise, and by ((36)) this implies that

$$\begin{aligned} w(G, \{R\}) &\leq \delta + \sum_{f'' \in \mathcal{F}(G'')} (s(|f''|) - c(f'')) \\ &= w(G'', \{R\}) + \delta - \sum_{f'' \in \mathcal{F}(G'')} c(f'') \\ &= w(G'', \{R\}) - c(G''). \end{aligned}$$

By Lemma 68, G'' is not very exceptional, hence $w(G'', \{R\}) \leq s(l - 5) + 5s(5)$ by induction. Note that $c(G'') \geq 10s(5)$ by Lemma 69, thus

$$w(G, \{R\}) \leq w(G'', \{R\}) - c(G'') \leq s(l - 5) - 5s(5),$$

which is a contradiction finishing the proof of the theorem. □

As a straightforward corollary, we obtain the following.

Theorem 74. *Let G be a graph of girth at least 5 drawn in plane and let C be a cycle in G. Suppose that there exists a precoloring ϕ of C by three colors that does not extend to a proper 3-coloring of G. Then there exists a subgraph $H \subseteq G$ such that $C \subseteq H$, $|V(H)| \leq 2501|C|$ and H has no proper 3-coloring extending ϕ.*

Proof. First, we choose values of $\epsilon = s(5)/2 \leq 1/1278$, $s(6)$, $s(7)$ and $s(8)$ so that (S0)–(S3) hold and s (with $s(l) = l - 8$ for $l \geq 9$) is an increasing convex function. A possible choice is $s(8) = 1/2$, $s(7) = 1/6$, $s(6) = 1/75$, $s(5) = 1/1500$ and $\epsilon = 1/3000$. Let G be a plane graph of girth at least five with a cycle C and ϕ a precoloring of C that does not extend to a 3-coloring of G. We can assume that G is ϕ-critical, and thus C is a face of G. By Theorem 73, we have $w(G, \{C\}) < w(|C|) < |C|$. Note that $3|V(G)| - 2|C| = \sum_f |f| \leq \sum_f 5s(|f|)/s(5) = 5w(G, \{C\})/s(5)$, where the sum goes over the faces of G distinct from C. Therefore, $|V(G)| \leq (5/s(5)+2)|C|/3$, and since $(5/s(5)+2)/3 < 2501$, Theorem 74 holds. □

7.8 Summary

In this section, we provide a summary of the results obtained so far, to simplify their usage in the rest of the chapter. Let Π be a surface with boundary and c a simple curve intersecting the boundary of Π exactly in its ends. The closed topological space obtained from Π by cutting along c (i.e., removing c and adding two new pieces of boundary corresponding to c) is a union of at most two surfaces. If $\Pi_1, \ldots, \Pi_k$ are obtained from Π by repeating this construction, we say that they are *fragments* of Π. Consider a graph H embedded in Π with rings $\mathcal{Q}$, and let f be an internal face of H. Let us recall that f is an open subset of $\hat{\Pi}$. For each facial walk t of f, we perform the following: if t consists only of a vertex ring incident with the cuff C, then we remove $\hat{C}$ from f. Otherwise, we add a simple closed curve tracing t (if an edge appears twice in t, then it will correspond to two disjoint parts of the curve). We define Π_f to be the resulting surface. Note that the cuffs of Π_f correspond to the facial walks of f.

Theorem 75. *Let G be a well-behaved graph embedded in a surface Σ with rings $\mathcal{R}$ satisfying* (I0)–(I9) *and let M be a subgraph of G that captures (≤ 4)-cycles. Let $\ell(\mathcal{R})$ be the sum of the lengths of the rings in $\mathcal{R}$ and g the genus of Σ, and assume that either $g > 0$ or $|\mathcal{R}| > 1$. Let ϵ be a real number satisfying $0 < \epsilon \leq 1/1278$, let $s : \{5, 6, \ldots\} \to \mathbb{R}$ be a function satisfying* (S0)–(S3)*, and suppose that $w(G, \mathcal{R}) > 8g + 8|\mathcal{R}| + (2/3 + 26\epsilon)\ell(\mathcal{R}) + 20|E(M)|/3 - 16$. If G is $\mathcal{R}$-critical, then there exists an $\mathcal{R}$-critical graph G' embedded in Σ with rings $\mathcal{R}$ such that $|E(G')| < |E(G)|$ and the following conditions hold.*

1. *If G has girth at least five, then there exists a set $Y \subseteq V(G')$ of size at most two such that $G' - Y$ has girth at least five.*

2. If C' is a (≤ 4)-cycle in G', then C' is non-contractible and G contains a non-contractible cycle C of length at most $|C'| + 3$ such that $C \not\subseteq M$. Furthermore, all ring vertices of C' belong to C, and if r_1, r_2 and r_3 are ring vertices adjacent to mutually distinct vertices of C', then r_1, r_2 and r_3 also have mutually distinct neighbors in C.

3. G' has an internal face that either is not closed 2-cell or has length at least 6.

4. There exists a collection $\{(J_f, S_f) : f \in F(G')\}$ of subgraphs J_f of G and sets S_f of faces of J satisfying (30) such that

   (a) no S_f is equal to the union of $\mathcal{R}$, for $f \in F(G')$,

   (b) for any $f \in F(G')$, the surfaces of the G-expansion of S_f are fragments of the surface Σ_f.

   (c) For every face $h \in F(G)$ but at most one, there exists unique $f \in F(G')$ such that h is a face of the G-expansion of S_f. If there exists a face $h \in F(G)$ without a corresponding face in the G-expansions of S_f for $f \in F(G')$, then h is a 6-face, and set $\delta = s(6)$. Otherwise, set $\delta = 0$.

   (d) For $f \in F(G')$, let $el(f) = \left(\sum_{h \in S_f} |h|\right) - |f|$ and if f is closed 2-cell or omnipresent, let its contribution $c(f)$ be defined as in Section 7.6. Then $\sum_{f \in F(G')} el(f) \leq 10$ and if G' has an omnipresent face, then $\sum_{f \in F(G')} el(f) \leq 5$. Furthermore, if every internal face of G' is closed 2-cell or omnipresent and G' satisfies (I6), then $\sum_{f \in F(G')} c(f) \geq \delta$.

   (e) if $f \in F(G')$ is closed 2-cell and $G_1, \ldots, G_k$ are the components of the G-expansion of S_f, where for $1 \leq i \leq k$, G_i is embedded in the disk with one ring R_i, then $\sum_{i=1}^{k} w(G_i, \{R_i\}) \leq s(|f|) - c(f)$.

Proof. By Lemma 62, a good configuration γ appears in G and does not touch M. By Lemma 64, we can assume that γ appears strongly in G. Let ϕ be a precoloring of $\mathcal{R}$ that does not extend to a 3-coloring of G, and let G_1 be a γ-reduction of G with respect to ϕ. By Lemma 52, ϕ does not extend to a 3-coloring of G_1, and thus G_1 contains a $\mathcal{R}$-critical subgraph G'. Clearly, $|E(G')| < |E(G)$. Let us now show that G' has the required properties:

1. Every (≤ 4)-cycle in G' contains a new vertex or a new edge, and thus they can all be intersected by at most two vertices.

2. Follows from Lemma 65.

3. If (I6) is false, then G' has an omnipresent face. Otherwise, the claim holds by Lemma 69.

4. For each $f \in F(G')$, we define S_f and J_f as in Section 7.6.

   (a) This follows by the construction of S_f, since G' is not equal to the union of $\mathcal{R}$.

   (b) The construction of J_f and S_f ensures that the surfaces corresponding to the faces of S_f are constructed from Σ_f by cutting along simple curves with ends in cuffs, as described in the definition of fragments.

   (c) The claim follows from Lemma 67.

   (d) The claim follows from Lemmas 66 and 69.

   (e) This was proved as ((36)) in Section 7.7.

$\square$

7.9 Narrow cylinder

In this section, we consider the graphs embedded in the cylinder with two rings of length at most 7. We start with the case that the rings have length at most 4. The following lemma links the weights of the graph and its reduction as needed in the inductive argument.

Lemma 76. *Let G be an $\mathcal{R}$-critical graph embedded in a surface Σ with rings $\mathcal{R}$ so that every (≤ 4)-cycle is non-contractible, let G' be another $\mathcal{R}$-critical graph embedded in Σ with rings $\mathcal{R}$ and let $X \subset F(G)$ and $\{(J_f, S_f) : f \in F(G')\}$ be a cover of G by faces of G'. Let f be an open 2-cell face of G' and let $G_1, \ldots, G_k$ be the components of the G-expansion of S_f, where for $1 \leq i \leq k$, G_i is embedded in the disk with one ring R_i. In this situation, $\sum_{i=1}^k w(G_i, \{R_i\}) \leq s(|f|) + el(f)$.*

Proof. By Theorem 73 and Lemma 72,

$$\sum_{i=1}^{k} w(G_i, \{R_i\}) \leq \sum_{i=1}^{k} s(|R_i|) \leq s\left(\sum_{i=1}^{k} |R_i|\right) = s(|f| + \mathrm{el}(f) \leq s(|f|) + \mathrm{el}(f).$$

$\square$

Let cyl be a function satisfying the following for all nonnegative integers x and y:

- $\mathrm{cyl}(x, y) = \mathrm{cyl}(y, x)$
- if $x > 0$, then $\mathrm{cyl}(x, y) \geq \mathrm{cyl}(0, y) + x + 13$
- if $x, y > 1$, then $\mathrm{cyl}(x, y) \geq \mathrm{cyl}(1, x) + \mathrm{cyl}(1, y) + 19$

- for any nonnegative integer $y' < y$, we have $\mathrm{cyl}(x, y) \geq \mathrm{cyl}(x, y') + s(y - y' + 8) \geq \mathrm{cyl}(x, y') + 1$
- $\mathrm{cyl}(x, y) \geq s(x + y + 11)$
- if $x \geq 4$, then $\mathrm{cyl}(x, y) \geq 886$
- $2\mathrm{cyl}(6, 7) \leq \mathrm{cyl}(7, 7)$
- if $x \leq 4$ and $5 \leq y \leq 6$, then $\mathrm{cyl}(x, y) \geq (2/3 + 52\epsilon)(x + y) + 20(10k + 5\mathrm{cyl}(4, 4)/s(5) + 144)/3$
- if $x \leq 7$, then $\mathrm{cyl}(x, 7) \geq 3/2(x + 7) + 20(10k + 5\mathrm{cyl}(6, 6)/s(5) + 144)/3$
- if $x, y \geq 5$, then $\mathrm{cyl}(x, y) \geq \mathrm{cyl}(4, x) + \mathrm{cyl}(4, y) + \mathrm{cyl}(4, 4)$

Theorem 77. *Let G be a graph embedded in the cylinder with rings R_1 and R_2 of length at most four, where R_2 is a facial ring. Suppose that every (≤ 4)-cycle in G is non-contractible. If G is $\{R_1, R_2\}$-critical and not a broken chain, then $w(G, \{R_1, R_2\}) \leq \mathrm{cyl}(|R_1|, |R_2|)$.*

Proof. We proceed by induction, and assume that the claim holds for all graphs with fewer than $|E(G)|$ edges. By Lemma 46, we can assume that $|R_2| = 4$. By Theorem 45, G is connected. By Lemma 70, G satisfies (I0), (I1) and (I2). Furthermore, we already observed that every critical graph satisfies (I9), and (I6) and (I8) hold trivially.

If (I3) is false, then G contains a cutvertex v. Let G_1 and G_2 be the subgraphs of G intersecting in v such that $G = G_1 \cup G_2$, $R_1 \subseteq G_1$ and $R_2 \subseteq G_2$. Note that G_1 is $\{R_1, v\}$-critical. By Lemma 46, we have $|R_1| \geq 1$. Furthermore, if $|R_1| \leq 3$, then G_1 consists of R_1 and an edge between R_1 and v. If that is the case, then G_2 is $\{v, R_2\}$-critical, where v is taken as a weak vertex ring. By induction, we have $w(G_2, \{v, R_2\}) \leq \mathrm{cyl}(0, |R_2|)$. Hence, if f is the cuff face of v in G_2, then $w(G, \{R_1, R_2\}) \leq \mathrm{cyl}(0, |R_2|) + s(|f| + 2 + |R_1|) - s(|f|) \leq \mathrm{cyl}(0, |R_2|) + 2 + |R_1| \leq \mathrm{cyl}(|R_1|, |R_2|)$. Similarly, if $|R_1| = 4$ and f_1 and f_2 are the cuff faces of R_1 and R_2, respectively, incident with v, then $w(G, \{R_1, R_2\}) \leq 2\mathrm{cyl}(1, 4) + s(|f_1| + |f_2|) - s(|f_1|) - s(|f_2|) \leq 2\mathrm{cyl}(1, 4) + 8 \leq \mathrm{cyl}(4, 4)$. Therefore, we can assume that (I3) holds.

If (I5) is false, then since (I3) holds, we can assume that the two vertices r_1 and r_2 belong to a ring of length four, say to R_2. If the internal face incident with $r_1 r_2$ has length five, then $r_1 r_2$ is a part of a triangle T separating R_1 from R_2. By applying induction to the subgraph of G drawn between R_1 and T, we conclude that $w(G, \{R_1, R_2\}) \leq \mathrm{cyl}(|R_1|, 3) + s(5) < \mathrm{cyl}(|R_1|, |R_2|)$. Otherwise, we apply induction to the graph obtained by contracting the edge $r_1 r_2$, and obtain $w(G, \{R_1, R_2\}) \leq \mathrm{cyl}(|R_1|, 3) + 1 \leq \mathrm{cyl}(|R_1|, |R_2|)$. Hence, assume that (I5) holds.

Suppose now that the distance between R_1 and R_2 is at most four. If R_1 is a vertex ring, then add a triangle R_1' forming a boundary of the cuff incident with

R_1 and note that this operation does not decrease the weight of G; otherwise, let $R_1' = R_1$. Next, we use Lemma 72 to the closed walk consisting of R_1', R_2 and the shortest path between R_1 and R_2 traversed twice. By Theorem 73, we have $w(G, \{R_1, R_2\}) \le s((|R_1| + 3) + |R_2| + 8) \le \mathrm{cyl}(|R_1|, |R_2|)$. Therefore, assume that (I7) holds.

Furthermore, if P is a path of length at most four with both ends being ring vertices, then both ends belong to the same ring R. By (I5), P has length at least three. Since G is embedded in the cylinder, there exists a subpath Q of R such that $P \cup Q$ is a contractible cycle. Note that $|P \cup Q| \le |R| + 3 \le 7$, and by Theorem 41, $P \cup Q$ bounds a face. Therefore, G is well-behaved and satisfies (I4).

Let M be the subgraph of G consisting of edges incident with a (≤ 4)-cycle. Since G is not a broken chain, Lemma 47 implies that $|E(M)| \le 132$. Note that M captures (≤ 4)-cycles of G. If the assumptions of Theorem 75 are not satisfied, then $w(G, \{R_1, R_2\}) \le (2/3 + 26\epsilon)\ell(\{R_1, R_2\}) + 20|E(M)|/3 < 886 \le \mathrm{cyl}(|R_1|, |R_2|)$. Therefore, assume the contrary.

Then, there exists an $\{R_1, R_2\}$-critical graph G' embedded in the cylinder with rings R_1 and R_2 such that $|E(G')| < |E(G)|$, satisfying the conditions of Theorem 75. By (b), all (≤ 4)-cycles in G' are non-contractible. By Theorem 45, G' is connected, and thus all its faces are open 2-cell. Let $X \subset F(G)$ and $\{(J_f, S_f) : f \in F(G')\}$ be the cover of G by faces of G' as in (d). For $f \in F(G')$, let $G_1^f, \ldots, G_{k_f}^f$ be the components of the G-expansion of S_f, where for $1 \le i \le k_f$, G_i^f is embedded in the disk with one ring R_i^f. We have

$$w(G, \{R_1, R_2\}) = \sum_{f \in F(G)} w(f) = \sum_{f \in X} w(f) + \sum_{f \in F(G')} \sum_{i=1}^{k_f} w(G_i^f, \{R_i^f\}).$$

Suppose first that G' is 2-connected, and thus it satisfies (I3). By Theorem 75(c), G' has a face of length at least 6, hence G' is not a broken chain. Therefore, by induction we have $w(G', \{R_1, R_2\}) \le \mathrm{cyl}(|R_1|, |R_2|)$. Since each internal face of G' is closed 2-cell, Theorem 75(e) implies that

$$\sum_{i=1}^{k_f} w(G_i^f, \{R_i^f\}) \le s(|f|) - c(f)$$

for every $f \in F(G')$, and consequently

$$\begin{aligned}
\sum_{f \in F(G')} \sum_{i=1}^{k_f} w(G_i^f, \{R_i^f\}) &\le \sum_{f \in F(G')} s(|f|) - c(f) \\
&= w(G', \{R_1, R_2\}) - \sum_{f \in F(G')} c(f) \\
&\le w(G', \{R_1, R_2\}) - |X| s(6).
\end{aligned}$$

Putting the inequalities together, we obtain

$$\begin{aligned}
w(G, \{R_1, R_2\}) &\leq w(G', \{R_1, R_2\}) + \left(\sum_{f \in X} w(f) \right) - |X|s(6) \\
&= w(G', \{R_1, R_2\}) \leq \mathrm{cyl}(|R_1|, |R_2|).
\end{aligned}$$

Finally, let us consider the case that G' is not 2-connected, let v be a cutvertex in G' and let G_1 and G_2 be the subgraphs of G' intersecting in v such that $G' = G_1 \cup G_2$, $R_1 \subseteq G_1$ and $R_2 \subseteq G_2$. As in the analysis of the property (I3), we show that $|R_1| \geq 1$, if $|R_1| \leq 3$, then $w(G', \{R_1, R_2\}) \leq \mathrm{cyl}(0, |R_2|) + 2 + |R_1| \leq \mathrm{cyl}(|R_1|, |R_2|) - 11$, and if $|R_1| = 4$, then $w(G', \{R_1, R_2\}) \leq 2\mathrm{cyl}(1, 4) + 8 \leq \mathrm{cyl}(4, 4) - 11$. By Lemma 76, we have

$$\sum_{f \in F(G')} \sum_{i=1}^{k_f} w(G_i^f, \{R_i^f\}) \leq w(G', \{R_1, R_2\}) + \sum_{f \in F(G')} \mathrm{el}(f) \leq w(G', \{R_1, R_2\}) + 10.$$

Combining the inequalities, we have

$$\begin{aligned}
w(G, \{R_1, R_2\}) &\leq w(G', \{R_1, R_2\}) + 10 + \sum_{f \in X} w(f) \\
&\leq \mathrm{cyl}(|R_1|, |R_2|) - 1 + \sum_{f \in X} w(f) \\
&< \mathrm{cyl}(|R_1|, |R_2|).
\end{aligned}$$

□

Let us now state an auxiliary result that will also be useful in the case of general surfaces. Consider a graph embedded in a surface Σ. If K_1 and K_2 are two cycles surrounding a cuff C and Δ_1 and Δ_2 are the open disks bounded by K_1 and K_2, respectively, in $\Sigma + \hat{C}$, then we say that K_1 and K_2 are *incomparable* if $\Delta_1 \not\subseteq \Delta_2$ and $\Delta_2 \not\subseteq \Delta_1$.

Lemma 78. *Let G be a graph in a surface Σ with rings $\mathcal{R}$, such that G is $\mathcal{R}$-critical and every (≤ 4)-cycle is non-contractible. Let K_0 be a cycle in G of length at most seven surrounding a ring R, let C be the cuff incident with R and let Δ be the closed disk in $\Sigma + \hat{C}$ bounded by K_0. In this situation, at most $10|K_0|$ edges of G drawn outside of Δ are incident with a (≤ 7)-cycle surrounding R that is incomparable with K_0.*

Proof. Let X be the set of edges drawn outside of Δ that are incident with (≤ 7)-cycles surrounding R and incomparable with K_0. Initially, we give each edge of X charge 1 and all the edges of K_0 charge 0. Next, we aim to move the charge from X to K_0.

For an edge $x \in X$, choose an incident (≤ 7)-cycle K surrounding R incomparable with K_0. Note that at least one edge of $E(K) \setminus E(K_0)$ is drawn in Δ. Let $K = P_1 \cup P_2$, where P_1 and P_2 are paths intersecting only in their endvertices such that P_1 is drawn in the interior of Δ. Let $K_0 = P_3 \cup P_4$, where P_3 and P_4 are paths sharing endvertices with P_1 and P_2 and the closed walk $P_1 \cup P_3$ surrounds R. For $1 \leq i \leq 4$, let m_i be the length of P_i. Since all (≤ 4)-cycles are non-contractible, we have $m_2+m_3 \geq 5$ and $m_1+m_4 \geq 5$. Since $m_1+m_2+m_3+m_4 = |K_0|+|K| \leq 14$, it follows that $m_2 + m_3 \leq 9$ and $m_1 + m_4 \leq 9$. Note that $P_2 \cup P_3$ is contractible, and consequently the symmetric difference K' of P_2 and P_3 is a cycle (otherwise, G would contain a contractible cycle of length at most four). We distribute the charge of x among the edges of $K' \cap K_0$ evenly.

Let us consider the case that K' does not bound a face of G. By Theorem 41, it follows that $|K'| \geq 8$ and at most one vertex is contained in the open disk bounded by K'. Since $m_2 + m_3 \leq 9$, we have that P_2 and P_3 are edge-disjoint and $K' = P_2 \cup P_3$. Since $m_1 \geq 1$, we have $m_2 \leq 6$ and thus $m_3 \geq 2$. Suppose first that only one edge e is drawn in the open disk bounded by K'. Let f_1 and f_2 be the faces of G in this disk, and assume that f_1 shares no edge with P_3. In this situation, we say that the charge sent from the edges of $f_1 \cap K$ to the edges of P_3 passes through e. Note that if $m_3 = 2$, then $m_1 = 1$, $m_2 = 6$, $|K'| = 8$ and $|f_1| = |f_2| = 5$, thus in this case 4 units of charge pass through e.

Suppose now that a vertex is contained in the open disk bounded by K', and let f_1, f_2 and f_3 be the faces of G in this disk. Note that $|f_1| = |f_2| = |f_3| = 5$ and $m_3 \geq 3$. For $1 \leq i < j \leq 3$, let e_{ij} be the common edge of f_i and f_j. Let us first consider the case that two of these faces, say f_2 and f_3, share an edge with P_3. We let the amount of charge of the edges of $f_1 \cap K$ proportional to $|E(f_i) \cap E(P_3)|$ pass through e_{1i}, for $i \in \{2, 3\}$. The other possible case is that only one face, say f_3, shares edges with P_3; in this situation, the charge of $f_1 \cap K$ passes through f_{13} and the charge of $f_2 \cap K$ passes through f_{23}.

Let us now analyze the final charge of the edges of K_0. Consider $e \in K_0$, let f be the face of G incident with e not drawn in Δ and let $m = |E(f) \cap E(K_0)|$. If $|f| \geq 10$, then no charge is sent to e. If $7 \leq |f| \leq 9$, then no charge passes through edges of f, and thus e has charge at most 8. Similarly, if $|f| = 6$ and $m = 1$, then no charge passes through edges of f (since $m_3 = 1$ implies $m_1 \geq 2$ and consequently $m_2 \leq 5$, hence $m_2+m_3 \leq 6$), and thus e has charge at most 5. If $|f| = 6$ and $m \geq 2$, then at most four units pass through each edge of $E(f) \setminus E(K_0)$ and this charge is evenly divided between the edges of $E(f) \cap E(K_0)$, hence e has charge at most 10. If $|f| = 5$ and $m = 1$, then at most one unit passes through each edge of $E(f) \setminus E(K_0)$, and e has charge at most 8. If $|f| = 5$ and $m \geq 2$, then at most five units of charge pass through each edge of $E(f) \setminus E(K_0)$, and e has charge at most 9.

Therefore, each edge of K_0 has charge at most 10, and $|X| \leq 10|K_0|$. $\square$

We will also need several related claims regarding cycles near a ring.

Lemma 79. *Let G be a graph embedded in the cylinder with rings R_1 and R_2, such that G is $\{R_1, R_2\}$-critical and every (≤ 4)-cycle is non-contractible. If $|R_1| = 4$, then at most 93 edges of G are incident with a non-contractible (≤ 7)-cycle that shares a vertex with R_1.*

Proof. Let C_1 be the cuff incident with R_1. Let K_0 be a non-contractible (≤ 7)-cycle that shares a vertex with R_1 such that the closed disk Δ bounded by K_0 in $\Sigma + \hat{C}$ is as large as possible. Observe that every edge of G incident with a non-contractible (≤ 7)-cycle that shares a vertex with R_1 is either drawn in Δ or is incident with a non-contractible (≤ 7)-cycle that is incomparable with K_0. Theorem 41 implies that at most 23 edges of G are drawn in Δ. Together with Lemma 78, the claim follows. □

If R is a 6-cycle and C is a (≤ 6)-cycle, we say that C is *bound* to R if either $|V(C) \cap V(R)| \geq 3$ or three internal vertices of C have distinct non-adjacent neighbors in R.

Lemma 80. *Let G be a graph embedded in the cylinder with rings R_1 and R_2, such that every (≤ 4)-cycle is non-contractible. Suppose that $|R_1| = 6$ and R_1 is an induced cycle, and let $X \subseteq E(G)$ be the set of edges incident with (≤ 6)-cycles bound to R_1. If G is $\{R_1, R_2\}$-critical, then $|X| \leq 51$.*

Proof. Let C be a non-contractible (≤ 6)-cycle in G that is bound to R_1. If $|V(C) \cap V(R_1)| \geq 3$, then for every edge $e \in E(C) \setminus E(R_1)$, there exists a contractible cycle K in $C \cup R$ of length at most 8 that contains e, sharing at least $|K| - 4$ edges with R_1. By Theorem 41, K either bounds a face or a disk consisting of two 5-faces. Since R is an induced cycle, observe that in the latter case, both 5-faces share an edge with R. Similarly, if three internal vertices of C have distinct non-adjacent neighbors in R_1, then observe that every edge $e \in E(C) \setminus E(R_1)$ is contained in a contractible cycle K of length at most 8 that shares two edges with R.

Assign the edges of $X \setminus E(R_1)$ to the edges of R_1 in the following way: any (≤ 8)-face K that shares edges with R_1 divides its edges between the edges of $E(K) \cap E(R_1)$ evenly. If K is a 5-face that shares two edges with R_1, we additionally assign edges of all other 5-faces sharing an edge with K to the edges of $E(K) \cap E(R_1)$, divided evenly. Note that at most $15/2$ edges of $X \setminus E(R_1)$ are assigned to each edge of R_1, and thus $|X| \leq 51$. □

Similarly, we can prove the following.

Lemma 81. *Let G be a graph embedded in the cylinder with rings R_1 and R_2, such that every (≤ 4)-cycle is non-contractible. Suppose that $|R_1| = 7$ and R_1 is an induced cycle, and let $X \subseteq E(G)$ be the set of edges incident with 7-cycles that share at least four vertices with R_1. If G is $\{R_1, R_2\}$-critical, then $|X| \leq 35$.*

The first paper of this series [25] together with the results of Gimbel and Thomassen [41] and Aksenov et al. [1] implies implies the following.

Theorem 82. *Let G be a graph embedded in the cylinder with one ring R of length at most 7 (the other cuff does not correspond to a ring). Suppose that all (≤ 4)-cycles in G are non-contractible and that G has girth at least $|R|-3$. If G is R-critical and R is an induced cycle, then $|R|=6$ and G contains a triangle C such that all vertices of C are internal and have mutually distinct and non-adjacent neighbors in R.*

Finally, we can prove the main result of this section.

Lemma 83. *Let G be a graph embedded in the cylinder with rings R_1 and R_2, where $|R_1| \leq |R_2|$ and $5 \leq |R_2| \leq 7$. Suppose that every (≤ 4)-cycle in G is non-contractible. Furthermore, assume that the following conditions hold:*

- *if $|R_1| = 4$, then all other 4-cycles in G are vertex-disjoint with R_1,*
- *if $|R_1| \geq 5$, then G contains no (≤ 4)-cycle,*
- *if $|R_2| = 7$, then G contains no triangle distinct from R_1 and R_2 is an induced cycle, and*
- *if $|R_i| = 6$ for some $i \in \{1,2\}$, then R_i is an induced cycle and G contains no triangle T such that all vertices of T are internal and they have mutually distinct and non-adjacent neighbors in R_i.*

If G is $\{R_1,R_2\}$-critical, then $w(G,\{R_1,R_2\}) \leq \mathrm{cyl}(|R_1|,|R_2|)$.

Proof. By induction, we can assume that the claim holds for all graphs with fewer than $|E(G)|$ edges. By Theorems 45 and 82, G is connected. Note that G satisfies (I0), (I1), (I2), (I6), (I8) and (I9).

If (I3) is false, then G contains a cutvertex v. Let G_1 and G_2 be the subgraphs of G intersecting in v such that $G = G_1 \cup G_2$, $R_1 \subseteq G_1$ and $R_2 \subseteq G_2$. Note that G_1 is $\{R_1, v\}$-critical. By Lemma 46, we have $|R_1| \geq 1$. Furthermore, if $|R_1| \leq 3$, then G_1 consists of R_1 and an edge between R_1 and v. If that is the case, then G_2 is $\{v, R_2\}$-critical, where v is taken as a weak vertex ring. By induction, we have $w(G,\{R_1,R_2\}) \leq \mathrm{cyl}(0,|R_2|) + s(|f|+2+|R_1|) - s(|f|) \leq \mathrm{cyl}(0,|R_2|) + 2 + |R_1| \leq \mathrm{cyl}(|R_1|,|R_2|)$. Similarly, if $|R_1| \geq 4$, then $w(G,\{R_1,R_2\}) \leq \mathrm{cyl}(1,|R_1|) + \mathrm{cyl}(1,|R_2|) + 8 \leq \mathrm{cyl}(|R_1|,|R_2|)$. Therefore, we can assume that (I3) holds.

If the distance between R_1 and R_2 is at most four, then we use Lemma 72 to the closed walk consisting of R_1, R_2 and the shortest path between R_1 and R_2 traversed twice (replacing R_1 by a triangle if R_1 is a vertex ring), and by Theorem 73, we have $w(G,\{R_1,R_2\}) \leq s((|R_1|+3)+|R_2|+8) \leq \mathrm{cyl}(|R_1|,|R_2|)$. Therefore, assume that (I7) holds. Furthermore, if P is a path of length at most four with both ends being ring vertices, then both ends belong to the same ring

R_i for some $i \in \{1, 2\}$. Since G is embedded in the cylinder, there exists a subpath Q of R_i such that $P \cup Q$ is a contractible cycle. Let Q' be the path with edge set $E(R_i) \setminus E(Q)$. If $|Q| > |P|$, then $Q' \cup P$ is a non-contractible cycle shorter than $|R_i|$, and by applying the induction to the subgraph of G between R_{3-i} and $Q' \cup P$ and Theorem 73 to the subgraph drawn in the disk bounded by $Q \cup P$, we conclude that $w(G, \{R_1, R_2\}) \leq \mathrm{cyl}(|R_{3-i}|, |Q' \cup P|) + s(|Q \cup P|) = \mathrm{cyl}(|R_{3-i}|, |Q' \cup P|) + s(|R_i| - |Q' \cup P| + 2|P|) \leq \mathrm{cyl}(|R_{3-i}|, |R_i|)$. Therefore, we may assume that $|Q| \leq |P|$ for each such path P. This implies that (I4) holds. Furthermore, $|P \cup Q| \leq 8$, and by Theorem 41, at most two faces of G are in the disk bounded by $P \cup Q$.

Suppose that (I5) is false, and the ring R_i for some $i \in \{1, 2\}$ contains adjacent vertices r_1 and r_2 of degree two. By (I3), we have $|R_i| \geq 4$, and by the previous paragraph, the internal face incident with r_1r_2 has length at least 6. We apply induction to the graph obtained by contracting the edge r_1r_2, and obtain $w(G, \{R_1, R_2\}) \leq \mathrm{cyl}(|R_1|, |R_2|-1)+1 \leq \mathrm{cyl}(|R_1|, |R_2|)$. Hence, assume that (I5) holds. Together with the observations from the previous paragraph, this implies that G is well-behaved.

If $|R_1| = |R_2| = 7$ and G contains a non-contractible (≤ 6)-cycle, then by induction we have $w(G, \{R_1, R_2\}) \leq 2\mathrm{cyl}(6, 7) \leq \mathrm{cyl}(7, 7)$, hence we can assume that if $|R_1| = |R_2| = 7$, then all non-contractible cycles have length at least seven.

Let M be the subgraph of G containing

- edges incident with non-contractible $(\leq k)$-cycles, where $k = 6$ if $|R_2| = 7$ and $k = 4$ otherwise;
- if $|R_1| = 4$, then also include edges incident with non-contractible (≤ 7)-cycles sharing a vertex with R_1,
- if $|R_i| = 6$ for some $i \in \{1, 2\}$, then include all edges of non-contractible (≤ 6)-cycles bound to R_i, and
- if $|R_i| = 7$ for some $i \in \{1, 2\}$, then include all edges of non-contractible 7-cycles that share at least four vertices with R_i.

Let us bound the number of edges of M. Note that if $|R_2| < 7$, then $k = 4$ and by the assumptions, if G contains a $(\leq k)$-cycle, then $|R_1| \leq 4$. Suppose that there exists a non-contractible $(\leq k)$-cycle C, and choose C so that the closed subset Σ' of Σ between R_1 and C is as large as possible. Note that $|R_1| \leq k < |R_2|$. If $|R_1| = k = 4$, then no 4-cycle distinct from R_1 contains a vertex of $V(R_1)$, hence the subgraph G' of G drawn in Σ' is not a broken chain. All non-contractible $(\leq k)$-cycles in G are either drawn in Σ' or are incomparable with C. By Theorems 73 and 77 and by induction, we conclude that the total weight of the internal faces of G' is at most $\max(\mathrm{cyl}(k, k), s(2k)) = \mathrm{cyl}(k, k)$. In particular, at most $5\mathrm{cyl}(k, k)/s(5)$ edges of G are drawn in Σ', and by Lemma 78, at most $10k + 5\mathrm{cyl}(k,k)/s(5)$ edges of G are incident with non-contractible $(\le k)$-cycles. By Lemmas 79, 80 and 81, we have $|E(M)| \le 10k + 5\mathrm{cyl}(k,k)/s(5) + 144$.

Note that M captures (≤ 4)-cycles of G, and that $(2/3 + 26\epsilon)(|R_1| + |R_2|) + 20(10k + 5\mathrm{cyl}(k,k)/s(5) + 144)/3 \le \mathrm{cyl}(|R_1|, |R_2|)$; therefore, we can assume that we can apply Theorem 75. Let G' be the $\{R_1, R_2\}$-critical graph embedded in the cylinder with rings R_1 and R_2 such that $|E(G')| < |E(G)|$, satisfying the conditions of Theorem 75. In particular, (b) implies that all (≤ 4)-cycles in G' are non-contractible; and furthermore, using the choice of M we have

- if $|R_2| = 7$, then G' contains no triangle distinct from R_1,
- if $|R_1| = 4$, then no 4-cycle in G' distinct from R_1 contains a vertex of $V(R_1)$,
- if $6 \le |R_i| \le 7$ for some $i \in \{1, 2\}$, then R_i is an induced cycle in G', and
- if $|R_i| = 6$ for some $i \in \{1, 2\}$, then G' contains no triangle T such that all vertices of R_i are internal and have non-adjacent neighbors in R_i.

By Theorem 82, we conclude that G' is connected, and thus all its faces are open 2-cell.

Let $X \subset F(G)$ and $\{(J_f, S_f) : f \in F(G')\}$ be the cover of G by faces of G' as in Theorem 75(d). For $f \in F(G')$, let $G^f_1, \ldots, G^f_{k_f}$ be the components of the G-expansion of S_f, where for $1 \le i \le k_f$, G^f_i is embedded in the disk with one ring R^f_i. We have

$$w(G, \{R_1, R_2\}) = \sum_{f \in F(G)} w(f) = \sum_{f \in X} w(f) + \sum_{f \in F(G')} \sum_{i=1}^{k_f} w(G^f_i, \{R^f_i\}).$$

Suppose first that G' is not 2-connected, let v be a cutvertex in G' and let G_1 and G_2 be the subgraphs of G' intersecting in v such that $G' = G_1 \cup G_2$, $R_1 \subseteq G_1$ and $R_2 \subseteq G_2$. As in the analysis of the property (I3), we show that $|R_1| \ge 1$, if $|R_1| \le 3$, then $w(G', \{R_1, R_2\}) \le \mathrm{cyl}(0, |R_2|) + 2 + |R_1| \le \mathrm{cyl}(|R_1|, |R_2|) - 11$, and otherwise $w(G', \{R_1, R_2\}) \le \mathrm{cyl}(1, |R_1|) + \mathrm{cyl}(1, |R_2|) + 8 \le \mathrm{cyl}(|R_1|, |R_2|) - 11$. Using Lemma 76, we have

$$\sum_{f \in F(G')} \sum_{i=1}^{k_f} w(G^f_i, \{R^f_i\}) \le w(G', \{R_1, R_2\}) + \sum_{f \in F(G')} \mathrm{el}(f) \le w(G', \{R_1, R_2\}) + 10.$$

Combining the inequalities, we conclude that

$$\begin{aligned} w(G, \{R_1, R_2\}) &\le w(G', \{R_1, R_2\}) + \sum_{f \in F(G')} \mathrm{el}(f) + \sum_{f \in X} w(f) \\ &\le \mathrm{cyl}(|R_1|, |R_2|) - 1 + \sum_{f \in X} w(f) \\ &< \mathrm{cyl}(|R_1|, |R_2|). \end{aligned}$$

Suppose now that G' is 2-connected. If G' does not satisfy the assumptions of Lemma 83, then $|R_1| \geq 5$ and G' contains a (≤ 4)-cycle. Let C_1 and C_2 be the (≤ 4)-cycles in G' such that the closed subset $\Sigma' \subseteq \Sigma$ between C_1 and C_2 is as large as possible, and observe that all (≤ 4)-cycles in G' belong to the subgraph G_c of G' drawn in Σ'. By Theorem 75(a), if G_c is a broken chain, then it has at most four internal faces. Therefore, Theorem 77 implies that the total weight of the internal faces of G_c is at most $\mathrm{cyl}(4,4)$. Applying induction to the subgraphs of G' between R_1 and C_1 and between R_2 and C_2, we have $w(G', \{R_1, R_2\}) \leq \mathrm{cyl}(4, |R_1|) + \mathrm{cyl}(4, |R_2|) + \mathrm{cyl}(4,4) \leq \mathrm{cyl}(|R_1|, |R_2|)$.

If G' satisfies the assumptions of Lemma 83, then the same inequality $w(G', \{R_1, R_2\}) \leq \mathrm{cyl}(|R_1|, |R_2|)$ follows by induction. Since each face of G' is closed 2-cell, we conclude that $w(G, \{R_1, R_2\}) \leq w(G', \{R_1, R_2\}) \leq \mathrm{cyl}(|R_1|, |R_2|)$ as in the proof of Theorem 77. □

7.10 Graphs on surfaces

Let $\mathrm{gen}(g, t, t_0, t_1)$ be a function defined for non-negative integers g, t, t_0 and t_1 such that $t \geq t_0 + t_1$ as

$$\mathrm{gen}(g, t, t_0, t_1) = 120g + 48t - 4t_1 - 5t_0 - 120.$$

Let $\mathrm{surf}(g, t, t_0, t_1)$ be a function defined for non-negative integers g, t, t_0 and t_1 such that $t \geq t_0 + t_1$ as

- $\mathrm{surf}(g, t, t_0, t_1) = \mathrm{gen}(g, t, t_0, t_1) + 116 - 42t = 8 - 4t_1 - 5t_0$ if $g = 0$ and $t = t_0 + t_1 = 2$,
- $\mathrm{surf}(g, t, t_0, t_1) = \mathrm{gen}(g, t, t_0, t_1) + 114 - 42t = 6t - 4t_1 - 5t_0 - 6$ if $g = 0$, $t \leq 2$ and $t_0 + t_1 < 2$, and
- $\mathrm{surf}(g, t, t_0, t_1) = \mathrm{gen}(g, t, t_0, t_1)$ otherwise.

Consider a graph H embedded in a surface Π with rings $\mathcal{Q}$, and let f be an internal face of H. Let a_0 and a_1 be the number of weak and non-weak rings, respectively, that form one of the facial walks of f by themselves. Let a be the number of facial walks of f. We define $\mathrm{surf}(f) = \mathrm{surf}(g(\Pi_f), a, a_0, a_1)$.

Let G_1 be a graph embedded in Σ_1 with rings $\mathcal{R}_1$ and G_2 a graph is embedded in Σ_2 with rings $\mathcal{R}_2$. Let $m(G_i)$ denote the number of edges of G_i that are not contained in the boundary of Σ_i. We write $(G_1, \Sigma_1, \mathcal{R}_1) \prec (G_2, \Sigma_2, \mathcal{R}_2)$ if $(g(\Sigma_1), |\mathcal{R}_1|, m(G_1), |E(G_1)|)$ is lexicographically smaller than $(g(\Sigma_2), |\mathcal{R}_2|, m(G_2), |E(G_2)|)$.

A graph G embedded in a surface Σ with rings $\mathcal{R}$ has *internal girth at least five* if every (≤ 4)-cycle in G is equal to one of the rings. Let $t_0(\mathcal{R})$ and $t_1(\mathcal{R})$ be the number of weak and non-weak vertex rings in $\mathcal{R}$, respectively. Finally, we are ready to prove Theorem 51, in the following more general setting.

Theorem 84. *There exists a constant C with the following property. Let G be a graph embedded in a surface Σ with rings $\mathcal{R}$. If G is $\mathcal{R}$-critical and has internal girth at least five, then $w(G, \mathcal{R}) \le C\mathrm{surf}(g(\Sigma), |\mathcal{R}|, t_0(\mathcal{R}), t_1(\mathcal{R})) + \ell(\mathcal{R})$.*

Proof. Let $C = 1867 + 67\mathrm{cyl}(7, 7)/s(5)$. We proceed by induction and assume that the claim holds for all graphs G' embedded in surfaces Σ' with rings $\mathcal{R}'$ such that $(G', \Sigma', \mathcal{R}') \prec (G, \Sigma, \mathcal{R})$. Let $g = g(\Sigma)$, $t_0 = t_0(\mathcal{R})$ and $t_1 = t_1(\mathcal{R})$. By Theorem 73, the claim holds if $g = 0$ and $|\mathcal{R}| = 1$, hence assume that $g > 0$ or $|\mathcal{R}| > 1$. Similarly, if $g = 0$ and $|\mathcal{R}| = 2$, then we can assume that $t_0 + t_1 \le 1$ by Lemma 46. By Lemma 70, Lemma 71 and Theorem 41, G satisfies (I0), (I1), (I2), (I6) and (I9).

Suppose now that there exists a path P of length at most four with ends in distinct rings $R_1, R_2 \in \mathcal{R}$. By choosing the shortest such path, we can assume that P intersects no other rings. If R_1 or R_2 is a vertex ring, first replace it by a facial ring of length three by adding new vertices and edges in the incident cuff. Let $J = P \cup \bigcup_{R \in \mathcal{R}} R$ and let $S = \{f\}$, where f is the face of J incident with edges of P. Let $\{(G', \Sigma')\}$ be the G-expansion of S and let $\mathcal{R}'$ be the natural rings of G'. Note that $g(\Sigma') = g$, $|\mathcal{R}'| = |\mathcal{R}| - 1$, $\ell(\mathcal{R}') \le \ell(\mathcal{R}) + 14$ and $t_0(\mathcal{R}') + t_1(\mathcal{R}') \ge t_0 + t_1 - 2$. Since $(G', \Sigma', \mathcal{R}') \prec (G, \Sigma, \mathcal{R})$, by induction we have $w(G, \mathcal{R}) = w(G', \mathcal{R}') \le C\mathrm{surf}(g, |\mathcal{R}| - 1, t_0(\mathcal{R}'), t_1(\mathcal{R}')) + \ell(\mathcal{R}) + 14 < C\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) + \ell(\mathcal{R})$. Therefore, we can assume that no such path exists, and in particular, (I7) holds.

Next, we aim to prove property (I3). For later use, we will consider a more general setting.

(37) *Let H be a graph embedded in Π with rings $\mathcal{Q}$ such that at least one internal face of H is not open 2-cell and no face of H is omnipresent. If H is $\mathcal{Q}$-critical, has internal girth at least five and $(H, \Pi, \mathcal{Q}) \preceq (G, \Sigma, \mathcal{R})$, then*

$$w(H, \mathcal{Q}) \le \ell(\mathcal{Q}) + C\left(\mathrm{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 7 - \sum_{h \in F(H)} \mathrm{surf}(h)\right).$$

Proof. We prove the claim by induction. Consider for a moment a graph H' of girth at least 5 embedded in a surface Π' with rings $\mathcal{Q}'$ with $(H', \Pi', \mathcal{Q}') \prec (H, \Pi, \mathcal{Q})$, such that H' is $\mathcal{Q}'$-critical. We claim that

$$w(H', \mathcal{Q}') \le \ell(\mathcal{Q}') + C\left(\mathrm{surf}(g(\Pi'), |\mathcal{Q}'|, t_0(\mathcal{Q}'), t_1(\mathcal{Q}')) - \sum_{h \in F(H')} \mathrm{surf}(h)\right). \tag{7.1}$$

If at least one internal face of H' is not open 2-cell and no face of H' is omnipresent, then this follows by induction (we could even strengthen the inequality by $3C$). If all internal faces of H' are open 2-cell, then note that $\mathrm{surf}(h) = 0$ for every $h \in H'$, and since $(H', \Pi', \mathcal{Q}') \prec (G, \Sigma, \mathcal{R})$, we can apply Theorem 84 to obtain (7.1). Finally, suppose that H' has an omnipresent face f, let $\mathcal{Q}' = \{Q_1, \ldots, Q_t\}$ and for $1 \leq i \leq t$, let f_i be the boundary walk of f such that f_i and Q_i are contained in a closed disk $\Delta_i \subset \Pi' + \hat{Q}_i$. Since all components of H' are planar and contain only one ring, Lemma 71 implies that all internal faces of H' distinct from f are closed 2-cell. Furthermore, each vertex ring forms component of the boundary of f, hence $\mathrm{surf}(f) = \mathrm{surf}(g(\Pi'), |\mathcal{Q}'|, t_0(\mathcal{Q}'), t_1(\mathcal{Q}'))$. If Q_i is a facial ring, then by applying Theorem 73 to the subgraph of H' drawn in Δ_i, we conclude that its weight is at most $s(|Q_i)|)$ and that $|f_i| \leq |Q_i|$. Note that $s(|Q_i|) - s(|f_i|) \leq |Q_i| - |f_i|$ by (S0). Therefore, we again obtain (7.1):

$$\begin{aligned} w(H', \mathcal{Q}') &\leq |f| + \sum_{i=1}^{t} s(|Q_i|) - s(|f_i|) \\ &\leq \ell(\mathcal{Q}') \\ &= \ell(\mathcal{Q}') + C\left(\mathrm{surf}(g(\Pi'), |\mathcal{Q}'|, t_0(\mathcal{Q}'), t_1(\mathcal{Q}')) - \sum_{h \in F(H')} \mathrm{surf}(h)\right). \end{aligned}$$

Let us now return to the graph H. Since H is $\mathcal{Q}$-critical, no component of H is a planar graph without rings. Let f be a face of H which is not open 2-cell. Since H has such a face and f is not omnipresent, we have $g(\Pi) > 0$ or $|\mathcal{Q}| > 2$. Let c be a simple closed curve in f infinitesimally close to a facial walk W of f. Cut Π along c and cap the resulting holes by disks (c is always a 2-sided curve). Let Π_1 be the connected surface obtained this way that contains W, and if c is separating, then let Π_2 be the other surface. Since f is not omnipresent, we can choose W so that either $g(\Pi_1) > 0$ or Π_1 contains at least two rings of $\mathcal{Q}$. Let us discuss several cases:

- *c is separating and H is contained in* Π_1. In this case f has only one facial walk, and since f is not open 2-cell, Π_2 is not the sphere. It follows that $g(\Pi_1) = g(\Pi) - g(\Pi_2) < g(\Pi)$, and thus $(H, \Pi_1, \mathcal{Q}) \prec (H, \Pi, \mathcal{Q})$. Note that the weights of the faces of the embedding of H in Π and in Π_1 are the same, with the exception of f whose weight in Π is $|f|$ and in Π_1 is $s(|f|) \geq |f| - 8$. By (7.1), we have

$$w(H, \mathcal{Q}) \leq \ell(\mathcal{Q}) + 8 + C\left(\mathrm{surf}(g(\Pi_1), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) + \mathrm{surf}(f) - \sum_{h \in F(H)} \mathrm{surf}(h)\right).$$

  Note that $\mathrm{surf}(f) = 120g(\Pi_2) - 72$. Since f is not omnipresent, we have either $|\mathcal{Q}| \geq 2$ or $g(\Pi_1) > 0$. Observe that

$$\mathrm{surf}(g(\Pi_1), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) = \mathrm{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 120g(\Pi_2) + \delta,$$

where $\delta = 0$ if $|\mathcal{Q}| \geq 3$ or $g(\Pi_1) > 0$ and $\delta \leq 116 - 42|\mathcal{Q}| = 32$ otherwise. We conclude that

$$w(H,\mathcal{Q}) \leq \ell(\mathcal{Q}) + C\left(\mathrm{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 40 - \sum_{h\in F(H)} \mathrm{surf}(h)\right).$$

- *c is separating and Π_2 contains a nonempty part H_2 of H.* Let H_1 be the part of H contained in Π_1. Let $\mathcal{Q}_i$ be the subset of $\mathcal{Q}$ belonging to Π_i and f_i the face of H_i corresponding to f, for $i \in \{1,2\}$. Note that f_1 is an open disk, hence $\mathrm{surf}(f_1) = 0$. Using (7.1), we get

$$\begin{aligned} w(H,\mathcal{Q}) \;\leq\;& w(f) - w(f_1) - w(f_2) + \ell(\mathcal{Q}_1) + \ell(\mathcal{Q}_2) + \\ & + C\sum_{i=1}^{2} \mathrm{surf}(g(\Pi_i), |\mathcal{Q}_i|, t_0(\mathcal{Q}_i), t_1(\mathcal{Q}_i)) + \\ & + C\left(\mathrm{surf}(f) - \mathrm{surf}(f_2) - \sum_{h\in F(H)} \mathrm{surf}(h)\right). \end{aligned}$$

  Note that $w(f) - w(f_1) - w(f_2) \leq 16$ and $\ell(\mathcal{Q}_1) + \ell(\mathcal{Q}_2) = \ell(\mathcal{Q})$. Also, $\mathrm{surf}(f) - \mathrm{surf}(f_2) \leq 48$, and when $g(\Pi_f) = 0$ and f has only two facial walks, then $\mathrm{surf}(f) - \mathrm{surf}(f_2) \leq 6$.

  Recall that $g(\Pi_1) > 0$ or $|\mathcal{Q}_1| \geq 2$, and that H_2 is not a planar graph without rings. For $i \in \{1,2\}$, let $\delta_i = |\mathcal{Q}_i|$ if $g(\Pi_i) = 0$ and $|\mathcal{Q}_i| \leq 2$, and let $\delta_i = 116/42$ otherwise. We have

$$\begin{aligned} \sum_{i=1}^{2} \mathrm{surf}(g(\Pi_i), |\mathcal{Q}_i|, t_0(\mathcal{Q}_i), t_1(\mathcal{Q}_i)) \;&\leq\; \sum_{i=1}^{2} \mathrm{gen}(g(\Pi_i), |\mathcal{Q}_i|, t_0(\mathcal{Q}_i), t_1(\mathcal{Q}_i)) + 116 - 42\delta_i \\ &=\; \mathrm{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) + 112 - 42\delta_1 - 42\delta_2 \\ &\leq\; \mathrm{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - \delta, \end{aligned}$$

  where $\delta = 14$ if $g(\Pi_2) = 0$ and $|\mathcal{Q}_2| = 1$ and $\delta = 56$ otherwise. Note that if $g(\Pi_2) = 0$ and $|\mathcal{Q}_2| = 1$, then $g(\Pi_f) = 0$ and f has only two facial walks. We conclude that $\mathrm{surf}(f) - \mathrm{surf}(f_2) - \delta \leq -8$. Therefore, $w(H,\mathcal{Q}) \leq \ell(\mathcal{Q}) + 16 + C\left(\mathrm{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 8 - \sum_{h\in F(H)} \mathrm{surf}(h)\right)$.

- *c is not separating.* Let f_1 be the face of H (in the embedding in Π_1) bounded by W and f_2 the other face corresponding to f. Again, note that $\mathrm{surf}(f_1) = 0$. By (7.1) applied to H embedded in Π_1, we obtain the following for the weight of H in Π:

$$w(H,\mathcal{Q}) \;\leq\; w(f) - w(f_1) - w(f_2) + \ell(\mathcal{Q}) +$$

$$+C\mathrm{surf}(g(\Pi_1), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) +$$
$$+C\left(\mathrm{surf}(f) - \mathrm{surf}(f_2) - \sum_{h\in F(H)} \mathrm{surf}(h)\right).$$

Since c is two-sided, $g(\Pi_1) = g(\Pi) - 2$, and

$$\mathrm{surf}(g(\Pi_1), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) = \mathrm{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 240 + \delta,$$

where $\delta = 0$ if $g(\Pi_1) > 0$ or $|\mathcal{Q}| > 2$ and $\delta \le 32$ otherwise. Since $\mathrm{surf}(f) - \mathrm{surf}(f_2) \le 48$ and $w(f) - w(f_1) - w(f_2) \le 16$, we have $w(H, \mathcal{Q}) \le \ell(\mathcal{Q}) + 16 + C\left(\mathrm{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 160 - \sum_{h\in F(H)} \mathrm{surf}(h)\right)$.

The results of all the subcases imply (37). □

(38) *Let H be a graph embedded in Σ with rings $\mathcal{R}$ and let f be an omnipresent face of H. If H is $\mathcal{R}$-critical, has internal girth at least five and no component of H satisfies (E1), (E2) or (E3), then*

$$w(H, \mathcal{R}) \le \ell(\mathcal{R}) - \kappa = \ell(\mathcal{R}) - \kappa + C\left(\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - \sum_{h\in F(H)} \mathrm{surf}(h)\right),$$

where $\kappa = 5 - 5s(5)$ if H has exactly one component not equal to a ring and this component is exceptional, $\kappa = 5 + 5s(5)$ if H has exactly one component not equal to a ring and this component is not exceptional, and $\kappa = 6$ otherwise.

Proof. Since H is $\mathcal{R}$-critical and f is an omnipresent face, each component of H is planar and contains exactly one ring. In particular, all internal faces of H distinct from f are closed 2-cell. For $R \in \mathcal{R}$, let H_R be the component of H containing R. Exactly one boundary walk W of f belongs to H_R. Cutting along W and capping the hole by a disk, we obtain an embedding of H_R in a disk with one ring R. Let f_R be the face of this embedding bounded by W. Note that either $H_R = R$ or H_R is $\{R\}$-critical. If R is a vertex ring, then we have $H_R = R$; hence, every vertex ring in $\mathcal{R}$ forms a facial walk of f, and $\mathrm{surf}(f) = \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1)$. Consequently, $\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) = \sum_{h\in F(H)} \mathrm{surf}(h)$, and it suffices to prove the first inequality of the claim.

Suppose that $H_R \ne R$ for a ring $R \in \mathcal{R}$. Since H_R does not satisfy (E1), (E2) or (E3), Theorem 73 implies that $w(H_R, \{R\}) \le s(|R| - 5) + \alpha$, where $\alpha = 5s(5)$ if H_R satisfies (E4) or (E5) and $\alpha = -5s(5)$ otherwise. Since f_R is a face of H_R and $s(y) - s(x) > 5s(5)$ for every $y > x \ge 5$ by (S0), we have $|f_R| \le |R| - 5$.

Furthermore, (S0) implies $w(H_R, \{R\}) - w(f_R) \leq s(|R) - 5) + \alpha - s(|f_R|) \leq |R| - |f_R| - 5 + \alpha$. Summing over all the rings, we obtain

$$\begin{aligned} w(H, \mathcal{R}) &= w(f) + \sum_{R \in \mathcal{R}} (w(H_R, \{R\}) - w(f_R)) \\ &\leq |f| + \sum_{R \in \mathcal{R}} (|R| - |f_R|) - \kappa \\ &= \ell(\mathcal{R}) - \kappa, \end{aligned}$$

□

(39) *Let H be a $\mathcal{R}$-critical graph embedded in Σ with rings $\mathcal{R}$ so that all internal faces of H are open 2-cell. If H is $\mathcal{R}$-critical, has internal girth at least five, $|E(H)| \leq |E(G)|$ and an internal face f of H is not closed 2-cell, then $w(H, \mathcal{R}) \leq \ell(\mathcal{R}) + C\,(\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 1/2)$.*

Proof. Since f is not closed 2-cell, there exists a vertex v appearing at least twice in the facial walk of f. There exists a simple closed curve c going through the interior of f and joining two of the appearances of v. Cut the surface along c and cap the resulting holes by disks. If c is separating, then let H_1 and H_2 be the resulting graphs embedded in the two surfaces Σ_1 and Σ_2 obtained by this construction; if c is not separating, then let H_1 be the resulting graph embedded in a surface Σ_1. Let v_1 and v_2 be the two vertices to that v is split, such that $v_1 \in V(H_1)$. Let f_1 and f_2 be the faces to that f is split by c, where f_1 is a face of H_1. Note that $\ell(f_1) + \ell(f_2) = \ell(f)$, and thus $w(f) - w(f_1) - w(f_2) \leq 16$. If c is separating, then for $i \in \{1, 2\}$, let $\mathcal{R}_i$ consist of the rings of $\mathcal{R}$ contained in Σ_i, and if none of these rings contains v, then also of a vertex ring v_i (we drill a new cuff next to it). Here, v_i is weak if c is separating, Σ_{3-i} is a disk and the ring incident with its cuff is a vertex ring. If c is not separating, then let $\mathcal{R}_1 = \mathcal{R} \cup \{v_1, v_2\}$ if v is internal and let $\mathcal{R}_1 = \mathcal{R} \cup \{v_i\}$ if v is a ring vertex, where $i \in \{1, 2\}$ is chosen so that v_i is not incident with a cuff of Σ_1. We treat v_1 and v_2 as non-weak vertex rings and drill cuffs next to them.

Suppose first that c is not separating. Note that H_1 has at most two more rings (of length 1) than H and $g(\Sigma_1) \in \{g-1, g-2\}$ (depending on whether c is one-sided or not), and that H_1 has at least two rings. If H_1 has only one more ring than H, then

$$\begin{aligned} \mathrm{surf}(g(\Sigma_1), |\mathcal{R}_1|, t_0(\mathcal{R}_1), t_1(\mathcal{R}_1)) &\leq \mathrm{surf}(g-1, |\mathcal{R}|+1, t_0, t_1+1) \\ &\leq \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 44. \end{aligned}$$

Let us now consider the case that H_1 has two more rings than H. If $g(\Sigma_1) = 0$ and $|\mathcal{R}_1| = 2$, then note that both rings of H_1 are vertex rings. Lemma 46 implies that H_1 has only one edge; but the corresponding edge in H would form a loop, which is a contradiction. Consequently, we have $g > 1$ or $|\mathcal{R}| > 0$, and

$$\begin{aligned}\mathrm{surf}(g(\Sigma_1), |\mathcal{R}_1|, t_0(\mathcal{R}_1), t_1(\mathcal{R}_1)) &\leq \mathrm{surf}(g-1, |\mathcal{R}|+2, t_0, t_1+2)\\ &= \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 32.\end{aligned}$$

By induction, we can apply Theorem 51 to H_1 and we conclude that $w(H,\mathcal{R}) = w(H_1,\mathcal{R}_1) + w(f) - w(f_1) - w(f_2) \leq \ell(\mathcal{R}) + 18 + C\,(\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 32)$, and the claim follows.

Next, we consider the case that c is separating. Suppose that for $i \in \{1,2\}$, we have $g(\Sigma_i) > 0$ or $|\mathcal{R}_i| > 2$, or that $g(\Sigma_i) = 0$, $|\mathcal{R}_i| = 2$ and v is incident with a cuff belonging to Σ_i (so that v_i was not added to $\mathcal{R}_i$ as a vertex ring). We apply Theorem 51 to H_1 and H_2 and obtain

$$\begin{aligned}w(H,\mathcal{R}) &= w(H_1,\mathcal{R}_1) + w(H_2,\mathcal{R}_2) + w(f) - w(f_1) - w(f_2)\\ &\leq \ell(\mathcal{R}) + 18 + C\sum_{i=1}^{2}\mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i))\\ &\leq \ell(\mathcal{R}) + 18 + C\,(\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 32)\,,\end{aligned}$$

and the claim follows.

Therefore, we can assume that say $i = 1$ violates the assumptions of the previous paragraph, i.e., either $g(\Sigma_1) = 0$, $|\mathcal{R}_1| = 2$ and $\mathcal{R}_1 = \{v_1, R_1\}$ for some ring R_1, or $g(\Sigma_1) = 0$, $|\mathcal{R}_1| = 1$ and v_1 is incident with the ring R_1 of H_1. If $i = 2$ violates the assumptions as well, then let R_2 be the ring of H_2 distinct from v_2 and by symmetry assume that $|R_1| \leq |R_2|$. By Lemma 46, neither R_1 nor R_2 is a weak vertex ring, since f contains at least one edge in H_1 and in H_2. Note that $g(\Sigma_2) = g$ and $|\mathcal{R}_2| = |\mathcal{R}|$.

If R_1 is not a vertex ring, then we have either $|\mathcal{R}_1| = 1$ and $t_0(\mathcal{R}_1) = t_1(\mathcal{R}_1) = 0$ or $|\mathcal{R}_1| = 2$, $t_0(\mathcal{R}_1) = 0$ and $t_1(\mathcal{R}_1) = 1$, hence

$$\mathrm{surf}(g(\Sigma_1), |\mathcal{R}_1|, t_0(\mathcal{R}_1), t_1(\mathcal{R}_1)) \leq 2.$$

We apply Theorem 51 to H_1 and H_2 and obtain $w(H,\mathcal{R}) \leq \ell(\mathcal{R}_1) + \ell(\mathcal{R}_2) + 16 + C\,(\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1+1) + 2) = \ell(\mathcal{R}) + 18 + C\,(\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 2)$, implying the claim of the lemma.

Finally, if R_1 is a vertex ring, then note that v_1 is a ring of H_1, since H_1 has at least one edge. By Lemma 46, H_1 consists of a single edge joining R_1 with v_1. In this case, v_2 is a weak vertex ring in H_2 (note that H_2 is indeed $\mathcal{R}_2$-critical, as any precoloring of R_1 forbids exactly one color at v, thus giving a precoloring of v_2). By Theorem 51 applied to H_2, We get $w(H,\mathcal{R}) \leq \ell(\mathcal{R}) + 1 + C\mathrm{surf}(g, |\mathcal{R}|, t_0 + 1, t_1 - 1) \leq \ell(\mathcal{R}) + 1 + C\,(\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 1)$, and again the claim follows. $\square$

By (37), (38) and (39), we can assume that G satisfies (I3).

Suppose that G contains a path P of length at most six joining two distinct vertices u and v of a ring $R \in \mathcal{R}$, such that $V(P) \cap V(R) = \{u, v\}$ and $R \cup P$ contains no contractible cycle. Since the distance between any two rings in G is at least five, all vertices of $V(P) \setminus \{u, v\}$ are internal. Let J be the subgraph of G consisting of P and of the union of the rings, and let S be the set of internal faces of J. Clearly, S and J satisfy (30). Let $\{(G_1, \Sigma_1), \ldots, (G_k, \Sigma_k)\}$ be the G-expansion of S, and for $1 \leq i \leq k$, let $\mathcal{R}_i$ be the natural rings of G_i. Note that $\sum_{i=1}^{k} t_0(\mathcal{R}_i) = t_0$ and $\sum_{i=1}^{k} t_1(\mathcal{R}_i) = t_1$. Let $r = \left(\sum_{i=1}^{k} |\mathcal{R}_i|\right) - |\mathcal{R}|$ and observe that either $r = 0$ and $k = 1$, or $r = 1$ and $1 \leq k \leq 2$ (depending on whether the curve in $\hat{\Sigma}$ corresponding to a cycle in $R \cup P$ distinct from R is one-sided, two-sided and non-separating or two-sided and separating). Furthermore, $\sum_{i=1}^{k} g(\Sigma_i) = g + 2k - r - 3$. If say $g(\Sigma_1) = g$, then $k = 2$ and $r = 1$, and $g(\Sigma_2) = 0$. Since $R \cup P$ contains no contractible cycle, Σ_2 is not a disk, hence $|\mathcal{R}_2| \geq 2$ and $|\mathcal{R}_1| < |\mathcal{R}|$. Consequently, $(G_i, \Sigma_i, \mathcal{R}_i) \prec (G, \Sigma, \mathcal{R})$ and by induction, we have $w(G_i, \mathcal{R}_i) \leq \ell(\mathcal{R}_i) + C\mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i))$, for $1 \leq i \leq k$. Since every internal face of G is an internal face of G_i for some $i \in \{1, \ldots, k\}$ and $\sum_{i=1}^{k} \ell(\mathcal{R}_i) \leq \ell(\mathcal{R}) + 12$, we conclude that $w(G, \mathcal{R}) \leq \ell(\mathcal{R}) + 12 + C \sum_{i=1}^{k} \mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i))$. Note that for $1 \leq i \leq k$, we have that Σ_i is not a disk and $\mathcal{R}_i$ contains at least one facial ring. Since $R \cup P$ contains no contractible cycle, we have $g > 0$ or $t > 2$. For $1 \leq i \leq k$, let $\delta_i = 30$ if $g(\Sigma_i) = 0$ and $|\mathcal{R}_i| = 2$, and let $\delta_i = 0$ otherwise. Note that

$$
\begin{aligned}
&\sum_{i=1}^{k} \mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) \\
&\quad = \sum_{i=1}^{k} \mathrm{gen}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) + \delta_i \\
&\quad = \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) + 120(2k - r - 3) + 48r - 120(k - 1) + \sum_{i=1}^{k} \delta_i \\
&\quad = \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) + 120k - 72r - 240 + \sum_{i=1}^{k} \delta_i \\
&\quad \leq \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 12.
\end{aligned}
$$

The inequality of Theorem 84 follows; therefore, we can assume that

(40) *if P is a path of length at most six joining two distinct vertices of a ring R, then $R \cup P$ contains a contractible cycle.*

Let us note that since $g > 0$ or $|\mathcal{R}| \geq 2$, this contractible cycle is unique.

Consider now a path P of length at most four, such that its ends u and v are ring vertices and all other vertices of P are internal. Both ends of P belong to the same ring R; let P, P_1 and P_2 be the paths in $R \cup P$ joining u and v. By the previous paragraph, we can assume that $P \cup P_2$ is a contractible cycle. Suppose that the disk bounded by $P \cup P_2$ neither is a face nor consists of two 5-faces. By Theorem 41, we have $|P \cup P_2| \geq 9$. Let J, S, G_i, Σ_i and $\mathcal{R}_i$ (for $i \in \{1, 2\}$) be defined as in the previous paragraph, where Σ_2 is a disk and $\mathcal{R}_2$ consists of a single ring corresponding to $P \cup P_2$. Since $g(\Sigma_1) = g$, $|\mathcal{R}_1| = |\mathcal{R}|$ and $|E(G_1)| < |E(G)|$, by induction we have $w(G_1, \{\mathcal{R}_1\}) \leq \ell(\mathcal{R}_1) + C\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1)$. Note that $\ell(\mathcal{R}_1) = \ell(\mathcal{R}) + |P| - |P_2|$. Furthermore, Theorem 73 implies $w(G_2, \{\mathcal{R}_2\}) \leq s(|P| + |P_2|) = |P| + |P_2| - 8$. Therefore, $w(G, \mathcal{R}) \leq \ell(\mathcal{R}) + C\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) + 2|P| - 8$. Since $|P| \leq 4$, the claim of Theorem 84 follows. Therefore, we can assume that the disk bounded by $P \cup P_1$ is either a face or consists of two 5-faces. The same calculation also excludes the possibility that $|P| \leq 2$, since $s(|P| + |P_2|) \leq |P| + |P_2| - 4$ for any P and P_2 such that $|P| + |P_2| \geq 5$. In particular, we can assume that (I4) holds for G.

Suppose that G contains two adjacent vertices r_1 and r_2 of degree two. By (I7), both r_1 and r_2 are incident with the same facial ring R. By (I3), we have $|R| \geq 4$. By (I4), the internal face f incident with $r_1 r_2$ has length at least six. Let G' be the graph obtained from G by contracting $r_1 r_2$, let $\mathcal{R}'$ be the set of rings of G' obtained from $\mathcal{R}$ by contracting edge $r_1 r_2$ in R, and let f' be the face of G' corresponding to f. Observe that G' is $\mathcal{R}'$-critical. Suppose that G' contains a (≤ 4)-cycle C' distinct from the rings. Then G contains a (≤ 5)-cycle C distinct from the rings containing $r_1 r_2$. Since G has internal girth at least 5, we have $|C| = 5$, and we obtain a contradiction with (I4). Therefore, G' has internal girth at least 5. By induction, we have $w(G', \mathcal{R}') = \ell(\mathcal{R}') + C\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1)$, and since $\ell(\mathcal{R}) = \ell(\mathcal{R}') + 1$ and $w(f) \leq w(f') + 1$, G satisfies the inequality of Theorem 84. Therefore, assume that G satisfies (I5). Together with the previous paragraph, this implies that G is well-behaved.

Suppose that G contains a non-contractible cycle C of length at most seven that does not surround any of the rings. Observe that C intersects at most one ring, and by (40), C shares at most one vertex with this ring. Let $s = 1$ if C intersects a ring, and $s = 0$ otherwise. Let J be the subgraph of G consisting of C and of the union of the rings, and let S be the set of internal faces of J. Clearly, S and J satisfy (30). Let $\{(G_1, \Sigma_1), \ldots, (G_k, \Sigma_k)\}$ be the G-expansion of S, and for $1 \leq i \leq k$, let $\mathcal{R}_i$ be the natural rings of G_i. Let $r = \left(\sum_{i=1}^{k} |\mathcal{R}_i|\right) - |\mathcal{R}|$. Note that either $r + s = 1$ and $k = 1$, or $r + s = 2$ and $1 \leq k \leq 2$. Observe that $\sum_{i=1}^{k} g(\Sigma_i) = g - s - r + 2k - 2$. Furthermore, $\sum_{i=1}^{k} t_0(\mathcal{R}_i) + \sum_{i=1}^{k} t_1(\mathcal{R}_i) \geq t_0 + t_1 - s$ and $\sum_{i=1}^{k} \ell(\mathcal{R}_i) \leq \ell(\mathcal{R}) + 14$. If $g(\Sigma_1) = g$, then $k = 2$ and $g(\Sigma_2) = 0$; furthermore, Σ_2 has at least two cuffs, and if $s = 0$, then it has at least three cuffs, since C does not surround a ring. Thus, if $g(\Sigma_1) = g$, then $|\mathcal{R}_1| < |\mathcal{R}|$. The same argument can be applied to Σ_2 if $k = 2$, hence $(G_i, \Sigma_i, \mathcal{R}_i) \prec (G, \Sigma, \mathcal{R})$

for $1 \leq i \leq k$. By induction, we conclude that

$$w(G, \mathcal{R}) \leq \ell(\mathcal{R}) + 12 + C \sum_{i=1}^{k} \mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)).$$

For $1 \leq i \leq k$, let $\delta_i = 72$ if $g(\Sigma_i) = 0$ and $|\mathcal{R}_i| = 1$, let $\delta_i = 30$ if $g(\Sigma_i) = 0$ and $|\mathcal{R}_i| = 2$, and let $\delta_i = 0$ otherwise. Since C does not surround a ring, if $k = 2$ then $\delta_1 + \delta_2 \leq 30s$. Thomassen [68] proved that every projective planar graph of girth at least five is 3-colorable, hence if $g = 1$, then $|\mathcal{R}| \geq 1$; and if $k = 1$ then $\delta_1 \leq 30 + 42s$. We have

$$\begin{aligned}
&\sum_{i=1}^{k} \mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) \\
&= \sum_{i=1}^{k} \mathrm{gen}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) + \delta_i \\
&\leq \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) + 120(2k - r - s - 2) + 48r - 120(k-1) + 5s + \sum_{i=1}^{k} \delta_i \\
&= \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) + 120k - 72r - 115s - 120 + \sum_{i=1}^{k} \delta_i \\
&\leq \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 24.
\end{aligned}$$

This implies the inequality of Theorem 84. Therefore, assume that every non-contractible cycle of length at most 7 surrounds a ring. In particular, G satisfies (I8).

For each ring $R \in \mathcal{R}$, let M_R be the set of all edges incident with cycles of G of length at most 7 that surround R, and let C_R be such a cycle chosen so that the part Σ_R of Σ between R and C_R is as large as possible. By Lemma 78, at most 70 edges of M_R are drawn outside of Σ_R. Let K_R be a (≤ 7)-cycle in $G \cap \Sigma_R$ chosen so that the part Σ'_R of Σ between R and K_R is as small as possible. Analogically to Lemma 78, we see that at most 70 edges of $M_R \cap \Sigma_R$ are drawn outside of Σ'_R. We claim that at most $5\mathrm{cyl}(7,7)/s(5))$ edges of G are drawn in Σ'_R: When K_R and C_R are vertex-disjoint, this follows from Lemma 83. When K_R intersects C_R, this is implied by Lemma 72 and Theorem 73, since $\mathrm{cyl}(7,7) > s(14)$. We conclude that $|M_R| \leq 140 + 5\mathrm{cyl}(7,7)/s(5)$.

Let M consist of all rings of length at most four and of all non-contractible cycles in G of length at most 7. Observe that $M = \bigcup_{R \in \mathcal{R}} M_R$, and thus $|E(M)| \leq (140 + 5\mathrm{cyl}(7,7)/s(5))|\mathcal{R}|$. Note that M captures all (≤ 4)-cycles in G. If $w(G, \mathcal{R}) \leq 8g + 8|\mathcal{R}| + (2/3 + 26\epsilon)\ell(\mathcal{R}) + 20|E(M)|/3 - 16$, then $w(G, \mathcal{R}) \leq \ell(\mathcal{R}) + C\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1)$ by the choice of C, and Theorem 84 is true. Therefore, assume that this is not the case, and thus the assumptions of Theorem 75 are satisfied.

Let G' be the $\mathcal{R}$-critical graph embedded in Σ such that $|E(G')| < |E(G)|$, satisfying the conditions of Theorem 75. In particular, (b) together with the choice of M implies that G' has internal girth at least five. Let $X \subset F(G)$ and $\{(J_f, S_f) : f \in F(G')\}$ be the cover of G by faces of G' as in Theorem 75(d). For $f \in F(G')$, let $\{(G^f_1, \Sigma^f_1), \ldots, (G^f_{k_f}, \Sigma^f_{k_f})\}$ be the G-expansion of S_f and let $\mathcal{R}^f_i$ denote the natural rings of G^f_i. We have

$$w(G, \mathcal{R}) = \sum_{f \in F(G)} w(f) = \sum_{f \in X} w(f) + \sum_{f \in F(G')} \sum_{i=1}^{k_f} w(G^f_i, \mathcal{R}^f_i).$$

Consider a face $f \in F(G')$. We have $g(\Sigma_f) \leq g$. If $g(\Sigma_f) = g$, then every component of G' is planar, and since G' is $\mathcal{R}$-critical, each component of G' contains at least one ring of $\mathcal{R}$; consequently, f has at most $|\mathcal{R}|$ facial walks and Σ_f has at most $|\mathcal{R}|$ cuffs. Since the surfaces of the G-expansion of S_f are fragments of Σ_f, we have $(G^f_i, \Sigma^f_i, \mathcal{R}^f_i) \prec (G, \Sigma, \mathcal{R})$ for $1 \leq i \leq k_f$: otherwise, we would have $m(G^f_i) = m(G)$, hence by the definition of G-expansion, S_f would have to be equal to the union of rings in $\mathcal{R}$, contrary to the definition of a cover. Therefore, we can apply Theorem 84 to G^f_i and we get $w(G^f_i, \mathcal{R}^f_i) \leq \ell(\mathcal{R}^f_i) + C\mathrm{surf}(g(\Sigma^f_i), |\mathcal{R}^f_i|, t_0(\mathcal{R}^f_i), t_1(\mathcal{R}^f_i))$. Observe that since $\{\Sigma^f_1, \ldots, \Sigma^f_{k_f}\}$ are fragments of Σ_f,

$$\sum_{i=1}^{k_f} \mathrm{surf}(g(\Sigma^f_i), |\mathcal{R}^f_i|, t_0(\mathcal{R}^f_i), t_1(\mathcal{R}^f_i)) \leq \mathrm{surf}(f).$$

In case that f is open 2-cell, all fragments of f are disks and we can use Theorem 73 instead of Theorem 84. Combining the inequalities, we obtain

$$\sum_{i=1}^{k_f} w(G^f_i, \mathcal{R}^f_i) \leq w(f) + \mathrm{el}(f) + C\mathrm{surf}(f). \tag{7.2}$$

Combining the inequalities and using Theorem 75(d), we have

$$\begin{aligned} w(G, \mathcal{R}) &\leq |X|s(6) + \sum_{f \in F(G')} w(f) + \mathrm{el}(f) + C\mathrm{surf}(f) \\ &\leq w(G', \mathcal{R}) + s(6) + 10 + C \sum_{f \in F(G')} \mathrm{surf}(f). \end{aligned}$$

If G' has a face that is neither open 2-cell nor omnipresent, then (37) implies that

$$w(G', \mathcal{R}) \leq \ell(\mathcal{R}) + C \left(\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 7 - \sum_{f \in F(G')} \mathrm{surf}(f) \right),$$

and consequently G satisfies the outcome of Theorem 84. Therefore, we can assume that all internal faces of G' are either open 2-cell or omnipresent. Similarly, using (39) we can assume that if no face of G' is omnipresent, then all of them are closed 2-cell.

Suppose first that G has no omnipresent face. Using Theorem 75(d) and (e) and applying Theorem 84 to G', we have

$$\begin{aligned}
w(G,\mathcal{R}) &\leq |X|s(6) + \sum_{f\in F(G')} w(f) - c(f) \\
&= w(G',\mathcal{R}) + |X|s(6) - \sum_{f\in F(G')} c(f) \\
&\leq w(G',\mathcal{R}) \leq \ell(\mathcal{R}) + C\mathrm{surf}(g,|\mathcal{R}|,t_0,t_1).
\end{aligned}$$

It remains to consider the case that G' has an omnipresent face h. Then, every component of G is a plane graph with one ring, and by Lemma 71, we conclude that every internal face of G different from h is closed 2-cell and G' satisfies (I6). By Theorem 75(d), we have $c(h) \neq -\infty$, hence no component of G' satisfies (E1), (E2) or (E3). By Theorem 75(d) and (e) by (7.2) and by (38), we have

$$\begin{aligned}
w(G,\mathcal{R}) &\leq |X|s(6) + \sum_{f\in F(G'), f\neq h} (w(f) - c(f)) + \sum_{i=1}^{k_h} w(G_i^h, \mathcal{R}_i^h) \\
&= w(G',\mathcal{R}) + |X|s(6) + (c(h) - w(h)) - \sum_{f\in F(G')} c(f) + \sum_{i=1}^{k_h} w(G_i^h, \mathcal{R}_i^h) \\
&\leq w(G',\mathcal{R}) + c(h) - w(h) + \sum_{i=1}^{k_h} w(G_i^h, \mathcal{R}_i^h) \\
&\leq w(G',\mathcal{R}) + c(h) + \mathrm{el}(h) + C\mathrm{surf}(g,|\mathcal{R}|,t_0,t_1) \\
&\leq \ell(\mathcal{R}) + C\mathrm{surf}(g,|\mathcal{R}|,t_0,t_1) + c(h) + \mathrm{el}(h) - \kappa,
\end{aligned}$$

where κ is defined as in (38). By Theorem 75(d), we have $\mathrm{el}(h) \leq 5$, and thus $c(h) + \mathrm{el}(h) - \kappa \leq 0$. Therefore,

$$w(G,\mathcal{R}) \leq \ell(\mathcal{R}) + C\mathrm{surf}(g,|\mathcal{R}|,t_0,t_1)$$

as required. $\square$

Chapter 8

Distant perturbations in 5-list-colorability of planar graphs[1]

A well-known result by Thomassen [65] states that every planar graph is 5-list-colorable. This implies that planar graphs are 5-colorable. Since planar graphs are known to be 4-colorable [6, 7], a natural question is whether the result can be strengthened. Voigt [75] gave an example of a non-4-list-colorable planar graph; hence, the vertices with lists of size smaller than 5 must be restricted in some way. For example, Albertson [5] asked the following question.

Problem 85. *Does there exist a constant d such that whenever G is a planar graph with list assignment L that gives list of size one or five to each vertex and the distance between every two vertices with list of size one is at least d, then G is L-colorable?*

For usual colorings, Albertson [5] proved, solving a problem asked earlier by Thomassen [70], that having a set of vertices in a planar graph G that are precolored with colors $1, \ldots, 5$ and are at distance at least 4 from each other, then the precoloring can be extended to a 5-coloring of G. This result does not generalize to 4-colorings even if we have only two precolored vertices (arbitrarily far apart). Examples are given by triangulations of the plane that have precisely two vertices of odd degree. As proved by Ballantine [9] and Fisk [37], the two vertices of odd degree must have the same color in every 4-coloring. Thus, precoloring them with a different color, we cannot extend the precoloring to a 4-coloring of the whole graph.

Recently, there has been a significant progress towards the solution of Albertson's problem, see [8] and [30]. Let us remark that when the number of precolored

[1] The results of this chapter are based on Dvořák et al. [31].

vertices is also bounded by some constant, then the answer is positive by the results of Kawarabayashi and Mohar [46] on 5-list-coloring graphs on surfaces. We aim to prove that the answer is positive in general.

Theorem 86. *There exists a constant d with the following property. If G is a planar graph with list assignment L that gives list of size one or five to each vertex and the distance between every two vertices with list of size one is at least d, then G is L-colorable.*

In the proof, we will need the following result concerning the case that the precolored vertices form a connected subgraph, which is of an independent interest.

Theorem 87. *Suppose that G is a planar graph, H is a connected subgraph of G and L is an assignment of lists to the vertices of G such that $|L(v)| \geq 5$ for $v \in V(G) \setminus V(H)$. If G is not L-colorable, then G contains a subgraph F with at most $72|V(H)|^2$ vertices such that F is not L-colorable.*

Let us remark that the existence of such a subgraph of bounded size follows from [46], but our bound on the size of F is much better and gives a better estimate on the required distance in Problem 85.

Let G be a plane graph, P a subpath of its outer face H, and X a subset of $V(G)$. For a positive integer M, a list assignment L for G is *M-valid with respect to P and X* if

- $|L(v)| = 5$ for $v \in V(G) \setminus (V(H) \cup X)$,
- $3 \leq |L(v)| \leq 5$ for $v \in V(H) \setminus (V(P) \cup X)$,
- $|L(v)| = 1$ for $v \in X$,
- the subgraph of G induced by $V(P) \cup X$ is L-colorable, and
- for every $v \in X$, the vertices of $V(G) \setminus \{v\}$ at distance at most M from v do not belong to P and have lists of size 5.

If $X = \emptyset$ and L is 0-valid, we say that L is *valid*.

A key ingredient for our proofs is the following well-known result of Thomassen [65] regarding the coloring of planar graphs from lists of restricted sizes.

Theorem 88. *If G is a connected plane graph with outer face H, xy an edge of H and L a list assignment that is valid with respect to xy, then G is L-colorable.*

There exist arbitrarily large non-L-colorable graphs with this structure if we allow a path of length two to be precolored. Thomassen [73] gave their complete description, see Lemma 93. In Theorem 92, we deal with the more general case when P has fixed length k. In particular, we show that if G is a minimal non-L-colorable graph satisfying the assumptions of Theorem 92, then at most $k-2$ of its vertices incident with the outer face have list of size at least four. In conjunction with Theorem 87, this enables us to bound the size of such graphs with the additional assumption that no two vertices with list of size three are adjacent.

Next, we use the new approach to 5-list-colorability of planar graphs developed in [30] (see also Chapter 9) to show that we can reduce the problem to the case that only one internal vertex is precolored. Having established this fact, the following lemma gives the affirmative answer to Problem 85.

Lemma 89. *There exists a constant M with the following property. For every plane graph G with outer face H, any (possibly null) subpath P of H of length at most one, any $x \in V(G) \setminus V(P)$ and any list assignment L that is M-valid with respect to P and $\{x\}$ such that no two vertices with list of size three are adjacent, the graph G is L-colorable.*

We first prove Theorem 87, in Section 8.1. In Section 8.2, we prove Theorem 92. In Section 8.3, we show that Lemma 89 implies our main result, Theorem 86. The proof of Lemma 89 is omitted from this presentation (it can be found in the full paper [31]), as it is rather technical and would not help illustrating the concepts of this chapter.

Let us mention that we could also allow different kinds of "irregularities" other than just precolored vertices, for example, precolored triangles or crossings, as long as the irregularity satisfies the condition analogous to Lemma 89. To keep the presentation manageable, we do not give proofs in this full generality and focus on the case of precolored single vertices.

8.1 Critical graphs

To avoid dealing with irrelevant subgraphs, we define what a list-coloring critical graph means. Let G be a graph, $T \subseteq G$ a (not necessarily induced) subgraph of G and L a list assignment to the vertices of $V(G)$. For an L-coloring φ of T, we say that φ *extends to an L-coloring of G* if there exists an L-coloring of G that matches φ on $V(T)$. The graph G is *T-critical with respect to the list assignment L* if $G \neq T$ and for every proper subgraph $G' \subset G$ such that $T \subseteq G'$, there exists a coloring of T that extends to an L-coloring of G', but does not extend to an L-coloring of G. If the list assignment is clear from the context, we shorten this and say that G is *T-critical.* Note that G is list-critical for the usual definition of criticality if and only if it is $\emptyset$-critical. Let us also observe that every proper subgraph of a T-critical graph that includes T is L-colorable, and that it may happen that G is also L-colorable.

Let G be a T-critical graph (with respect to some list assignment). For $S \subseteq G$, a graph $G' \subseteq G$ is an *S-component* of G if S is a proper subgraph of G', $T \cap G' \subseteq S$ and all edges of G incident with vertices of $V(G') \setminus V(S)$ belong to G'. For example, if G is a plane graph with T contained in the boundary of its outer face and S is a cycle in G that does not bound a face, then the subgraph of G drawn inside the closed disk bounded by S (which we denote by $\mathrm{Int}_S(G)$) is an S-component of G.

Critical graphs have the following basic property.

Lemma 90. *Let G be a T-critical graph with respect to a list assignment L. Let G' be an S-component of G, for some $S \subseteq G$. Then G' is S-critical.*

Proof. If G contains an isolated vertex v that does not belong to T, then since G is T-critical, we have that $L(v) = \emptyset$ and $T = G - v$. Observe that if G' is an S-component of G, then $S \subseteq T$ and $G' - v = S$, and clearly G' is S-critical.

Therefore, we can assume that every isolated vertex of G belongs to T. Consequently, every isolated vertex of G' belongs to S. Suppose for a contradiction that G' is not S-critical. Then, there exists an edge $e \in E(G') \setminus E(S)$ such that every L-coloring of S that extends to $G' - e$ also extends to G'. Note that $e \notin E(T)$. Since G is T-critical, there exists a coloring ψ of T that extends to an L-coloring φ of $G - e$, but does not extend to an L-coloring of G. However, by the choice of e, the restriction of φ to S extends to an L-coloring φ' of G'. Let φ'' be the coloring that matches φ' on $V(G')$ and φ on $V(G) \setminus V(G')$. Observe that φ'' is an L-coloring of G extending ψ, which is a contradiction. □

Clearly, to prove Theorem 87, it suffices to bound the size of critical graphs. It is more convenient to bound the *weight* of such graphs, which is defined as follows. Let G be a plane graph, P a subgraph of the outer face H of G, and L a list assignment. For a face $f \neq H$, we set $\omega_{G,P,L}(f) = |f| - 3$, where $|f|$ denotes the number of edges appearing on the boundary of f (if an edge is incident with f on both sides, it contributes 2 to $|f|$). We set $\omega_{G,P,L}(H) = 0$. The weight is also defined for the vertices of G. If $v \in V(P)$, then $\omega_{G,P,L}(v) = 1$ if v is a cut-vertex of G, and $\omega_{G,P,L}(v) = 0$ otherwise. If $v \in V(H) \setminus V(P)$, then $\omega_{G,P,L}(v) = |L(v)| - 3$. If $v \in V(G) \setminus V(H)$, then $\omega_{G,P,L}(v) = 0$. In the cases where G, P or L are clear from the context, we drop the corresponding indices. We set

$$\omega_{P,L}(G) = \sum_{v \in V(G)} \omega_{G,P,L}(v) + \sum_{f \in F(G)} \omega_{G,P,L}(f),$$

where the sums go over the vertices and faces of G, respectively.

Given a graph G and a cycle $K \subseteq G$, an edge uv is a *chord* of K if $u, v \in V(K)$, but uv is not an edge of K. For an integer $k \geq 2$, a path $v_0v_1 \ldots v_k$ is a *k-chord* if $v_0, v_k \in V(K)$ and $v_1, \ldots, v_{k-1} \notin V(K)$.

Let $\mathcal{S}$ be a set of proper colorings of K. We say that $v \in V(K)$ is *relaxed in* $\mathcal{S}$ if there exist two distinct colorings in $\mathcal{S}$ that differ only in the color of v.

Lemma 91. *Let G be a plane graph with its outer face H bounded by a cycle and L a list assignment for G such that $|L(v)| \geq 5$ for $v \in V(G) \setminus V(H)$. If G is H-critical with respect to the list assignment L and G is not equal to H with one added chord, then*

$$\omega_{H,L}(G) + \frac{|V(G) \setminus V(H)|}{2|H|+2} \leq |H| - 9/2.$$

Proof. We proceed by induction. Assume that the lemma holds for all graphs having fewer edges than G. For a subgraph G' of G with outer face C, let

$$\theta(G') = \omega_{C,L}(G') + \frac{|V(G') \setminus V(C)|}{2|H|+2}.$$

Let $C \neq H$ be a cycle in G such that $|C| \leq |H|$. By Lemma 90, $\mathrm{Int}_C(G)$ is C-critical with respect to L if C is not a face boundary. If $\mathrm{Int}_C(G)$ has at least four faces, then the induction hypothesis applied to $\mathrm{Int}_C(G)$ implies that $\theta(\mathrm{Int}_C(G)) \leq |C| - 9/2$. Observe that if $\mathrm{Int}_C(G)$ has three faces (i.e., consists of C and its chord), then $\theta(\mathrm{Int}_C(G)) = \omega_{C,L}(\mathrm{Int}_C(G)) = |C| - 4$, and if C bounds a face, then $\theta(\mathrm{Int}_C(G)) = |C| - 3$.

We construct a sequence $G_0 \supset G_1 \supset \ldots \supset G_k$ of subgraphs of G with outer faces H_0, H_1, …, H_k such that for $0 \leq i \leq k$, G_i is H_i-critical and

$$\omega_{H_i,L}(G_i) = \omega_{H,L}(G) - (|H| - |H_i|). \tag{8.1}$$

We set $G_0 = G$ and $H_0 = H$. Suppose that G_i was already constructed. If H_i has a chord, or a vertex of G_i has at least four neighbors in H_i, then we set $k = i$ and stop. Otherwise, by Theorem 88, there is a vertex $v \in V(G_i)$ with three neighbors v_1, v_2 and v_3 in H_i. (If that is not the case, consider any L-coloring φ of H_i, remove the colors of the vertices of H_i from the lists of their neighbors and color $G_i - V(H_i)$ from the resulting lists. This shows that every L-coloring of H_i extends to an L-coloring of G_i, contradicting the H_i-criticality of G_i.) Let C_1, C_2 and C_3 be the three cycles of $H_i + \{v_1v, v_2v, v_3v\}$ distinct from H_i, where C_j does not contain the edge vv_j ($j = 1, 2, 3$). If at most one of these cycles bounds a face of G_i, then we set $k = i$ and stop. Otherwise, assume that C_1 and C_3 are faces of G_i. Let $\mathcal{S}_i$ be the set of L-colorings of H_i that do not extend to an L-coloring of G_i. If v_2 is relaxed in $\mathcal{S}_i$, then again set $k = i$ and stop. Otherwise, let $G_{i+1} = \mathrm{Int}_{C_2}(G_i)$ and let $H_{i+1} = C_2$ be the cycle bounding its outer face. Note that $|H_{i+1}| \leq |H_i|$ and that

$$|H_{i+1}| - |H_i| = (|C_1| - 3) + (|C_3| - 3). \tag{8.2}$$

Observe that if $w \in V(H_{i+1}) \setminus \{v\}$ is relaxed in $\mathcal{S}_i$, then it is also relaxed in $\mathcal{S}_{i+1}$. This is obvious if $w \neq \{v_1, v_3\}$. Suppose that say $w = v_1$ and that $\varphi_1, \varphi_2 \in \mathcal{S}_i$ differ only in the color of v_1. Since v has list of size at least 5, there exists a color $c \in L(v) \setminus \{\varphi_1(v_1), \varphi_2(v_1), \varphi_1(v_2), \varphi_1(v_3)\}$. Let φ_1' and φ_2' be the L-colorings of H_{i+1} that match φ_1 and φ_2 on H_i and $\varphi_1'(v) = \varphi_2'(v) = c$. Then neither φ_1' nor φ_2' extend to an L-coloring of G_{i+1}, showing that v_1 is relaxed in $\mathcal{S}_{i+1}$. Similarly, v is relaxed in $\mathcal{S}_{i+1}$, since for any $\varphi \in \mathcal{S}$, there exist at least two ways how to L-color v. We conclude that the number of non-relaxed vertices in $\mathcal{S}_{i+1}$ is smaller than the number of non-relaxed vertices in $\mathcal{S}_i$ for every $i < k$, and consequently, $k \le |H|$.

Lemma 90 implies that every G_i is H_i-critical. It is also easy to see by induction and using (8.2) that (8.1) holds for $0 \le i \le k$. In each step in the construction of the sequence $(G_i, H_i)_{i=0}^k$, the number $|V(G_i) \setminus V(H_i)|$ is decreased by 1. Thus, (8.1) implies that

$$\theta(G) - \theta(G_k) = |H| - |H_k| + \frac{k}{2|H| + 2}. \tag{8.3}$$

Suppose that there exist a proper subgraph $G' \supset H_k$ of G_k and a coloring $\varphi \in \mathcal{S}_k$ that does not extend to an L-coloring of G'. We may choose G' to be H_k-critical. Note that

$$\theta(G) = \frac{k}{2|H| + 2} + (|H| - |H_k|) + \theta(G') + \sum_f (\theta(\operatorname{Int}_f(G)) - \omega(f)),$$

where the summation goes over the faces of G'. By induction, $\theta(G') \le |H_k| - 4$, since $G' \neq H_k$. This implies that all faces of G' are shorter than $|H|$. Since G' is a proper subgraph of G_k, we have $\theta(\operatorname{Int}_f(G)) \le \omega(f) - 1$ for at least one face f of G' by induction. Therefore, $\theta(G) \le 1/2 + |H| - 4 - 1 = |H| - 9/2$, as required. Therefore, we can assume that every coloring in $\mathcal{S}_k$ extends to every proper subgraph of G_k that includes H_k.

Let us now consider various possibilities in the definition of G_k. If $v \in V(G_k) \setminus V(H_k)$ has exactly three neighbors v_1, v_2 and v_3 in H_k and v_2 is relaxed, then consider the colorings $\varphi_1, \varphi_2 \in \mathcal{S}_k$ that differ only in the color of v_2. The coloring φ_1 extends to an L-coloring ψ of $G_k - vv_2$. But $\psi(v) \neq \varphi_1(v_2)$ or $\psi(v) \neq \varphi_2(v_2)$, hence either φ_1 or φ_2 extends to an L-coloring of G_k. This is a contradiction.

Suppose now that H_k has a chord $e = xy$ in G_k. If $G_k = H_k + e$, then since G is not H with a single chord, we have $k > 0$. However, that implies that a vertex of G_{k-1} has degree at most four and list of size 5, which is impossible in a critical graph. This implies that $G_k \neq H_k + e$. Since G_k is H_k-critical, there exists a coloring $\varphi \in \mathcal{S}_k$ that extends to an L-coloring of $H_k + e$, i.e., $\varphi(x) \neq \varphi(y)$. However, every coloring in $\mathcal{S}_k$ extends to every proper subgraph of G_k that includes H_k, and it follows that φ extends to an L-coloring of $G_k - e$. This gives an L-coloring of G_k extending φ, contradicting the assumption that $\varphi \in \mathcal{S}_k$. Therefore, we can assume that H_k is an induced cycle in G_k.

It follows that a vertex $v \in V(G_k) \setminus V(H_k)$ either has at least four neighbors in H_k, or three neighbors v_1, v_2 and v_3 in H_k such that at most one of the cycles of $H_k + \{v_1v, v_2v, v_3v\}$ bounds a face distinct from H_k. Then H_k has a 2-chord Q such that neither of the cycles K_1 and K_2 of $H_k \cup Q$ distinct from H_k bounds a face. For $i \in \{1,2\}$, let $G'_i = \mathrm{Int}_{K_i}(G)$. Suppose first that it is not possible to choose Q so that neither G'_1 nor G'_2 is a cycle with one chord. Since the middle vertex v of Q has degree at least 5, this can only happen if $V(G_k) \setminus V(H_k) = \{v\}$ and v has degree exactly 5. But then $k = 0$, since otherwise G_{k-1} would contain a vertex of degree at most four with list of size 5, and we have $\theta(G) = |H| - 5 + \frac{1}{2|H|+2} < |H| - 9/2$.

Finally, suppose that neither G'_1 nor G'_2 is a cycle with a chord. By induction, we have $\theta(G) \leq \frac{k+1}{2|H|+2} + (|H| - |H_k|) + \theta(G'_1) + \theta(G'_2) \leq 1/2 + (|H| - |H_k|) + |K_1| + |K_2| - 9 = 1/2 + (|H| - |H_k|) + |H_k| - 5 = |H| - 9/2$, as required. $\square$

Lemma 91 gives rise to a natural algorithm to enumerate all such H-critical graphs: we proceed by the length k of the cycle H, thus assume that we already know, up to isomorphism, the set $\mathcal{G}$ of all planar graphs with precolored outer cycle of length at most $k-1$, such that the internal vertices have lists of size at least five. Let $\mathcal{H}_A$ be all graphs consisting of a cycle of length $\leq k$ with a chord and $\mathcal{H}_B$ the graphs consisting of a cycle of length $\leq k$ and a vertex with at least three neighbors in the cycle. Let $\mathcal{H}'_0$ be the set of all graphs that can be obtained from the graphs in $\mathcal{H}_A \cup \mathcal{H}_B$ by pasting the graphs of $\mathcal{G}$ in some of the faces. Let $\mathcal{H}_0$ be the subset of $\mathcal{H}'_0$ consisting of the graphs that are critical with respect to their outer face. For each graph in $\mathcal{H}_0$, keep adding a vertex of degree three adjacent to three consecutive vertices of H, as long as the resulting graph is critical with respect to its outer face. This way, we will obtain all graphs critical with respect to the outer face of length ℓ. Lemma 91 guarantees that this algorithm will finish. Note also that by omitting $\mathcal{H}_A$ in the first step of the algorithm, we can generate such critical graphs whose outer cycle is chordless.

The main difficulty in the implementation is the need to generate all the possible lists in order to test the criticality, which makes the time complexity impractical. However, sometimes it is sufficient to generate a set of graphs that is guaranteed to contain all graphs that are critical (for some choice of the lists), but may contain some non-critical graphs as well. To achieve this, one may replace the criticality testing by a set of simple heuristics that prove that a graph is not critical. For example, in an H-critical graph G, each vertex $v \in V(G) \setminus V(H)$ has degree at least $|L(v)|$, and the vertices whose degrees match the sizes of the lists induce a subgraph G' such that each block of G' is either a complete graph or an odd cycle [74]. There are similar claims forbidding other kinds of subgraphs with specified sizes of lists. On the positive side, to prove that a graph is H-critical, it is usually sufficient to consider the case that all lists are equal. By combining these two tests, we were able to generate graphs critical with respect to the outer face of length at most 9. If the outer face is an induced cycle, then there are three of them for length 6, six for length 7, 34 for length 8 and 182 for length 9. The program that we used can be found at

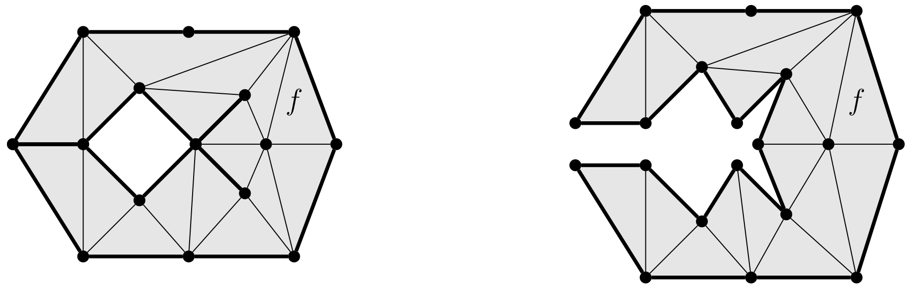


Figure 8.1: Splitting the boundary of a face of H. The boundaries of f and the split cycle C are shown by bold edges.

`http://atrey.karlin.mff.cuni.cz/~rakdver/5choos/`.

Theorem 87 is an easy corollary of Lemma 91.

Proof of Theorem 87. Let F be a minimal subgraph of G including H that is not L-colorable. If $F = H$, then the conclusion of Theorem 87 clearly holds. Hence, assume that $F \neq H$, and thus F is H-critical. Let f be a face of H and let F'_f be the subgraph of F drawn in f. In F'_f, split the vertices of f so that the interior of f is unchanged and f becomes a cycle. The notion of "splitting" should be clear from a generic example shown in Figure 8.1. Let F_f be the resulting graph and C the cycle corresponding to f, and note that the length of C is $|f|$. Observe that if $V(F_f) \neq V(C)$, then F_f is C-critical, and by Lemma 91,

$$|V(F'_f) \setminus V(f)| = |V(F_f) \setminus V(C)| \leq (2|f| + 2)(|f| - 9/2) \leq 2|f|^2. \tag{8.4}$$

Let f_i ($i = 1, \ldots, k$) be the faces of H which contain at least one vertex of F in their interior, and note that

$$\sum_{i=1}^{k} |f_i| \leq 2|E(H)| \leq 6|V(H)| - 12, \tag{8.5}$$

where the last inequality is a well-known consequence of Euler's formula for planar graphs (which holds if $|V(H)| \geq 3$; this is true by Theorem 88). Thus,

$$\sum_{i=1}^{k} |f_i|^2 \leq \Big(\sum_{i=1}^{k} |f_i|\Big)^2 \leq 36|V(H)|^2 - 144|V(H)| + 144 < 36|V(H)|^2 - |V(H)|/2. \tag{8.6}$$

Finally, by using (8.4) and (8.6), we bound the order of F as follows:

$$|V(F)| \leq |V(H)| + \sum_{i=1}^{k} |V(F'_{f_i}) \setminus V(f_i)| \leq |V(H)| + 2\sum_{i=1}^{k} |f_i|^2 \leq 72|V(H)|^2.$$

This completes the proof of Theorem 87. □

8.2 Extending a coloring of a path

For a path P, we let $\ell(P)$ denote its length (the number of its edges). A vertex of P is an *inside* vertex if it is not an endvertex of P. The main result of this section follows by using the same basic strategy as in Thomassen's proof of Theorem 88 [65].

Theorem 92. *Let G be a plane graph and P a subpath of its outer face H. Let L be a list assignment valid with respect to P. If G is P-critical with respect to L, then $\omega_{P,L}(G) \leq \ell(P) - 2$.*

Proof. Suppose for a contradiction that G is a counterexample with the smallest number of edges, and in particular that $\omega_{P,L}(G) \geq \ell - 1$, where $\ell = \ell(P)$. By Theorem 88, we have $\ell \geq 2$. Furthermore, Theorem 88 also implies that if either a vertex or two adjacent vertices form a vertex-cut R in G, then each component of $G - R$ contains a vertex of P. Let $P = p_0p_1 \dots p_\ell$. If p_i is a cut-vertex for some $1 \leq i \leq \ell - 1$, then $G = G_1 \cup G_2$, where $G_1, G_2 \neq \{p_i\}$ and $G_1 \cap G_2 = \{p_i\}$. Let $P_1 = P \cap G_1$ and $P_2 = P \cap G_2$. Since $G \neq P$, we can assume that $G_1 \neq P_1$. Note that if $G_2 = P_2$, then $\omega_{P_2,L}(G_2) = \ell(P_2) - 1$. If $G_i \neq P_i$, then G_i is P_i-critical by Lemma 90, for $i \in \{1, 2\}$. By the minimality of G, we have $\omega_{P,L}(G) = \omega_{P_1,L}(G_1) + \omega_{P_2,L}(G_2) + 1 \leq (\ell(P_1) - 2) + (\ell(P_2) - 1) + 1 = \ell - 2$. Since $\omega_{P,L}(G) \geq \ell - 1$, we conclude that G is 2-connected.

Suppose that there exists a proper subgraph $G' \supseteq P$ of G and an L-coloring ψ of P does not extend to an L-coloring of G'. We may choose G' to be P-critical. By the minimality of G, we have $\omega_{P,L}(G') \leq \ell - 2$. Let H' be the outer face of G' and let W' be the walk such that the concatenation of W' and P is the boundary walk H' of G'. Since G' is P-critical, Theorem 88 implies that W' is a path. Let $q_0, \dots, q_m$ be the vertices of $V(H) \cap V(W')$ that are not inside vertices on the path P, listed in the order as they appear in W', where q_0 and q_m are the endvertices of P. Observe that $q_0, \dots, q_m$ appear in the same order also in H. Each subwalk Q_i of W' from q_{i-1} to q_i ($i = 1, \dots, m$) is called a *span*. Note that W' is the union of spans $Q_1, \dots, Q_m$, and each of the spans is a path. For $1 \leq i \leq m$, let R_i be the segment of H from q_{i-1} to q_i, and let G_i be the subgraph of G drawn inside the closed disk bounded by $R_i \cup Q_i$. Note that if $G_i = Q_i$, then Q_i is an edge of H. Observe that $\omega_{G',P,L}(v) \geq 1$ for each inside vertex v of Q_i, since v either has list of size 5 or it is a cut-vertex in G'; hence, their total weight is at least $\ell(Q_i) - 1$. Note that $\omega_{G,P,L}(v) = 0$. By the minimality of G, we have $\omega_{Q_i,L}(G_i) \leq \ell(Q_i) - 2$ if Q_i is not equal to an edge of H. If Q_i is an edge of H, then $\omega_{Q_i,L}(G_i) = 0 = \ell(Q_i) - 1$. Furthermore, if f is an internal face of G', then Lemma 91 implies that $\omega_{f,L}(\mathrm{Int}_f(G)) \leq \omega_{G',P,L}(f)$. It follows that

$$\omega_{P,L}(G) \leq \omega_{P,L}(G') + \sum_{i=1}^{m}(\omega_{Q_i,L}(G_i) - (\ell(Q_i) - 1)) +$$

$$\sum_{f\in F(G')} (\omega_{f,L}(\operatorname*{Int}_f(G)) - \omega_{G',P,L}(f))$$
$$\leq \omega_{P,L}(G') \leq \ell - 2.$$

This is a contradiction which proves the following:

(41) *For every proper subgraph G' of G, every L-coloring ψ of P extends to an L-coloring of G'.*

Let ψ be an L-coloring of P that does not extend to G. If L' is the list assignment such that $L'(v) = L(v)$ for $v \notin V(P)$ and $L'(v) = \{\psi(v)\}$ for $v \in V(P)$, (41) implies that G is P-critical with respect to L'. Note that $\omega_{P,L}(G) = \omega_{P,L'}(G)$ as the sizes of the lists of the vertices of P are not affecting ω. Consequently, we can assume henceforth that $|L(v)| = 1$ for every $v \in V(P)$. If $V(H) = V(P)$, then by Lemma 91, $\omega_{P,L}(G) = \omega_{H,L}(G) \leq \ell - 2$. This is a contradiction, hence p_0 has a neighbor $w \in V(H) \setminus V(P)$.

If $|L(w)| \geq 4$, then let L' be the list assignment obtained from L by setting $L'(w) = L(w) \setminus L(p_0)$. Note that $G' = G - p_0w$ is P-critical with respect to L', and by the minimality of G, $\omega_{P,L'}(G') \leq \ell - 2$. Let f be the internal face of G incident with p_0w. Suppose that $u \in V(f) \setminus \{w, p_0\}$. If u belongs to $V(H)$, then u is a cutvertex in G', and as shown at the beginning of the proof, u is an inside vertex of P. Therefore, $\omega_{G',P,L'}(u) = 1$ and $\omega_{G,P,L}(u) = 0$. On the other hand, if $u \notin V(H)$, then $\omega_{G',P,L'}(u) = 2$ and $\omega_{G,P,L}(u) = 0$. Using these facts we obtain a contradiction:

$$\begin{aligned}\omega_{P,L}(G) &= \omega_{P,L'}(G') + \omega_{G,P,L}(f) + 1 - \sum_{u\in V(f)\setminus\{w,p_0\}} (\omega_{G',P,L'}(u) - \omega_{G,P,L}(u)) \\ &\leq \omega_{P,L'}(G') + (|f| - 3) + 1 - (|f| - 2) = \omega_{P,L'}(G') \leq \ell - 2.\end{aligned}$$

Next, consider the case that $|L(w)| = 3$ and w is adjacent to a vertex p_i for some $1 \leq i \leq \ell - 1$. Let C be the cycle composed of p_0wp_i and a subpath of P and let G' be the subgraph of G obtained by removing all vertices and edges of $\mathrm{Int}_C(G)$ except for p_iw. Let $P' = (P \cap G') + p_iw$. Note that G' is P'-critical with respect to L. By the minimality of G and Lemma 91, we have

$$\omega_{P,L}(G) = \omega_{P',L}(G') + \omega_{C,L}(\operatorname*{Int}_C(G)) \leq \ell(P') - 2 + |C| - 3 = \ell - 2.$$

Suppose now that w is adjacent to p_ℓ. Note that wp_ℓ is an edge of H and $G \neq H$, hence Lemma 91 implies that $\omega_{P,L}(G) = \omega_{H,L}(G) \leq |H| - 4 = \ell - 2$. This is a contradiction.

Finally, suppose that p_0 is the only neighbor of w in P. Note that $L(p_0) \subset L(w)$, since G is P-critical. Furthermore, w has only one neighbor $z \in V(H)$ distinct from p_0. Let $S = L(w) \setminus L(p_0)$, $G' = G - w$ and let L' be defined by $L'(v) = L(v)$ if v is not a neighbor of w or if $v = p_0$ or $v = z$, and $L'(v) = L(v) \setminus S$ otherwise. Since $|S| = 2$, L' is a valid list assignment with respect to P. Note

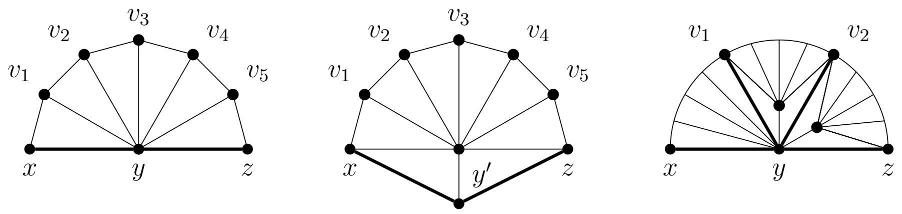


Figure 8.2: A fan, a fat fan, and a fan procession

that G' is not L'-colorable, as every L'-coloring of G' can be extended to an L-coloring of G by coloring w using a color from S different from the color of z. Let G'' be a P-critical subgraph of G'. Let $Q_1, \ldots, Q_m$ be the spans in the outer face of G'' and let G_i be defined as in the proof of (41), for $1 \leq i \leq m$, where $w \in V(G_1)$. The path Q_1 is an edge-disjoint union of paths $M_1, \ldots, M_t$, where the endvertices of M_j are neighbors of w and the inside vertices of M_j are non-adjacent to w for $1 \leq j \leq t$ (with the exception that one of the endvertices of M_t does not have to be adjacent to w). For $1 \leq j \leq t$, let C_j be the cycle or path formed by M_j and the edges between w and M_j and let H_j be the subgraph of G split off by C_j. Note that if v is an inside vertex of M_j, then $\omega_{G,P,L}(v) = 0$ and $\omega_{G'',P,L'}(v) \geq 1$, while endvertices of M_j have the same weight in G and in G''. Furthermore, $\omega_{G,P,L}(w) = 0$. By the minimality of G and Lemma 91, we have

$$\omega_{Q_1,L}(G_1) \leq \sum_{j=1}^{t} \omega_{C_j,L}(H_j) \leq \sum_{j=1}^{t} (\ell(M_j) - 1).$$

Furthermore,

$$\sum_{v \in V(Q_1)} \omega_{G'',P,L'}(v) - \omega_{G,P,L}(v) \geq \sum_{j=1}^{t} (\ell(M_i) - 1) \geq \omega_{Q_1,L}(G_1).$$

We analyze the weights of the other pieces of $G - G'$ in the same way as in the proof of (41) and conclude that $\omega_{P,L}(G) \leq \omega_{P,L'}(G'')$. This contradicts the minimality of G. $\square$

We shall need a more precise description of critical graphs in the case that $\ell(P) = 2$. There are infinitely many such graphs. However, their structure is relatively simple and it is described in the sequel.

For an integer $n \geq 0$, a *fan of order n* with *base xyz* is the graph consisting of the path xyz, a path $xv_1 \ldots v_n z$ and edges yv_i for $1 \leq i \leq n$. For an integer $n \geq 1$, a *fat fan of order n with base xyz* is the graph consisting of the path xyz, a vertex y' adjacent to x, y and z, and a fan of order n with base $xy'z$. A *fan procession with base xyz* is a graph consisting of the path xyz, vertices v_1,

…, v_{k-1} (for some $k \geq 1$) adjacent to y, and subgraphs G_1, …, G_k where for $1 \leq i \leq k$, G_i is either a fan or a fat fan with base $v_{i-1}yv_i$ (where we set $v_0 = x$ and $v_k = z$). Each fan or fan procession is a planar near-triangulation, and we consider its unique face of size ≥ 4 to be the outer face. See Figure 8.2. A fan procession is *even* if all constituent fat fans have even order. A list assignment L for a fan procession G with base xyz and outer face H is *dangerous* if $|L(v)| = 3$ for all $v \in V(H) \setminus \{x, y, z\}$ and $|L(v)| = 5$ for all $v \in V(G) \setminus V(H)$.

Consider a fat fan G of order $n > 0$ with base xyz and a valid list assignment L (with respect to the path xyz). Let y' be the common neighbor of x, y and z, and let $v_1v_2 \ldots v_n$ be the subpath of the outer face from the definition of a fat fan. Suppose that G is not L-colorable, and let φ be an L-coloring of xyz. It is easy to see that the list assignment L must be dangerous. Let $S = L(y') \setminus \{\varphi(x), \varphi(y), \varphi(z)\}$. If there exists $c \in S$ and $1 \leq i \leq n$ such that $c \notin L(v_i)$, then φ extends to an L-coloring of G assigning the color c to y'. Therefore, we have $S \subseteq L(v_i)$ for $1 \leq i \leq n$. Similarly, we have $\varphi(x) \in L(v_1)$ and $\varphi(z) \in L(v_n)$. Since $\varphi(x) \notin S$ and $S \cup \{\varphi(x)\} \subseteq L(v_1)$, we have $|S| = 2$, $\{\varphi(x), \varphi(y), \varphi(z)\} \subset L(y')$ and $\varphi(x) \neq \varphi(z)$. Observe also that $n \geq 2$, as otherwise y' has degree four. Therefore, $\{\varphi(x)\} = L(v_1) \setminus L(v_n)$, $\{\varphi(z)\} = L(v_n) \setminus L(v_1)$ and $\{\varphi(y)\} = L(y') \setminus (L(v_1) \cup L(v_n))$. Therefore, there exists at most one precoloring of xyz that does not extend to an L-coloring of G. Furthermore, if the order n of G is odd, then we can color y' by a color from S and the vertices v_1, v_3, …, v_n by the other color from S and extend this to an L-coloring of G. Therefore, the order of the fat fan G is even.

Using this analysis, it is easy to see that the following holds:

(42) *Let G be a fan procession with base xyz and L a dangerous list assignment for G. If φ_1 and φ_2 are precolorings of xyz that do not extend to an L-coloring of G, and $\varphi_1(x) = \varphi_2(x)$ and $\varphi_1(y) = \varphi_2(y)$, then $\varphi_1 = \varphi_2$. Furthermore, if there exist two different precolorings of xyz that do not extend to an L-coloring of G, then G is a fan.*

Conversely, Thomassen [73] essentially showed that even fan processions with dangerous list assignments are the only plane graphs with valid list assignments that are P-critical for a path P of length two.

Lemma 93. *Let G be a plane graph with outer face H and P a subpath of H of length two. Let L be a list assignment valid with respect to P. If G is P-critical with respect to L, then G is an even fan procession with base P and L is dangerous.*

Proof. By Theorem 92, G is 2-connected, all faces other than H are triangles and all vertices in $V(H) \setminus V(P)$ have list of size three. Since G is P-critical, there exists an L-coloring of P that does not extend to an L-coloring of G. By Theorem 3 of [73], there exists a fan procession $G' \subseteq G$ with base P and L is a dangerous list assignment for G'. By Lemma 91, every triangle in G bounds a face. Furthermore, Theorem 88 implies that every chord of H is incident with the middle vertex of P. We conclude that $G = G'$, and thus G is a fan procession with base P. Furthermore, since an L-coloring of P does not extend to an L-coloring of G, the fan procession is even, as we have argued before. $\square$

8.3 Reducing the precolored vertices

One could hope that the proof of Theorem 92 can be modified to deal with the situation that G contains sufficiently distant precolored vertices. Most of the inductive applications deal with the situations which reduce the length of the precolored path, and if the distance between the new precolored path (one of the spans) from the old one is guaranteed to be bounded by a constant, we could ensure that the distance between P and the precolored vertices is at least some function of $\ell(P)$. However, the fact that there are infinitely many critical graphs makes it difficult to prove such a constraint on the distance.

To avoid this problem, we restrict ourselves to working with list assignments such that the vertices with list of size three form an independent set. In this setting, we easily conclude by combining Theorem 92 with Lemma 91 that the size of critical graphs is bounded.

Lemma 94. *Let G be a plane graph whose outer face is H, let P be a subpath of H and let L be a list assignment valid with respect to P, such that no two vertices with lists of size three are adjacent. If G is P-critical, then $|V(G)| \leq 8\ell(P)^2$.*

Proof. By induction, we can assume that no cut-vertex belongs to P, and thus G is 2-connected. The claim is true if $V(G) = V(P)$, thus assume that $V(G) \neq V(P)$. For $i \in \{3, 4, 5\}$, let n_i denote the number of vertices with list of size i in $V(H) \setminus V(P)$. We have $\omega_{P,L}(G) \geq n_4 + 2n_5$. Let Q be a path of length $n_3 + 2$ whose endvertices coincide with the endvertices of P, but is otherwise disjoint from G, and let G' be the graph obtained from $G \cup Q$ by joining each vertex $v \in V(H) \setminus V(P)$ with $5 - |L(v)|$ vertices of Q in the planar way. Let L_Q be the list assignment to the inside vertices of Q such that each such vertex has a single color that does not appear in any other list (including the lists of vertices of G). Let L' be the list assignment for G' that matches L_Q on the inside vertices of Q and the list of each vertex $v \in V(G) \setminus V(P)$ consists of $L(v)$ and the colors of the adjacent inside vertices of Q. Observe that G' is $(P \cup Q)$-critical, and by Lemma 91,

$$\frac{|V(G) \setminus V(P)|}{2|P \cup Q| + 2} = \frac{|V(G') \setminus V(P \cup Q)|}{2|P \cup Q| + 2} \leq |P \cup Q| - 9/2.$$

This implies that $|V(G) \setminus V(P)| \leq 2(|P \cup Q| - 1)^2 - |P \cup Q|$, and therefore $|V(G)| \leq 2(|P \cup Q| - 1)^2$. Since L is valid, since no two vertices with list of size three are adjacent, and since G is 2-connected, we have $n_3 \leq n_4 + n_5 + 1$.

Consequently, $\ell(Q) \leq n_4 + n_5 + 3 \leq \omega_{P,L}(G) + 3$. Since $\omega_{P,L}(G) \leq \ell(P) - 2$ by Theorem 92, we have that $|P \cup Q| \leq 2\ell(P) + 1$, and the claim follows. $\square$

Let us remark that a converse of the transformation described in the proof of Lemma 94 can be used to generate all critical graphs satisfying the assumptions of the lemma with the length of P fixed.

Our aim in this section is to show that Lemma 89 implies a positive answer to Problem 85. We need to introduce several technical definitions.

Let G be a plane graph with outer face H and Q a path in G. Suppose that $Q = q_0q_1 \dots q_k$ and $q_0 \in V(H)$. For $0 < i < k$, let L_i and R_i be the sets of edges of G incident with q_i such that the cyclic clockwise order (according to the drawing of G in the plane) of the edges incident with q_i is q_iq_{i+1}, R_i, q_iq_{i-1}, L_i. We define L_0 and R_0 similarly, except that we consider the face H instead of the edge q_iq_{i-1}. We define G^Q as the graph obtained from G by splitting the vertices along Q in the natural way, i.e., so that Q corresponds to paths $Q_L = q_0^Lq_1^L \dots q_{k-1}^Lq_k$ and $Q_R = q_0^Rq_1^R \dots q_{k-1}^Rq_k$ and for $0 \leq i < k$, the vertex q_i^L is incident with the edges in L_i and the vertex q_i^R is incident with the edges in R_i. If G is given with a list assignment L, then let L^Q be the list assignment for G^Q such that $L^Q(q_i^L) = L^Q(q_i^R) = L(q_i)$ for $0 \leq i < k$ and $L^Q(v) = L(v)$ for other vertices of G^Q. We say that G^Q is *obtained by cutting along* Q.

For integers M and k, let $D(M,k) = M + 1$ if $k \leq 1$ and $D(M,k) = D(M,k-1) + 16k^2$ if $k \geq 2$. Note that there is a simple explicit formula for the values $D(M,k)$, but we shall only use its recursive description. A set X of vertices is *M-scattered* if the distance between any two elements of X is at least $\max\{D(M,2M+11), D(M,2) + D(M,6) + 1\}$.

Let $Q = q_0q_1 \dots q_k$ be a path of length k. If k is even, then $q_{k/2}$ is said to be the *central vertex* of Q; if k is odd, then each of the two vertices $q_{(k-1)/2}$ and $q_{(k+1)/2}$ is a *central vertex* of Q.

The aim of this section is to show that Lemma 89 implies our main result (Theorem 86).

Lemma 95. *Suppose that there is a positive integer M such that the conclusion of Lemma 89 holds. Let G be a plane graph, let P be a subpath of its outer face H and let p be a central vertex of P. Let X be an M-scattered subset of $V(G)$ such that the distance between p and X is at least $D(M,\ell(P))$. Let L be a list assignment for G that is M-valid with respect to P and X. Furthermore, assume that there is at most one edge $uv \in E(G)$ such that $u,v \in V(G) \setminus V(P)$ and $|L(u)| = |L(v)| = 3$, and if such an edge exists, then $\ell(P) \leq 1$, u or v is adjacent to p and the distance between p and X is at least $D(M,2) - 1$. If G is P-critical with respect to L, then $X = \emptyset$.*

Proof. For a contradiction, suppose that G is a counterexample to Lemma 95 with the smallest number of edges that do not belong to P, subject to that, with the smallest number of vertices, and subject to that, with the largest number of vertices in P. Since G is P-critical, every vertex $v \in V(G) \setminus V(P)$ satisfies $\deg(v) \geq |L(v)|$. Let $\ell = \ell(P)$ and $P = p_0p_1 \ldots p_\ell$. If ℓ is odd, choose the labels so that $p = p_{(\ell+1)/2}$.

Suppose that G is disconnected. Since it is P-critical, it has two components: one is equal to P and the other one, G', is not L-colorable. Choose $v \in V(H) \cap V(G')$ arbitrarily, and let P' be the path consisting of v. Note that G' is P'-critical. If G' with the path P' satisfies the assumptions of Lemma 95, then by the minimality of G we have $X \cap V(G') = \emptyset$, and thus $X = \emptyset$. This is a contradiction, and thus the distance from v to the closest vertex $x \in X$ is at most M. Let Q be the shortest path between v and x and let G^Q, Q_L and Q_R with the list assignment L^Q be obtained from G by cutting along Q. Let $Q' = Q_L \cup Q_R$ and $X' = X \setminus \{x\}$. Note that x is the central vertex of Q' and its distance to any $u \in X'$ is at least $D(M, \ell(Q'))$, since X is M-scattered and $\ell(Q') \leq 2M$. In particular, L^Q is M-valid with respect to Q' and X'. Furthermore, G^Q is Q'-critical with respect to L^Q. To see this, consider an arbitrary edge $e \in E(G') \setminus E(Q)$. Since G is P-critical, there exists an L-coloring of P that extends to $G - e$ but not to G. The coloring of $G - e$ induced on Q gives rise to an L^Q-coloring of Q' that extends to $G^Q - e$ but not to G^Q. This shows that G^Q is Q'-critical. Since the distances in G^Q are not shorter than those in G, the graph G^Q satisfies all assumptions of Lemma 95. By the minimality of G, we conclude that $X' = \emptyset$. But then $|X| = 1$ and G' contradicts Lemma 89.

Therefore, G is connected. In particular, if $\ell = 0$, then we can include another vertex of H in P; therefore, we will assume that $\ell \geq 1$. Since G is connected, its outer face H has a facial walk, which we write as $p_\ell \ldots p_1p_0v_1v_2v_3 \ldots v_s$.

Suppose that the distance between P and X is at most $M + 5$. Then the distance from p to X is at most $M + \ell + 5$. By the assumptions of the lemma, this distance is at least $D(M, \ell)$, which is only possible if $\ell \leq 1$. As assumed above, this means that $\ell = 1$. Moreover, the assumptions of the lemma imply that no two vertices with list of size three are adjacent. Let Q be a shortest path between P and a vertex $x \in X$. Let G^Q, Q_L and Q_R with the list assignment L^Q be obtained from G by cutting along Q. Let Q' be the path consisting of $Q_L \cup Q_R$ and of the edge of P, and let $X' = X \setminus \{x\}$. Note that $\ell(Q') \leq 2M + 11$. Since X is M-scattered, so is X', and the distance in G^Q from the central vertex x of Q' to X' is at least $D(M, 2M + 11) \geq D(M, \ell(Q'))$. As in the previous paragraph, we conclude that since G^Q is Q'-critical with respect to L^Q, we have $X' = \emptyset$. Then $|X| = 1$ and, consequently, G contradicts the postulated property of the constant M. Therefore, we conclude:

(43) *The distance between P and X is at least $M + 6$.*

Let $T = t_1 \ldots t_k$ be a separating k-cycle in G that is distinct from H and where $k \leq 4$. Suppose that the distance from t_1 to P is at most $6 - k$. Let us choose such a cycle with $\mathrm{Int}_T(G)$ minimal; it follows that T is an induced cycle. By Lemma 90, $\mathrm{Int}_T(G)$ is T-critical, and thus there exists an L-coloring ψ of T

that does not extend to an L-coloring of $\mathrm{Int}_T(G)$. Let $G' = \mathrm{Int}_T(G) - \{t_3, \ldots, t_k\}$. Let L' be the list assignment for G' such that $L'(t_1) = \{\psi(t_1)\}$, $L'(t_2) = \{\psi(t_2)\}$ and $L'(v) = L(v) \setminus \{\psi(t_i) \mid vt_i \in E(G), 3 \leq i \leq k\}$ for other vertices $v \in V(G')$. Note that no two vertices with list of size three are adjacent in G', as otherwise we have $k = 4$ and t_3t_4 is incident with a separating triangle contradicting the choice of T. Note that G' is not L'-colorable, hence it contains a t_1t_2-critical subgraph G''. By (43), L' is an M-valid list assignment for G'', and the distance between t_1t_2 and $X \cap V(G')$ is at least $M + 1$. By the minimality of G, it follows that $X \cap V(G'') = \emptyset$. However, then G'' contradicts Theorem 88. We conclude that the following holds:

(44) *If $T \neq H$ is a separating k-cycle in G, where $k \leq 4$, then the distance between T and P is at least $7 - k$.*

Similarly, by applying induction, we obtain the following property.

(45) *If R is either a chord of H that does not contain an internal vertex of P, or R is a cut-vertex of G, then the distance between R and P is at least* 4.

Proof. Suppose first that R does not contain an internal vertex of P. Let G' be an R-component in G that contains no edges of P. By Lemma 90, G' is R-critical, and Theorem 88 implies that $X \cap V(G') \neq \emptyset$. If the distance from P to R is at most 3, then by (43), the distance between R and X is at least $M + 1 = D(M, \ell(R))$. If $G' - V(R)$ does not contain two adjacent vertices with list of size three, this contradicts the minimality of G. If $uv \in E(G' - V(R))$ and $|L(u)| = |L(v)| = 3$, then by the assumptions, we have $\ell = 1$ and u is adjacent to p. Consequently, $p \in V(R)$, and thus the distance between a central vertex p of R and X is at least $D(M, 2) - 1$. Again, we have a contradiction with the minimality of G.

Suppose now that P contains a cut-vertex v of G, and let G_1 and G_2 be the two maximal connected subgraphs of G that intersect in v. For $i \in \{1, 2\}$, let $P_i = P \cap G_i$ and note that either $P_i = G_i$ or G_i is P_i-critical by Lemma 90. By the minimality of G, we conclude that neither G_1 nor G_2 contains a vertex of X, and thus $X = \emptyset$. This contradiction completes the proof. □

Next, we claim the following.

(46) *Let $C \subset G$ be a cycle of length at most $\ell + 1$ such that $C \neq H$ and the distance between C and p is at most $8\ell^2$. Then $\mathrm{Int}_C(G)$ contains no vertices of X.*

Proof. The length of C is at least three, and thus $\ell \geq 2$. If $x \in X$ belongs to C, then the distance from x to p is less than $8\ell^2 + \ell < D(M, \ell)$, a contradiction. Thus, we may assume that $V(C) \cap X = \emptyset$ and, in particular, that C does not bound a face. If $\ell(C) \leq \ell$, then the claim holds even under a weaker assumption that the distance between C and P is at most $16\ell^2$. Indeed, consider a spanning subpath Q of C of length $\ell(C)-1$ such that the distance between p and a central vertex q of Q is at most $16\ell^2$. The distance of every vertex of X in $\mathrm{Int}_C(G)$ from q is at least $D(M,\ell)-16\ell^2 \geq D(M,\ell(Q))$. By Lemma 90, we have that $\mathrm{Int}_C(G)$ is Q-critical, and the claim follows by the minimality of G.

Suppose now that $\ell(C)=\ell+1$ and let $C=c_0c_1\ldots c_\ell$, where $c_{\lceil \ell/2\rceil}$ is the vertex of C nearest to p. There exists an L-coloring φ of C that does not extend to an L-coloring of $\mathrm{Int}_C(G)$. Let d be a new color that does not appear in any of the lists and let L' be the list assignment obtained from L by replacing $\varphi(c_\ell)$ by d in the lists of c_ℓ and its neighbors and by setting $L'(c_0)=\{\varphi(c_0),\varphi(c_1),d\}$. Let φ' be the coloring of the path $C'=c_1c_2\ldots c_\ell$ such that $\varphi'(c_\ell)=d$ and φ' matches φ on the rest of the vertices. The coloring φ' does not extend to an L'-coloring of $\mathrm{Int}_C(G)$; hence, $\mathrm{Int}_C(G)$ contains a subgraph $F\supset C'$ that is C'-critical with respect to L'. The distance of $X\cap V(F)$ from the central vertex $c_{\lceil \ell/2\rceil}$ of C' is at least $D(M,\ell)-8\ell^2 > D(M,\ell(C'))$. By the minimality of G, we conclude that F contains no vertex of X. By Theorem 92, we have $\omega_{C',L'}(F)\leq \ell-3$, and in particular, every face of F has length at most ℓ. By Lemma 94, the distance from $c_{\lceil \ell/2\rceil}$ to every vertex of F is less than $8\ell^2$, thus the distance between every vertex of F and p is at most $16\ell^2$. By the previous paragraph, we conclude that no vertex of X appears in the interior of any face of F. Let Q be the path in the outer face of F, distinct from C', joining c_1 with c_ℓ. If $v\neq c_0$ is an inside vertex of Q, then $\omega_{F,C',L'}(v)\geq 1$, hence Q contains at most $\ell-3$ such inside vertices. It follows that $Q+c_1c_0c_\ell$ is either a cycle of length at most ℓ (if $c_0\notin V(Q)$) or a union of two cycles of total length at most $\ell+1$ (if $c_0\in V(Q)$). In both cases, the interiors of the cycles do not contain any vertex of X by the previous paragraph. Consequently, $X\cap V(\mathrm{Int}_C(G))=\emptyset$ as claimed. □

Let ψ be an L-coloring of P that does not extend to an L-coloring of G. Suppose that there exists a proper subgraph $F\subset G$ such that $P\subset F$ and ψ does not extend to an L-coloring of F. Let F be minimal subject to this property. Then F is a P-critical graph. If F does not satisfy the assumptions of Lemma 95, then $\ell=1$ and there exist adjacent vertices $u,v\in V(F)\setminus V(P)$ with lists of size three such that neither of them is adjacent to p_1 in F, while say u is adjacent to p_1 in G. Let c be a new color that does not appear in any of the lists. Let L' be the list assignment for F obtained from L by replacing $\psi(p_1)$ by c in the lists of all vertices adjacent to p_1 in F and by setting $L'(p_0)=\{\psi(p_0)\}$, $L'(p_1)=\{c\}$, and $L'(u)=L(u)\cup\{c\}$. Note that each L'-coloring of $F+up_1$ corresponds to an L-coloring of F extending ψ, hence $F+up_1$ is not L'-colorable and it contains a P-critical subgraph F'. Note that $|L'(u)|=4$ and hence no two vertices with list of size three are adjacent in F'. However, the minimality of G implies that F' contains no vertices of X, and we obtain a contradiction with Theorem 88.

Therefore, we can assume that F satisfies the assumptions of Lemma 95, and the minimality of G implies that F contains no vertices of X. By Theorem 92, it follows that $\omega_{P,L}(F) \leq \ell - 2$, and in particular, $\ell \geq 2$. Let f be a face of F distinct from the outer one such that $\mathrm{Int}_f(G) \neq f$. Since $\omega(f) \leq \ell - 2$, we have $\ell(f) \leq \ell + 1$. Furthermore, by Lemma 94, the distance in F between f and p is at most $8\ell^2$. By (46), no vertex of X appears in $\mathrm{Int}_f(G)$.

Consider now a span[2] Q forming a subpath of the outer face of F. Each internal vertex $v \in V(Q)$ satisfies $\omega_{F,P,L}(v) \geq 1$, hence $\ell(Q) \leq \omega_{P,L}(F)+1 \leq \ell-1$. Let G_Q be the Q-component of G split off by Q and let q be a central vertex of Q. By Lemma 94, the distance between p and q in F is at most $8\ell^2$, and thus the distance between q and X in G_Q is at least $D(M, \ell) - 8\ell^2 \geq D(M, \ell(Q))$. Observe that G_Q is Q-critical if $G_Q \neq Q$, and by the minimality of G, G_Q contains no vertices of X.

Since G is the union of $\mathrm{Int}_f(G)$ over the faces of F and G_Q over the spans Q contained in the boundary of the outer face of F, we conclude that $X = \emptyset$. This is a contradiction; therefore, ψ extends to all proper subgraphs of G that contain P. Therefore, we can assume that the following holds.

(47) *The vertices of P have lists of size one, G is not L-colorable and every proper subgraph of G that contains P is L-colorable.*

Let us fix ψ as the unique L-coloring of P.

Consider a chord $e = uv$ of H at distance at most three from P. By (45), we can assume that u is an inside vertex of P, and in particular $\ell \geq 2$. If v belonged to P as well, then by (47) we have $G = P + e$, implying $X = \emptyset$. Hence, v does not belong to P.

Let G_1 and G_2 be the maximal connected subgraphs of G intersecting in uv, such that $G_1 \cup G_2 = G$ and $p_0 \in V(G_2)$. Let $P_i = (P \cap G_i) + e$. For $i \in \{1, 2\}$, Lemma 90 implies that the graph G_i is P_i-critical. Note that either $\ell(P_i) < \ell(P)$, or $\ell(P_i) = \ell$, in which case p is a central vertex of P_i. We conclude that the distance between a central vertex of P_i and X is at least $D(M, \ell(P_i))$. By the minimality of G, we have $X \cap V(G_i) = \emptyset$ for $i \in \{1, 2\}$. Therefore, $X = \emptyset$, which is a contradiction. Therefore, we have:

(48) *The distance of any chord of H from P is at least four.*

In particular, we have $s = |V(H) \setminus V(P)| \geq 1$. Another consequence of (47) is the following:

(49) *If $|L(v_1)| > 3$, then $|L(v_1)| = 4$, $s \geq 2$ and $|L(v_2)| = 3$.*

Otherwise, suppose that $|L(v_1)| = 5$, or $|L(v_1)| = 4$ and either $s = 1$ or $|L(v_2)| \geq 4$. Let $G' = G - p_0v_1$ and let L' be the list assignment obtained from L by removing $\psi(p_0)$ from the list of v_1. The assumptions together with (48) imply that no two vertices with list of size three are adjacent in G'. By (47), G' is P-critical with respect to L', contradicting the minimality of G.

[2]Recall that a span, as defined in the proof of Theorem 92, is a subwalk of F and starts and ends with a vertex in H.

Suppose now that $\ell \geq 2$ and a vertex v is adjacent to p_0, p_1 and p_2. By (48), we have $v \notin V(H)$. Let $P' = p_0 v p_2 p_3 \dots p_\ell$ and $G' = G - p_1$. By Lemma 90, G' is P'-critical. If $\ell \geq 3$, then p is a central vertex of P' and by the minimality of G, we have $X \cap V(G') = \emptyset$. Furthermore, (44) implies that $p_0 p_1 v$ and $p_1 p_2 v$ are faces of G, and thus $X = \emptyset$. This contradiction shows the following.

(50) *If $\ell \geq 2$ and p_0, p_1 and p_2 have a common neighbor, then $\ell = 2$.*

For a vertex $v \in V(G) \setminus V(P)$, let

$$S(v) = L(v) \setminus \{\psi(r) : r \in V(P), vr \in E(G)\}.$$

If v is a vertex of $V(G) \setminus V(P)$ with k neighbors in P, then $|S(v)| = |L(v)| - k$. To see this, suppose v is adjacent to a vertex $r \in V(P)$ and $\psi(r) \notin L(v)$, or v is adjacent to two vertices $r, r' \in V(P)$ with $\psi(r) = \psi(r')$. Then we can remove the edge vr and obtain a contradiction to the last assertion in (47).

Consider a nonempty set $Y \subseteq V(G) \setminus V(P)$ and a partial coloring φ of the subgraph of G induced by Y from the lists given by S. We define L_φ as the list assignment such that

$$L_\varphi(z) = L(z) \setminus \{\varphi(y) : y \in \mathrm{dom}(\varphi), yz \in E(G), \varphi(y) \in S(z)\}$$

for every $z \in V(G - Y)$.

We now define a set $Y \subseteq V(F) \setminus V(P)$ and a partial L-coloring φ of Y as follows:

(Y1) If $|L(v_1)| = 3$ and one of the following holds:

- $s = 1$, or
- $s \geq 2$ and $|L(v_2)| = 5$, or
- $s = 2$ and $|L(v_2)| = 4$, or
- $s \geq 3$, $|L(v_2)| = 4$ and $|L(v_3)| \neq 3$,

then $Y = \{v_1\}$ and $\varphi(v_1) \in S(v_1)$ is chosen arbitrarily.

(Y2) If $|L(v_1)| = 3$, $s \geq 3$, $|L(v_2)| = 4$ and $|L(v_3)| = 3$, then $Y = \{v_1, v_2\}$ and φ is chosen so that $\varphi(v_2) \in L(v_2) \setminus L(v_3)$ and $\varphi(v_1) \in S(v_1) \setminus \{\varphi(v_2)\}$.

(Y2a) If $s = 2$ and $|L(v_1)| = |L(v_2)| = 3$, then $Y = \{v_1, v_2\}$ and $\varphi(v_1) \in S(v_1)$ and $\varphi(v_2) \in S(v_2)$ are chosen arbitrarily so that $\varphi(v_1) \neq \varphi(v_2)$.

(Y3) If $|L(v_1)| = 4$, $s \geq 2$, $|L(v_2)| = 3$, and one of the following holds:

- $s \leq 3$, or
- $s \geq 4$ and $|L(v_3)| = 5$, or
- $s \geq 4$ and $|L(v_4)| \neq 3$,

then $Y = \{v_2\}$. If $s = 3$ and $|L(v_3)| = 3$, then $\varphi(v_2)$ is chosen in $L(v_2) \setminus S(v_3)$, otherwise $\varphi(v_2) \in S(v_2)$ is chosen arbitrarily.

(Y4) If $s \geq 4$, $|L(v_1)| = 4$, $|L(v_2)| = 3$, $|L(v_3)| = 4$ and $|L(v_4)| = 3$, then:

(Y4a) If v_1, v_2 and v_3 do not have a common neighbor, then $Y = \{v_2, v_3\}$ and φ is chosen so that $\varphi(v_3) \in L(v_3) \setminus L(v_4)$ and $\varphi(v_2) \in L(v_2) \setminus \{\varphi(v_3)\}$.

(Y4b) If v_1, v_2 and v_3 have a common neighbor, then $Y = \{v_1, v_2, v_3\}$ and φ is chosen so that $\varphi(v_3) \in L(v_3) \setminus L(v_4)$, $\varphi(v_1) \in S(v_1)$ and either at least one of $\varphi(v_1)$ and $\varphi(v_3)$ does not belong to $L(v_2)$, or $\varphi(v_1) = \varphi(v_3)$. The vertex v_2 is left uncolored.

By using (49) it is easy to see that Y and φ are always defined. We remark that the following is true.

(51) *Every L_φ-coloring of $G - Y$ extends to an L-coloring of G.*

Indeed, this is obviously true if $\mathrm{dom}(\varphi) = Y$. The only case when $\mathrm{dom}(\varphi) \neq Y$ is (Y4b), where $Y = \{v_1, v_2, v_3\}$ and $\mathrm{dom}(\varphi) = \{v_1, v_3\}$. However, $\deg(v_2) = 3$ by (44), and $|L_\varphi(v_2)| \geq 2$ by the choice of $\varphi(v_1)$ and $\varphi(v_3)$. This implies that any L_φ-coloring of $G - Y$ extends to v_2 and proves (51). Consequently, $G - Y$ is not L_φ-colorable. We let G_φ be a P-critical subgraph of $G - Y$.

Using (43) and (48), it is easy to verify that the choice of Y and φ ensures that L_φ is M-valid with respect to P and X. Let us now distinguish two cases depending on whether G_φ contains two adjacent vertices with list of size three (that did not have lists of size three in G as well) or not.

- Suppose first that **no two vertices** $u, v \in V(G_\varphi)$ **such that** $|L_\varphi(u)| = |L_\varphi(v)| = 3$ **and** $\max(|L(u)|, |L(v)|) > 3$ **are adjacent.** If G_φ with the list assignment L_φ does not satisfy the assumptions of Lemma 95, then $\ell = 1$, $|L(v_s)| = |L(v_{s-1})| = 3$, $v_s v_{s-1} \in E(G_\varphi)$ and $p_1 v_s \notin V(G_\varphi)$. Let c be a new color that does not appear in any of the lists and let L' be the list assignment obtained from L_φ by replacing $\psi(p_1)$ with c in the lists of vertices adjacent to p_1 in G_φ and by setting $L'(p_1) = \{c\}$ and $L'(v_s) = L(v_s) \cup \{c\}$. Observe that $G_\varphi + p_1 v_s$ is not L'-colorable and thus it contains a P-critical subgraph G'. By the minimality of G, we have $X \cap V(G') = \emptyset$. However, then G' contradicts Theorem 88.

  Therefore, G_φ with the list assignment L_φ satisfies the assumptions of Lemma 95. By the minimality of G, we conclude that G_φ does not contain any vertex of X. By Theorem 92, we have $\ell \geq 2$ and $\omega_{P,L_\varphi}(G_\varphi) \leq \ell - 2$. Let Q be the span contained in the outer face of G_φ such that its Q-component G_Q contains Y. Analogically to the proof of (47), we argue that if f is a face of G_φ, then $\mathrm{Int}_f(G)$ contains no vertex of X, and that if Q' is a span different from Q, then the subgraph of G split off by Q' contains no vertex of X. Since $X \neq \emptyset$, it follows that G_Q contains a vertex of X. By the minimality of G, we conclude that $\ell(Q) \geq \ell$.

If v is an inside vertex of Q, then $\omega_{P,L_\varphi}(v) \geq 1$, unless $|L_\varphi(v)| = 3$. Since the sum of the weights of the inside vertices of Q is at most $\omega_{P,L_\varphi}(G_\varphi) \leq \ell - 2$, we conclude that at least one inside vertex of Q has list of size three. This is only possible in the cases (Y2), (Y4a) and (Y4b). Furthermore, observe that only one inside vertex of Q has list of size three by (44); let v denote this vertex. It follows that $\ell(Q) = \ell$ and that all inside vertices of Q other than v either belong to P or have list of size four.

Let y_1 and y_2 be the neighbors of v in $\mathrm{dom}(\varphi)$, where y_1 is closer to p_0 than y_2. Let Q_1 and Q_2 be the subpaths of Q intersecting in v (where Q_1 is closer to p_0 than Q_2) and let Q'_1 and Q'_2 be the paths obtained from them by adding the edge y_1v. For $i \in \{1,2\}$, if $\ell(Q'_i) < \ell - 1$, then as in (47), we conclude that the subgraph of G split off by Q'_i does not contain any vertex of X. Since $X \neq \emptyset$ and $\ell(Q'_1) + \ell(Q'_2) = \ell(Q) + 2 = \ell + 2$, it follows that $\ell(Q_1) = 1$ or $\ell(Q_2) = 1$.

If for some $i \in \{1,2\}$, we have $\ell(Q_i) > 1$ and an inside vertex z of Q_i is adjacent to y_i, then Q_i is an edge-disjoint union of paths Q'_i and Q''_i such that Q'_i together with vy_iz forms a cycle C of length at most ℓ and $Q''_i + zy_i$ is a k-chord of H for some $k \leq \ell - 1$. By considering the interior of C and the subgraph of G split off by $Q''_i + zy_i$ separately, we again conclude that the subgraph of G split off by Q does not contain any vertex of X. This is a contradiction. It follows that no inside vertex of Q_i is adjacent to y_i, and thus no inside vertex of Q_i has a list of size four. Therefore, all inside vertices of Q except for v belong to P.

If $\ell(Q_1) > 1$, then let $Q_2 = vw$ and consider the subgraph F of G split off by y_1vw. Note that $\ell = \ell(Q) \geq 3$ and the distance between v and X is at least $D(M,\ell) - \lceil \ell/2 \rceil - 3 \geq D(M,2)$. By the minimality of G, it follows that F contains no vertex of X. By Theorem 92, we have $\omega_{y_1vw,L}(F) = 0$. This is a contradiction, since in each of the cases (Y2), (Y4a) and (Y4b), $F - \{y_1, v, w\}$ contains a vertex with list of size four.

Therefore, $\ell(Q_1) = 1$. In case (Y4a), v is not adjacent to v_1, and thus v is adjacent to p_0. Similarly, in case (Y4b), v is adjacent to p_0, since v_1 belongs to Y. Since v_1 has list of size four, it has degree at least four, and thus at least one vertex of G is drawn inside the 4-cycle $v_1v_2vp_0$. This contradicts (44). Suppose now that (Y2) holds. Since $\ell(Q_2) = \ell - 1$, we conclude that $Q_2 = vp_2p_3 \ldots p_\ell$. By Lemma 93, we have that G_φ consists of P and v adjacent to p_0, p_1 and p_2. By (50), $\ell = 2$. Let us postpone the discussion of this case,

(52) *(Y2) holds, $\ell = 2$ and p_0, p_1, p_2, v_1 and v_2 have a common neighbor.*

- Let us now consider the case that **two vertices** $u, v \in V(G_\varphi)$ **with** $|L_\varphi(u)| = |L_\varphi(v)| = 3$ **and** $|L(v)| > 3$ **are adjacent.** By (44), at most one of u and v has two neighbors in $\mathrm{dom}(\varphi)$. If neither u nor v has two neighbors in $\mathrm{dom}(\varphi)$, then $u, v \in V(H)$ and the choice of Y and φ ensures that uv is a chord of H. However, that contradicts (48). Thus, we can assume that v has two neighbors in $\mathrm{dom}(\varphi)$ and $v \notin V(H)$; and in particular, Y was chosen according to one of the cases (Y2), (Y4a) or (Y4b) (the case (Y2a) is excluded, since in that case G_φ would contain at most one vertex with list of size three). Since u has at most one neighbor in $\mathrm{dom}(\varphi)$ and $|L_\varphi(u)| = 3$, we have $u \in V(H)$. If $|L(u)| = 4$, then u has a neighbor $y \in \mathrm{dom}(\varphi)$, and by (48), we have $uy \in V(H)$. This is not possible (in the case (Y4a), the vertex v_1 has list of size four, but it is not adjacent to v). Therefore, $|L(u)| = 3$. Furthermore, observe that u has no neighbor in Y, as otherwise G would contain a 4-cycle $y'yuv$ with $y, y' \in Y$ and $|L(y)| = 4$; hence, y would have degree at least four, contradicting (44).

  Let $y_1, y_2 \in \mathrm{dom}(\varphi)$ be the neighbors of v, where y_1 is closer to p_0 than y_2. Let F be the uvy_1-component of G that does not contain P, and assume that u was chosen so that F is as small as possible. Note that $\omega_{uvy_1,L}(F) \geq 1$, as $|L(y_2)| = 4$. The minimality of G and Theorem 92 implies that a vertex $x \in X \cap V(F)$ is in distance at most $D(M, 2) - 1$ from v. In particular, we have $\ell \leq 2$.

  Let Q be the path consisting of P, the subpath of H between p_0 and y_1 and the path y_1vu. If $|L(v_s)| = |L(v_{s-1}| = 3$ and $u \neq v_s$, include also the edge p_1v_s in Q. Let G_Q be the uvv_1-component of G different from F. Note that $\ell(Q) \leq 6$. Since the distance between v and a vertex of $X \cap V(F)$ is at most $D(M, 2) - 1$ and X is M-scattered, the distance between a central vertex of Q and $X \cap V(G_Q)$ is at least $D(M, 6)$. By the minimality of G, we conclude that G_Q contains no vertex of X.

  Consider now the graphs $G'_Q = G_Q - Y$ and $F' = F - Y$ with list assignment L_φ. By the choice of u (so that F is minimal), no two adjacent vertices of F' other than u and v have lists of size three. Furthermore, the distance between v and X is at least $M + 3 > D(M, 1)$ by (43). By the minimality of G, no uv-critical subgraph of F' (with respect to the list assignment L_φ) contains a vertex of X, and by Theorem 88 we conclude that every L_φ-coloring of uv extends to an L_φ-coloring of F'. Since G_φ is not L_φ-colorable, it follows that G'_Q is not L_φ-colorable. By Theorem 88 this is not possible if $\ell = 1$, and thus $\ell = 2$.

  Note that if xy is an edge of G'_Q and $|L_\varphi(x)| = |L_\varphi(y)| = 3$, then x or y is equal to v. Lemma 93 and (48) implies that either v is adjacent to all vertices of P, or v is adjacent to p_0 and v_s, $|L(v_s)| = 3$ and p_0, p_1, p_2, v and v_s have a common neighbor. This is not possible in the cases (Y4a) and (Y4b), since v_1 cannot have degree less than four. We are left with the case that either the configuration described in (52) appears, or

(53) *(Y2) holds, $\ell = 2$, the common neighbor v of v_1 and v_2 is adjacent to p_0 and v_s, $|L(v_s)| = 3$, v_s is adjacent to p_2 and there exists a vertex w adjacent to $V(P) \cup \{v, v_s\}$.*

Since either (52) or (53) holds, we have $\ell = 2$ and there exists a vertex w adjacent to all vertices of P, where $w = v$ if (52) holds. In particular, no two vertices with list of size three are adjacent in G and P has a unique central vertex. Therefore, by symmetry we also have

(54) *$|L(v_s)| = 3$, $|L(v_{s-1})| = 4$ and w is either adjacent to v_{s-1} and v_s, or adjacent to v_1 and the common neighbor v' of v_{s-1}, v_s and p_2.*

Observe that (53) contradicts (54), as w does not have a neighbor with list of size four (thus w is not adjacent to v_{s-1}) and v is the only neighbor of w with list of size five and v is not adjacent to p_2 (excluding the existence of v').

Therefore, (52) holds and v is also adjacent to v_{s-1} and v_s. Let us choose $c_1 \in S(v)$ and $c_2 \in S(v_1)$ arbitrarily so that $c_1 \neq c_2$. Let L' be the list assignment such that $L'(v_2) = L(v_2) \setminus \{c_2\}$, $L'(v) = \{c_1\}$ and $L'(z) = L(z)$ for any other vertex z. Let $G' = G - \{p_1, p_0, v_1\}$ and $P' = p_2 v$. Note that each L'-coloring of G' corresponds to an L-coloring of G, and thus G' is not L'-colorable. Let G'' be a P'-critical subgraph of G'. The only adjacent vertices of G'' with lists of size three are v_2 and v_3. Also, the distance between v and X is at least $D(M, 2) - 1$. If $vv_2 \in E(G'')$, then G'' satisfies the assumptions of Lemma 95, and by the minimality of G, we have $X \cap V(G'') = \emptyset$. However, then G'' contradicts Theorem 88.

Finally, suppose that $vv_2 \notin E(G'')$. Let d be a new color that does not appear in any of the lists, and let L'' be the list assignment obtained from L' by replacing c_1 with d in the lists of v and its neighbors in G'' and by setting $L''(v_2) = L(v_2) \cup \{d\}$. Observe that $G'' + vv_2$ is not L''-colorable, and since no two vertices of $G'' + vv_2$ with lists of size three are adjacent, we again obtain a contradiction with the minimality of G and Theorem 88. □

Part III

Dealing with distant perturbations

How robust are the results on the colorability of say planar graphs? That is, suppose that we perform "perturbations" on graphs in one of the considered classes of graphs which are sufficiently far apart (e.g., we precolor mutually distant vertices, or add edges forming mutually distant crossings). Suppose furthermore that these perturbations do not prevent coloring locally (e.g., any subgraph of the graph that contains only one of the perturbations is still colorable by the original number of colors). Is the resulting graph still colorable?

- For 4-colorability of planar graphs the answer is negative in general—there are known examples of planar graphs where a precoloring of arbitrarily distant vertices (only two suffice) cannot be extended to a 4-coloring.

- For 5-colorability (and even 5-list-colorability), the answer is likely to be always positive, as indicated by the results of Chapters 8 and 9. In particular, in Dvořák et al. [31] we build upon the results of Chapter 8 and prove that any precoloring of sufficiently distant vertices in a planar graph extends to its 5-list-coloring. For 5-colorability, this is true even for distance four between the precolored vertices by a reduction to the Four Color Theorem (Albertson [5]). Similarly, a graph drawn in plane with crossings is 5-colorable if no two crossings are incident with the same edge (Král' and Stacho [50]) and it is 5-list-colorable if the distance between crossings is at least 15 (Dvořák et al. [30], Chapter 9).

- The case of 3-colorability and 3-list-colorability of planar graphs of girth at least five was studied much less. However, there are again indications that the answer is positive, as the methods used to deal with 5-list-colorability are very similar to those used to deal with 3-list-colorability and should extend to this case. An example is given in Chapter 10, where we show that every planar graph with (≤ 4)-cycles sufficiently far apart is 3-list-colorable.

- The case of 3-colorability of triangle-free planar graphs is perhaps the most interesting. It is possible to construct such graphs with distant perturbations that prevent 3-colorability. For example, consider a plane graph with exactly two triangles and all other faces of length four—a standard winding number argument shows that there exists a precoloring of the triangles (namely the ones where the colors 1, 2 and 3 appear widdershins on one of the triangles and deasil on the other one). Even without the winding number issue, separating 4-cycles can allow an interaction between distant perturbations, as demonstrated by the broken chains of Chapter 6. However, these seem to be the only principal obstructions; in Dvořák et al. [24], we exploit this to prove Havel's conjecture that every planar graph with triangles sufficiently far apart is 3-colorable. A similar approach can be used to show that any precoloring of sufficiently distant vertices in a triangle-free planar graph extends to a 3-coloring.

Applying the method of reducible configurations to deal with distant perturbations is rather problematic. Although proving that a reducible configuration far from the perturbations exists by discharging is routine, most reductions decrease the distance between vertices. This could decrease the distance between the perturbations and prevent us from using the induction hypothesis. One way to deal with the problem is to use only reductions that do not decrease distances, e.g., consisting only of removing some of the vertices or edge of the configuration. However, a list of such configurations that would be guaranteed to appear in every planar graph is not known for any of the major colorability problems, and even if it existed, it would be impractically large.

Another way is to use the weight technique presented in the previous part: if we can show that the weight of every critical graph with k perturbations is $O(k)$, it is clear that the distance between some of the perturbations in a critical graph must be bounded by a constant (as otherwise the graph would contain $\omega(k)$ elements contributing to the weight). We use this idea in our proof of Havel's conjecture (Dvořák et al. [24]), in a somewhat more complicated form (in the setting of 4-critical triangle-free graphs, 4-faces have zero weight and need to be dealt with specially). Let us note that for this idea to succeed, the bound on the weight must be linear in the number of perturbations, which is fortunately quite usual.

The precoloring extension method turns out to be well suitable for the setting, as all the graphs that appear in it throughout the proof are subgraphs of the original graph, and thus the distance condition cannot get violated (of course, this assumes that we actually can prove the basic claim by the precoloring extension method—no such proof is known for 3-colorability of triangle-free planar graphs, as even those based on precoloring extension first get rid of 4-faces by reducing them). To apply the method, we prove the following stronger claim:

(55) *Let G be any plane graph in the considered class (with perturbations far apart), f its outer face and P a subpath of f of length at most k. Let L be the list assignment such that vertices of P have lists of size one, giving a proper coloring of the subgraph induced by $V(P)$, and the sizes of the lists of other vertices satisfy the assumptions of the basic claim (e.g., all those in F have list of size at least three and all others have list of size at least five, when we aim to prove 5-list-colorability). In this situation, G is L-colorable.*

A problem is that such a statement is usually false in the case that some perturbation is near to P. Thus, a more common outcome is "either G contains one of finitely many configurations involving P and exactly one of the perturbations, or G is L-colorable". In either case, the proof can proceed as in the basic case unless one of the perturbations is close to the part of G that is being colored and removed on one side of P (or intersects with the boundary of F in such a way that none of the choices of the basic case is possible). In this situation, the reduction could be either violating the assumptions on the list sizes or result in one of the forbidden configurations. In the latter case, such an arising forbidden configuration often gives us enough information about the graph to enable a different reduction. In the former case, the perturbation X typically has to be adjacent to or intersect the removed part.

The key idea to dealing with this case is to try applying the same procedure on the other side of P. If we fail again, there is a perturbation X' close to this part of the graph as well. However, X and X' cannot be different perturbations, as they would be too close to each other, and thus we have $X = X'$. Consequently, there exists a path Q joining the parts of the boundary of F splitting G to two subgraphs G_1 and G_2, where G_1 contains P and the perturbation X. By induction, we can color G_1, obtaining a coloring of Q. Ideally, the length of Q is small enough that we can also apply induction to extend its coloring to G_2, thus obtaining a coloring of G.

This may fail in several ways. One of them is that G_2 contains one of the forbidden configurations preventing the extension of the coloring. In this case, we need to use the resulting information about the structure of G_2 to devise a different way of reducing G. Other possibility is that the length of Q is greater than k, and thus we cannot apply (55) inductively. One way how to deal with this issue is to increase the constant k in (55) sufficiently; however, this may result in an unacceptably large number of forbidden configurations. A simpler possibility is to use the fact that in G_2, all perturbations are far from Q. That is, we will be proving a claim of the following form.

(56) *Let G be any plane graph in the considered class (with perturbations far apart), f its outer face and P a subpath of f of length at most k'. Let L be the list assignment such that vertices of P have lists of size one, giving a proper coloring of the subgraph induced by $V(P)$, and the sizes of the lists of other vertices satisfy the assumptions of the basic claim. If $\ell(P) \leq k$ and G does not contain any of the configurations in a set O_1, or if the distance between P and the nearest perturbation is at least d and G does not contain any of the configurations in a set O_2, then G is L-colorable.*

Here, the configurations in the set O_1 may contain one perturbation, while the ones in O_2 contain no perturbations. Until the reductions in the proof interfere with one of the perturbations, we only need to exclude paths of length at most k between the vertices of F (so that the assumption on the sizes of the lists is preserved), and when we cut along these paths, we obtain subgraphs conforming to the first part of the claim (even if the graph G we consider only satisfies the second condition). On the other hand, when we cut along a path Q passing through a perturbation (in this case, G necessarily satisfies the first condition), then in G_2, we have a precolored path Q of length at most k' whose distance to a nearest perturbation is at least d (which is somewhat smaller than the distance between distinct perturbations). For technical reasons (dealing with chords incident with internal vertices of P), the distance d may also be specified as a function of the length of P (the longer the path, the greater distance we must require). Although technically slightly more difficult than just increasing the value of k to k' in (55), the set of forbidden configurations O_2 is usually just a small fraction of the forbidden configurations that would be necessary in that case.

We give two examples of the applications of this idea, in the cases of 5-list-colorability of planar graphs (Chapter 9) and of 3-list-colorability of planar graphs of girth at least five (Chapter 10). Let us note that the arguments in Chapter 9 could be simplified somewhat by using the idea of Chapter 8—we could only deal with graphs with one perturbation. We avoid this in order to obtain a better bound on the distance between the perturbations, and also to provide a self-contained example.

Chapter 9

5-list-colorability of graphs with crossings far apart[1]

We aim to show that every graph drawn in plane with crossings sufficiently far apart is 5-list-colorable. Let us recall that a *list assignment* L for G is a function that assigns to each vertex of G a set $L(v)$, called the *list of admissible colors* for v. An *L-coloring* is a choice of a color $c(v) \in L(v)$ for each $v \in V(G)$ such that no two adjacent vertices receive the same color. The graph is *k-list-colorable* if it admits an L-coloring for every list assignment L with $|L(v)| \geq k$ for every $v \in V(G)$.

Theorem 96. *Every graph drawn in the plane so that the distance between every pair of crossings is at least* 15 *is* 5*-list-colorable.*

Some distance condition on the crossings is necessary, even if we would allow only three crossings, as shown by K_6. On the other hand, it was proved in [33] and independently also in [14] that the distance requirement is not needed, if we have at most two crossings.

For the purposes of the induction, we need to prove a stronger claim.

Theorem 97. *Let G be a graph drawn in the plane with some crossings and let $N \subseteq V(G)$ be a set of vertices such that the distance between any pair of crossed edges is at least* 15*, the distance between any crossing and a vertex in N is at least* 13*, and the distance between any two vertices in N is at least* 11*. Then G is L-colorable for any list assignment L such that $|L(v)| = 4$ for $v \in N$ and $|L(v)| \geq 5$ for $v \in V(G) \setminus N$.*

The inductive proof of Theorem 97 involves a stronger inductive hypothesis that is stated later as Theorem 100 and in particular also implies the above-mentioned result from [14, 33].

[1] The results of this chapter are based on Dvořák et al. [30].

Theorem 98 ([33, 14])**.** *Every graph whose crossing number is at most two is* 5*-list-colorable.*

The proof of Theorem 98 is given at the end of this chapter.

9.1 5-list-colorability of planar graphs

In this section, we present a new proof of 5-list-colorability of planar graphs. While the proof of this claim by Thomassen [65] is short and beautiful, it has a disadvantage that it requires a strengthening (Theorem 2) of the claim that is rather tight, and introducing almost any kind of irregularity (distant crossings, increasing the length of the precolored path, ...) makes it fail with infinitely many counterexamples. The proof of this section (inspired by Thomassen's proof of 3-list-colorability of planar graphs of girth at least five [69]) avoids this by only showing a somewhat weaker generalization.

Let P be a path or a cycle. The *length* $\ell(P)$ of P is the number of its edges, i.e., a path of length l has $l+1$ vertices and a cycle of length l has l vertices. Given a graph G and a cycle $K \subseteq G$, an edge uv of G is a *chord* of K if $u, v \in V(K)$, but uv is not an edge of K. For an integer $k \geq 2$, a path $v_0v_1 \ldots v_k$ is a k*-chord* if $v_0, v_k \in K$ and $v_1, \ldots, v_{k-1} \notin V(K)$. We define a chord to be a 1*-chord*. If G is a plane graph, let $\mathrm{Int}_K(G)$ be the subgraph of G consisting of the vertices and edges drawn inside the closed disc bounded by K, and $\mathrm{Ext}_K(G)$ the subgraph of G obtained by removing all vertices and edges drawn inside the open disc bounded by K. In particular, $K = \mathrm{Int}_K(G) \cap \mathrm{Ext}_K(G)$. Note that each k-chord of K belongs to exactly one of $\mathrm{Int}_K(G)$ and $\mathrm{Ext}_K(G)$. If the cycle K is the outer face of G and Q is a k-chord of K, let C_1 and C_2 be the two cycles in $K \cup Q$ that contain Q. Then the subgraphs $G_1 = \mathrm{Int}_{C_1}(G)$ and $G_2 = \mathrm{Int}_{C_2}(G)$ are the Q*-components* of G.

As we have mentioned earlier, Thomassen's Theorem 2 does not easily extend to the case when we have a precolored path of length two. However, if we strengthen the condition on the list sizes of the other vertices on the outer face, such an extension is possible.

Theorem 99. *Let G be a plane graph with the outer face F, P a subpath of F of length at most two and L a list assignment such that the following conditions are satisfied:*

(i) $|L(v)| \geq 5$ *for* $v \in V(G) \setminus V(F)$,

(ii) $|L(v)| \geq 3$ *for* $v \in V(F) \setminus V(P)$,

(iii) $|L(v)| = 1$ *for* $v \in V(P)$,

(iv) *no two vertices with lists of size three are adjacent in* G,

(v) *L gives a proper coloring to the subgraph induced by $V(P)$, and*

(vi) *if* $P = uvw$ *has length two and* x *is a common neighbor of* u, v *and* w, *then* $L(x) \neq L(u) \cup L(v) \cup L(w)$.

Then G *is* L*-colorable.*

Proof. Suppose for a contradiction that the claim is false, and let G be a counterexample with $|V(G)|+|E(G)|$ the smallest possible, and subject to that, with the longest path P and with the minimum size of the lists (while satisfying (i)–(vi)). It is clear that G is connected and that every vertex $v \in V(G)$ satisfies $\deg(v) \geq |L(v)|$.

Furthermore, G is 2-connected: otherwise, let v be a cut-vertex and let G_1 and G_2 be subgraphs of G such that $G_1 \cup G_2 = G$, $V(G_1) \cap V(G_2) = \{v\}$ and $|V(G_1)|, |V(G_2)| > 1$. If $v \in V(P)$, then by the minimality of G there exist L-colorings of G_1 and G_2, and these colorings together give an L-coloring of G. Otherwise, we may assume by symmetry that $P \subseteq G_1$. Consider an L-coloring φ of G_1. Let L_2 be the list assignment for G_2 such that $L_2(u) = L(u)$ for $u \neq v$ and $L_2(v) = \{\varphi(v)\}$. By the minimality of G, G_2 is L_2-colorable, and this coloring together with φ gives an L-coloring of G.

Every triangle T in G bounds a face: otherwise, first color the subgraph $\mathrm{Ext}_T(G)$ and then extend the coloring to $\mathrm{Int}_T(G)$. A similar argument shows that G contains no separating 4-cycles; otherwise, consider such a 4-cycle $K = k_1k_2k_3k_4$, and let φ be an L-coloring of $\mathrm{Ext}_K(G)$. Let $G' = \mathrm{Int}_K(G)$. Since K is separating, we have $V(G') \neq V(K)$, and since every triangle bounds a face, we conclude that K has no chord in G'. Let L' be the list assignment for $G' - k_1$ such that $L'(z) = \{\varphi(z)\}$ for $z \in \{k_2, k_3, k_4\}$, $L'(z) = L(z) \setminus \{\varphi(k_1)\}$ if $z \notin \{k_2, k_4\}$ is a neighbor of k_1 and $L'(z) = L(z)$ if z is any other vertex. By the minimality of G, the graph $G' - k_1$ is L'-colorable, and this coloring together with φ gives an L-coloring of G.

Since G is 2-connected, its outer face is bounded by a cycle, which we denote by F as well. Next, we show that F has no chords. Otherwise, let uv be a chord of F and let G_1 and G_2 be the uv-components of G. If $P \subseteq G_1$, then we first color G_1 and then extend the coloring to G_2. The case that $P \subseteq G_2$ is symmetric. It follows that P has length two and all the chords of F are incident with its middle vertex. Let $P = z_1uz_2$, where $z_i \in V(G_i)$ for $i \in \{1, 2\}$. Let φ be an L-coloring of G_1 and let L_2 be the list assignment for G_2 such that $L_2(z) = L(z)$ for $z \neq v$ and $L_2(v) = \{\varphi(v)\}$. Since G is not L-colorable, G_2 is not L_2-colorable. By the minimality of G, either v is adjacent to z_2, or u, v and z_2 have a common neighbor w with list of size three (which means, in particular, that $w \in V(F)$). Since every chord of G is incident with u, the edge vz_2 or vw belongs to F. Since every triangle bounds a face, we conclude that v has degree two in G_2. By symmetry, v has degree two in G_1 as well, and thus v has degree three in G. It follows that $|L(v)| = 3$, and thus v cannot be adjacent to any other vertex with list of size three. In particular, we cannot have the case with the vertex w. We conclude that v is adjacent to z_1 and z_2 and $V(G) = \{u, v, z_1, z_2\}$.

However, $L(v) \neq L(u) \cup L(z_1) \cup L(z_2)$ by (vi), and thus G is L-colorable. This contradiction proves that F has no chords.

Similarly, we have the following property:

(57) *Let uvw be a 2-chord of F and let G_1 and G_2 be uvw-components of G. If $P \subseteq G_1$, then either u and w are adjacent and G_2 is equal to the triangle uvw, or there exists a vertex x such that $V(G_2) = \{u, v, w, x\}$, $|L(x)| = 3$ and x is adjacent to u, v and w.*

If $\ell(P) < 2$, then it is easy to see that we can precolor $2 - \ell(P)$ more vertices of F without violating (vi). Thus, we may assume that $\ell(P) = 2$. Let $P = p_0p_1p_2$. Suppose that p_0, p_1 and p_2 have a common neighbor v. If $v \in V(F)$, then $V(G) = \{p_0, p_1, p_2, v\}$ and G is L-colorable. If $v \notin V(F)$, then v has degree at most four in G by (57) and thus $\deg(v) < |L(v)|$, which is a contradiction. Therefore, p_0, p_1 and p_2 have no common neighbor.

Furthermore, $\ell(F) \geq 6$: If $\ell(F) = 3$, then we remove one vertex of F and remove its color from the lists of all its neighbors, and observe that the resulting graph is a smaller counterexample to Theorem 99. In the case when $\ell(F) = 4$, then similarly color and remove the vertex of $V(F) \setminus V(P)$. Finally, suppose that $\ell(F) = 5$. Let φ be an arbitrary L-coloring of $F = p_2p_1p_0v_1v_2$. Remove v_1 and v_2 from G and remove their colors according to φ from the lists of their neighbors, obtaining a graph G' with the list assignment L'. Since every triangle in G bounds a face, at most one vertex in G' has list of size three. Since p_0, p_1 and p_2 have no common neighbor and p_0 is not adjacent to p_2, G' with the list assignment L' is a smaller counterexample to Theorem 99, which is a contradiction.

Let $F = p_2p_1p_0v_1v_2v_3v_4\ldots$. If $\ell(F) = 6$, then we set $v_4 = p_2$. We may assume that $|L(v_1)| = 3$ or $|L(v_2)| = 3$, since otherwise we can remove a color from the list of v_1. Let us consider a set $X \subseteq V(F) \setminus V(P)$ and a partial L-coloring φ of X that are defined as follows:

(X1) If $|L(v_1)| = 3$ and $|L(v_3)| \neq 3$, then $X = \{v_1\}$ and $\varphi(v_1) \in L(v_1) \setminus L(p_0)$ is chosen arbitrarily.

(X2) If $|L(v_1)| = 3$ and $|L(v_3)| = 3$, then $X = \{v_1, v_2\}$ and φ is chosen so that $\varphi(v_2) \in L(v_2) \setminus L(v_3)$ and $\varphi(v_1) \in L(v_1) \setminus (L(p_0) \cup \{\varphi(v_2)\})$.

(X3) If $|L(v_2)| = 3$, and either $|L(v_4)| \neq 3$ or $|L(v_3)| \geq 5$, then $X = \{v_2\}$ and $\varphi(v_2) \in L(v_2)$ is chosen arbitrarily.

(X4) If $|L(v_2)| = 3$, $|L(v_3)| = 4$ and $|L(v_4)| = 3$, then:

 (X4a) If v_1, v_2 and v_3 do not have a common neighbor or $|L(v_1)| \geq 5$, then $X = \{v_2, v_3\}$ and φ is chosen so that $\varphi(v_3) \in L(v_3) \setminus L(v_4)$ and $\varphi(v_2) \in L(v_2) \setminus \{\varphi(v_3)\}$.

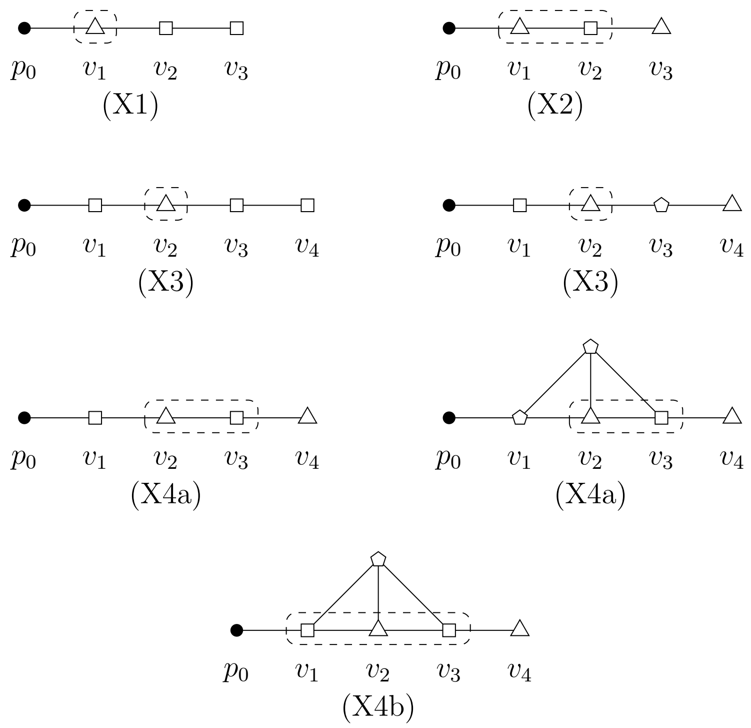


Figure 9.1: Subcases in the definition of X. Triangle vertices represent lists of size 3, square vertices list of size ≥ 4. Encircled vertices are in X.

(X4b) If v_1, v_2 and v_3 have a common neighbor and $|L(v_1)| = 4$, then $X = \{v_1, v_2, v_3\}$ and φ is chosen so that $\varphi(v_3) \in L(v_3) \setminus L(v_4)$, $\varphi(v_1) \in L(v_1) \setminus L(p_0)$ and either at least one of $\varphi(v_1)$ and $\varphi(v_3)$ does not belong to $L(v_2)$, or $\varphi(v_1) = \varphi(v_3)$. The vertex v_2 is left uncolored.

For later reference, Figure 9.1 shows the subcases used in the definition of X and φ.

Let $G' = G - X$ and let L' be the list assignment obtained from L by removing the colors of the vertices of X according to φ from the lists of their neighbors (if a vertex of X is not colored according to φ, we do not remove any colors for it). Observe that any L'-coloring of G' can be extended to an L-coloring of G, thus G' is not L'-colorable. By the minimality of G, this implies that G' violates the assumptions of Theorem 99. Since F has no chords, the choice of X and φ implies that every vertex of $V(G') \setminus V(P)$ has list of size at least three. Since p_0 is not adjacent to p_2 and p_0, p_1 and p_2 do not have a common neighbor in G, it follows that the conditions (v) and (vi) are satisfied by G' with the list assignment L'. We conclude that (iv) is false, i.e., G' contains adjacent vertices u and v such that $|L'(u)| = |L'(v)| = 3$.

Since F has no chords, the choice of X ensures that at most one of u and v

belongs to $V(F)$; hence, we can assume that $v \notin V(F)$ and v has two neighbors in X. In particular, X was chosen according to the cases (X2) or (X4). Since G contains no separating cycles of length at most 4, we conclude that u has at most one neighbor in X, and thus $u \in V(F)$. Let $x \in X$ be the neighbor of v such that the distance between u and x in $F - P$ is as large as possible. By (57) applied to the 2-chord xvu, we conclude that the xvu-component of G that does not contain P consists of xvu and a vertex z adjacent to x, v and u with $|L(z)| = 3$. It follows that $|L(u)| > 3$, and since $|L'(u)| = 3$, we have $z \in X$ and $|L(u)| = 4$. The inspection of the choice of X shows that (X4) holds, i.e., $u = v_1$, $z = v_2$ and $x = v_3$. However, note that the condition of (X4b) holds; hence $u \in X$, contrary to the assumption that $u \in V(G')$. This completes the proof of Theorem 99. □

9.2 Near-planar graphs

Our proof of Theorem 97 is a modification the proof of 5-list-colorability of planar graphs that we presented in the previous section. Our main goal is to show that graphs drawn in the plane with crossings far apart are 5-list-colorable. For the purposes of the induction, it will be useful to allow other kinds of irregularities (adjacent vertices with list of size three, as well as vertices with list of size four not incident with the outer face, which arise when some vertices incident with a crossing are colored and their color is removed from the lists of their neighbors), subject to distance constraints.

Let us first introduce some terminology. Let G be a graph. For two subgraphs $H_1, H_2 \subseteq G$, the *distance* $d(H_1, H_2)$ between H_1 and H_2 is the minimum of the distances between the vertices of H_1 and H_2, i.e., the minimum k such that there exists a path $v_0 v_1 \ldots v_k$ in G with $v_0 \in V(H_1)$ and $v_k \in V(H_2)$. A *drawing* $\mathcal{G}$ of G in the plane consists of a set $\mathcal{V} = \{p_v \mid v \in V(G)\}$ of distinct points in the plane and a set of simple polygonal curves $\mathcal{E} = \{c_e \mid e \in E(G)\}$ such that

- if $uv \in E(G)$, then p_u and p_v are the endpoints of c_{uv},
- no internal point of any $c_e \in \mathcal{E}$ belongs to $\mathcal{V}$, and
- at most two of the curves in $\mathcal{E}$ contain any point that does not belong to $\mathcal{V}$, and any two curves in $\mathcal{E}$ have at most one point in common.

A *crossing* of $\mathcal{G}$ is a point in the plane that belongs to two of the curves in $\mathcal{E}$, but not to $\mathcal{V}$. An edge e is *incident with the crossing* x if $x \in c_e$. An edge e is *crossed* if it is incident with some crossing, and *non-crossed* otherwise. For a crossing x, we define G_x to be the graph consisting of the two edges incident with x. Two vertices of G are *crossing-adjacent* if they belong to G_x for some crossing x and are not adjacent in G_x. Removal of $\bigcup \mathcal{E}$ splits the plane into several connected subsets, which we call *faces* of $\mathcal{G}$. By a slight abuse of terminology, we sometimes

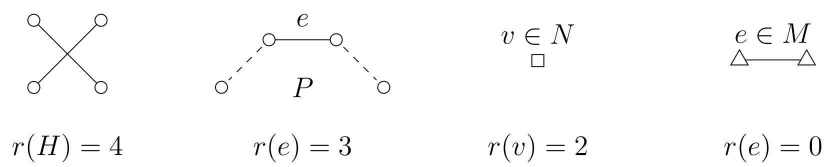


Figure 9.2: Special subgraphs and their ranks

identify a face with its boundary and hence speak about the vertices, edges and crossings of the face.

Let $\mathcal{G}$ be a drawing of a graph G, let P be a path of length at most three contained in the boundary of the outer face F of $\mathcal{G}$ (where in particular, no edge of P is crossed), N a subset of $V(G)$ and M a subset of $E(G)$, and let L be a list assignment for G. We say that L is (P, N, M)-*valid* if the following conditions are satisfied:

(S) $|L(v)| \geq 5$ for $v \in V(G) \setminus (V(F) \cup N)$, $|L(v)| \geq 3$ for $v \in V(F) \setminus V(P)$ and $|L(v)| = 1$ for $v \in V(P)$,

(N) $|L(v)| \geq 4$ for $v \in N \setminus V(F)$,

(M) if $|L(u)| = |L(v)| = 3$ and u and v are adjacent, then $uv \in M$,

(P) L gives a proper coloring to the subgraph induced by $V(P)$,

(T) if a vertex v has three neighbors w_1, w_2, w_3 in $V(P)$, then $L(v) \neq L(w_1) \cup L(w_2) \cup L(w_3)$, and

(C) if x is a crossing and G_x contains a vertex with list of size three, then all other vertices of G_x have lists of size 1 or ≥ 5.

We define some subgraphs H of G to be *special*, and assign a *rank* $r(H)$ to each such subgraph (see Figure 9.2). Specifically, H is *special* if it falls into one of the following cases:

- H consists of the two edges incident with a crossing. In this case, its rank is 4.
- P has length three and H consists of the middle edge of P; the rank of H is 3.
- H is equal to a vertex of N, and $r(H) = 2$.
- H is equal to an edge of M, and $r(H) = 0$.

The drawing $\mathcal{G}$ is (P, N, M)-*distant* if $d(H_1, H_2) \geq r(H_1) + r(H_2) + 7$ for every pair $H_1 \neq H_2$ of special subgraphs of G. We shall occasionally refer to the (P, N, M)-distant requirement as the *distance condition*. The purpose of the introduced

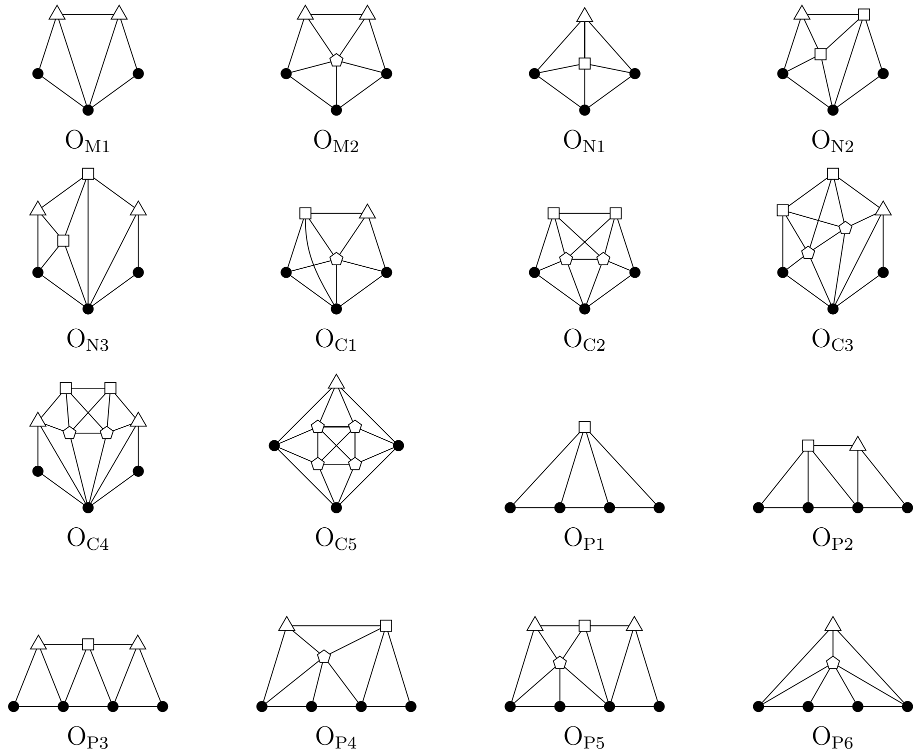


Figure 9.3: The obstructions used in Theorem 100

rank function is the following. In our inductive arguments, we will occasionally construct a smaller graph G' and introduce a new special subgraph H' in a vicinity of a special subgraph H that would no longer exist in G'. If H' has smaller rank than H, the distance condition for special subgraphs in G' would still hold, and the induction hypothesis can be applied.

A subgraph $O \subseteq G$ is an *obstruction* if O is isomorphic to one of the graphs drawn in Figure 9.3 and sizes of the lists of its vertices match those prescribed by the figure, where the full-circle vertices have list of size one, triangle vertices have list of size three, square vertices have list of size four and pentagonal vertices have list of size five. Let us remark that if the distance condition holds, then G can contain at most one of the obstructions. For further reference we exhibit in Figure 9.4 all possible list assignments for which the obstructions are not colorable. In particular, observe that the following holds:

(58) *Let H be one of the obstructions and let Q be the path in H consisting of full-circle vertices. Suppose that Q has length two and that H is neither* O_{M1} *nor* O_{C1}*. Let q be the middle vertex of Q and let L be a list assignment such that each vertex v drawn by a k-gon has $|L(v)| = k$, while the vertices of Q have lists consisting of all possible colors. Then there exists a color b such that every L-coloring ψ of Q with $\psi(q) \neq b$ extends to an L-coloring of H.*

We prove the following claim, which obviously implies our main result, Theorem 97.

Theorem 100. *Let G be a graph drawn in the plane, P a path of length at most three contained in the boundary of the outer face F of $\mathcal{G}$ and L a list assignment for G. Suppose that there exist sets $N \subseteq V(G)$ and $M \subseteq E(G)$ such that L is (P, N, M)-valid and the drawing of G is (P, N, M)-distant. If*

(O) *every obstruction in G is L-colorable,*

then G is L-colorable.

Before giving the proof of this statement, let us give a quick outline. Essentially, we follow the proof of Theorem 99. First, we show that the outer face of a hypothetical minimal counterexample G has no chords and then we also restrict its 2-chords. This is somewhat more complicated due to the presence of crossings and the condition (O). Next, we find the set X and its partial coloring φ defined in the same way as in the proof of Theorem 99, and use it to construct the graph G' with the list assignment L'. By the minimality of G, we conclude that G' violates one of the assumptions of the theorem. A straightforward case analysis shows that (O) holds, and the conditions (S), (P) and (T) follow in the same way as in the proof of Theorem 99; but (M), (N) and (C) can be violated in ways which do not enable us to obtain a contradiction directly. However, we observe that in such a case, there is a special subgraph S near to X. In this situation, we apply the symmetric argument on the other side of the path P, and obtain another set X' and a special subgraph S' close to it. By the distance condition, we have $S = S'$, and thus there exists a short path from X to X' passing through S. In this situation, we consider all the possible combinations of X, X' and their positions relatively to S, and obtain a contradiction similarly to the way we deal with 2-chords.

Let us note that the assumption (C) is a product of a somewhat delicate tradeoff. We believe the claim still essentially holds even without this assumption, and avoiding it would greatly reduce the number of possible bad cases and simplify the last part of the proof. However, the list of obstructions in (O) would be significantly larger, making the first part of the proof longer and more complicated. Moreover, if we omit (C) completely, then there exists an obstruction with a precolored path of length one (see Figure 9.5(a)), which would be a major problem (we could not easily get rid of chords of F). One could consider excluding Figure 9.5(a) by forbidding vertices with lists of sizes three or four joined by a crossed edge. This would still simplify the last part of the proof a lot. However, in addition to having more than 10 new obstructions, we do not see a way how to reduce the 2-chord depicted in Figure 9.5(b), which would need to be dealt with somehow.

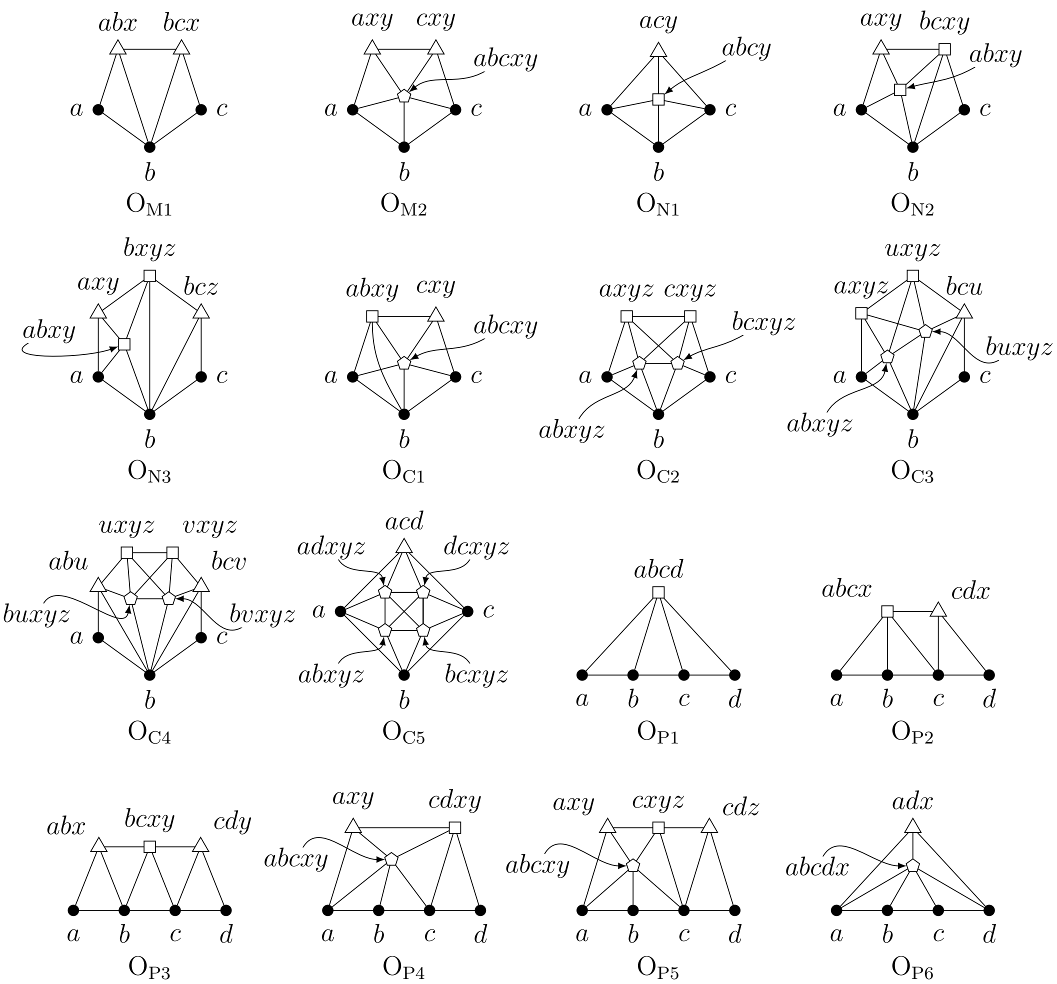
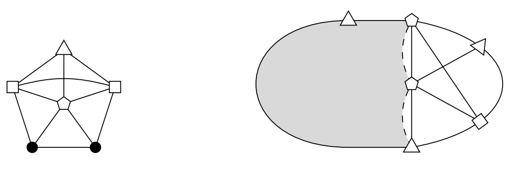


Figure 9.4: The lists for which the obstructions cannot be colored. Colors represented by different letters may be equal to each other if they do not occur in the same list for a particular obstruction.



(a) (b)

Figure 9.5: Why is condition (C) needed?

Proof of Theorem 100. We follow the outline of the proof of Theorem 99. We assume that G is a counterexample to Theorem 100 with $|V(G)| + |E(G)|$ as small as possible, and subject to that, with the minimum sizes of its lists. Let $k = \ell(P) \le 3$, and let $P = p_0p_1 \dots p_k$. By applying the same kind of reductions as used in the proof of Theorem 99 together with the minimality of G, one can show:

(59) *The following properties are satisfied:*

(a) *Every vertex $v \in V(G)$ satisfies* $\deg(v) \ge |L(v)|$.

(b) *G is 2-connected and $\ell(P) \ge 1$.*

(c) *Every non-crossed chord of F is incident with exactly one internal vertex of P.*

(d) *If K is a triangle in G and no edge of K is crossed, then K is not separating. If K is a separating 4-cycle without crossed edges, then* $\mathrm{Int}_K(G) - V(K)$ *is either a vertex in N or a complete graph on 4 vertices involving a crossing.*

(e) *Every vertex $v \in V(G)$ satisfies $|L(v)| \le 5$.*

(f) *If $v \in V(G) \setminus V(P)$ is adjacent to a vertex $p \in V(P)$, then $L(p) \subseteq L(v)$.*

Most properties in (59) are easy to argue about; they are left to the reader. Property (e) is achieved by removing colors from lists of size 6 or more. The only problem that may arise is that we obtain an obstruction; however, inspection of bad lists for the obstructions exhibited in Figure 9.4 shows that we can always remove one of the colors so that (O) still holds. The only remaining nontrivial claim is the property of separating 4-cycles in part (d). To prove that, we first color the subgraph of G consisting of $\mathrm{Ext}_K(G)$ and all chords of K and then consider $G' = \mathrm{Int}_K(G)$. Let $K = v_1v_2v_3v_4$, and let c_i be the color of v_i as used in the coloring of $\mathrm{Ext}_K(G)$. Suppose that $c_1 \ne c_3$. In that case we consider the list assignment L' for G' given by setting $L'(v) = L(v)$ if $v \notin V(K)$, $L'(v_i) = \{c_i\}$ for $i = 1, 2, 3$, and $L'(v_4) = \{c_1, c_3, c_4\}$. Since any L'-coloring of G' yields an L-coloring of G, we conclude that G' does not satisfy all assumptions of the theorem. It is easy to see that the only possible violation is that G' contains an obstruction. Note that this obstruction contains the whole path $v_1v_2v_3$ and that the only vertices whose lists have size 3 or 4 are v_4 and possibly a vertex in N. If a vertex in N is present, there is no crossing by the distance condition. The only obstructions with these properties are $\mathrm{O_{N1}}$ and $\mathrm{O_{C5}}$, yielding the outcome of the claim (a similar argument shows that $V(\mathrm{Int}_K(G))$ only consists of the vertices of the obstruction; see (60) below for details). The remaining case to consider is when $c_1 = c_3$. In this case we replace the color c_3 in the list of v_3 and in the lists of all its neighbors by a new color c_3' that does not occur elsewhere, and then apply the same argument as in the previous case. It is to be observed that the color c_3' will only be used for v_3, and the color c_3 will not be used on any of the neighbors of v_3. Thus a coloring with the revised lists gives rise to an L-coloring of G also in this case. This completes the proof of (59).

Let $T = v_1v_2v_3$ be a triangle in G. Suppose that the edge v_1v_2 is crossed by an edge uw, where w belongs to $G_2 = \mathrm{Int}_T(G)$ and $w \neq v_3$. Let φ be an L-coloring of $\mathrm{Ext}_T(G)$ and let L_2 be the list assignment such that $L_2(v_i) = \{\varphi(v_i)\}$ for $1 \leq i \leq 3$, $L_2(w) = L(w) \setminus \{\varphi(u)\}$ and $L_2(z) = L(z)$ otherwise. Note that $|L_2(w)| \geq 4$, that G_2 is not L_2-colorable and that it is $(v_1v_2v_3, (N \cap V(G_2)) \cup \{w\}, \emptyset)$-distant. Observe that G_2 satisfies the validity conditions (S)–(C), and also satisfies (O). Hence it is a counterexample to Theorem 100, contradicting the minimality of G. Similarly, if $w = v_3$, then we conclude that no vertex is drawn in the open disc bounded by T. Together with (59), we obtain the following conclusion:

(60) *If T is a triangle in G distinct from F, then $V(\mathrm{Int}_T(G)) = V(T)$.*

Suppose now that G contains one of the obstructions from Figure 9.3. Note that each of the obstructions contains a special subgraph. By the distance condition, none of them has further crossed edges and (60) implies that no such obstruction H appears in G, as otherwise we would have $G = H$ and G would be L-colorable by the assumptions.

Furthermore, analogous arguments as used in the proof of (59)(d) show that the following conditions hold:

(61) *If K is a 4-cycle in G distinct from F and $V(\mathrm{Int}_K(G)) \neq V(K)$, then either $\mathrm{Int}_K(G) - V(K)$ is K_4, or there exists a vertex z such that $V(\mathrm{Int}_K(G)) = V(K) \cup \{z\}$, z is adjacent to all vertices of K and z either belongs to N or is incident with an edge crossing an edge of K.*

(62) *If K is a 5-cycle in G distinct from F, $V(\mathrm{Int}_K(G)) \neq V(K)$, no edge of K is crossed and there exists a special subgraph $S \subseteq \mathrm{Ext}_K(G)$ such that $d(S, K) \leq 1$, then $V(\mathrm{Int}_K(G)) = V(K) \cup \{z\}$ for a vertex z adjacent to all vertices of K.*

Some explanation concerning the proof of (62) is needed: Again, we first color $\mathrm{Ext}_K(G)$ and then consider $\mathrm{Int}_K(G)$ with the 5-cycle precolored. By the previous results, K has no chords, since the outcomes of (61) would yield a special subgraph too close to S. Let uv be an edge of K, and let $G' = \mathrm{Int}_K(G) - \{u, v\}$. By removing the colors of u and v from the lists of their neighbors, we obtain another instance of a list coloring problem with a precolored path of length 2. Since any coloring of G' gives rise to a coloring of G, we conclude that one of the assumptions of the theorem is violated. By the distance condition, the only one that may not hold is the assumption (T). Since the common neighbor w of the three vertices on the path has list of size 3 (and it had list of size 5 in G), it is adjacent to u and v in G, thus proving the claim.

Our next goal is to show that F does not have chords. Let uv be a non-crossed chord of F. By (59)(c), u is an internal vertex of P, say $u = p_1$, while $v \notin V(P)$. Let G_1 and G_2 be the uv-components for G such that $p_0 \in V(G_1)$, and let $P_1 = p_0p_1v$ and $P_2 = vp_1 \dots p_k$. For each color $c \in L(v) \setminus L(u)$, let L_c be the list assignment such that $L_c(v) = \{c\}$ and $L_c(z) = L(z)$ if $z \neq v$. Since G is not L-colorable, either G_1 or G_2 is not L_c-colorable. Furthermore, since both G_1 and G_2 are L-colorable (by the minimality of G), there exist distinct colors c_1 and c_2 such that G_1 is not L_{c_1}-colorable and G_2 is not L_{c_2}-colorable. Since G is a minimal counterexample, the assumptions of our theorem fail for G_1 and G_2 with respect to these list assignments. In the sequel we discuss what can go wrong.

All special subgraphs in G that do not contain v remain special in G_1 or G_2 and no new special subgraphs arise. Thus, G_1 is $(P_1, N \cap V(G_1), M \cap E(G_1))$-distant and G_2 is $(P_2, N \cap V(G_2), M \cap E(G_2))$-distant. Clearly, validity conditions (S), (N), (M), and (C) hold for both graphs. Thus one of (P), (T), or (O) fails. If G_1 contains an obstruction, then it contains a special subgraph whose distance to p_1 is at most two. In that case, we conclude that $\ell(P) \leq 2$ and that G_2 contains no obstruction, since the distance between special subgraphs in G is more than four; also, no edge at distance at most two from p_1 in G_2 is crossed. Since $\ell(P) \leq 2$, we may in this case exchange the role of G_1 and G_2 and henceforth assume that G_1 contains no obstructions. Similarly, by exchanging the roles of G_1 and G_2 if necessary, we may assume that no edge in G_1 at distance at most 2 from p_1 is crossed. Furthermore, if G_1 violates (T), then since no edge in G_1 incident with p_1 is crossed, we could consider the chord p_1z instead of p_1v, where z is the common neighbor of p_0, p_1 and v. Therefore, we can assume that G_1 satisfies (T) and (O). Since no L-coloring of G_2 extends to an L-coloring of G_1, we conclude that G_1 violates (P), and thus v is adjacent to p_0. Since vp_0 is neither crossed nor incident with an internal vertex of P, we conclude that vp_0 is part of the boundary of F, and hence G_1 is equal to the triangle p_0p_1v. Suppose now that G_2 contains an obstruction H; by (59) and (60), we have $G_2 = H$. However, the inspection of the obstructions shows that G would either be L-colorable or an obstruction. Therefore, G_2 satisfies (O). Furthermore, by the absence of $\mathrm{O_{P1}}$ and property (T) of G, we conclude that there exists a color $c \in L(v) \setminus (L(p_0) \cup L(p_1))$ such that G_2 satisfies (P). Since this coloring does not extend to an L-coloring of G_2, it follows that G_2 violates (T), i.e., there exists a vertex w adjacent to v and to vertices $p, p' \in V(P) \setminus \{p_0\}$ such that $L(w) = L(p) \cup L(p') \cup \{c\}$. Since we cannot choose c so that G_2 satisfies both (P) and (T), it follows that either G contains $\mathrm{O_{P2}}$, or $vw \in M$ (in which case $\ell(P) = 2$), and G contains $\mathrm{O_{M1}}$. This is a contradiction, thus every chord of F is crossed.

Consider now a (crossed) chord uv of F that is not incident with an internal vertex of P. Let e be the edge crossing uv and let G_1 and G_2 be the uv-components of $G - e$ such that $P \subseteq G_1$. Let $e = x_1x_2$, where $x_1 \in V(G_1)$ and $x_2 \in V(G_2)$. By the minimality of G, there exists an L-coloring φ of G_1. Since $\varphi(u) \neq \varphi(v)$, we can assume that $\varphi(x_1) \neq \varphi(u)$. Let G' be the graph obtained from $G_2 - uv$ by adding new vertices y_1 and y_2, edges of the path $P' = uy_1y_2v$ and the edge y_1x_2. Let L' be the list assignment for G' such that

$L'(u) = \{\varphi(u)\}$, $L'(v) = \{\varphi(v)\}$, $L'(y_1) = \{\varphi(x_1)\}$, $L'(y_2) = \{c\}$ for a new color c that does not appear in any of the lists and $L'(z) = L(z)$ for any other vertex z. Note that G' has a new special subgraph consisting of the edge y_1y_2 and that G' is $(P', N \cap V(G'), M \cap E(G'))$-distant, since the crossing of G incident with x_2 does not belong to G' and any path from a special subgraph in G' to y_1y_2 passes through one of the vertices u, v, x_2 of the crossing in G. Furthermore, G' is not L'-colorable, and by the minimality of G, it violates (T) or (O). The latter is not possible, since y_2 has degree two, thus (T) does not hold in G'. This implies that x_2 has list of size three and it is adjacent to u and v. By (59) and (60), we have $V(G_2) = \{u, v, x_2\}$. Note that by (C), we conclude that each of $|L(u)|, |L(v)|, |L(x_1)|$ is either 1 or 5. Let a be a color in $L(x_2)$ distinct from the colors of its neighbors in P, which exists by (T). Let $G'' = G - x_2$ with the list assignment L'' such that $L''(z) = L(z) \setminus \{a\}$ for $z \in \{u, v, x_1\}$ and $L''(z) = L(z)$ otherwise. Note that G'' is $(P, N \cup \{x_1\}, M\}$-distant and L'' is $(P, N \cup \{x_1\}, M)$-valid. By the minimality of G, we have that G'' violates (O). The obstruction involved is L-colorable, so it must contain one of the vertices whose lists have been changed. Since none of these vertices has list of size 3 or 4 in L and all special subgraphs are far away from the crossing, we conclude that $\ell(P) = 2$, $|L''(x_1)| = 4$ and G'' contains one of $\mathrm{O_{N1}}$, $\mathrm{O_{N2}}$ or $\mathrm{O_{N3}}$, in which the interior vertex with list of size 4 is x_1. However, inspection of these graphs shows that $|L''(u)| = 3$ or $|L''(v)| = 3$, which is a contradiction.

Finally, consider a crossed chord uv of F incident with an internal vertex u of P. Since G is (P, N, M)-distant, we have $\ell(P) = 2$, thus $u = p_1$. Let e be the edge crossing uv and let G_1 and G_2 be the uv-components of $G - e$ such that $p_0 \in V(G_1)$ and $p_2 \in V(G_2)$. Let $P_1 = p_0p_1v$ and $P_2 = p_2p_1v$, and let $e = x_1x_2$, where $x_i \in V(G_i)$ for $i \in \{1, 2\}$. Note that G_i is $(P_i, (N \cap V(G_i)) \cup \{x_i\}, M \cap E(G_i))$-distant. If G_i contains an edge f different from p_0p_1, p_0v, p_1v, then by the minimality of G there exists an L-coloring φ_{3-i} of $G - f \supseteq G_{3-i} + x_1x_2$. If additionally $|L(x_i)| \in \{1, 5\}$, then define L_i to be the list assignment for G_i such that $L_i(v) = \{\varphi_{3-i}(v)\}$, $L_i(x_i) = L(x_i) \setminus \{\varphi_{3-i}(x_{3-i})\}$, and $L_i(z) = L(z)$ for any other vertex z. Observe that G_i is not L_i-colorable, and we conclude that it violates (P), (T) or (O). (For (S) to hold, we add x_i to N). Since φ_{3-i} is a coloring of $G-f$, (P) is satisfied for G_i. Since G is (P, N, M)-distant and contains no non-crossed chord, it follows that G_i satisfies (T). Thus, G_i violates (O). The corresponding obstruction is $\mathrm{O_{N1}}$ since all others either have a special subgraph of G that would violate the distance condition in G, or have a non-crossed chord incident with p_1. Together with (59) and (60), we have that for each $i \in \{1, 2\}$, one of the following holds:

- $x_i \in V(P_i)$ and either $G_i = P_i$ or G_i is the triangle on $V(P_i)$, or
- $|L(x_i)| \in \{3, 4\}$, or
- G_i is equal to $\mathrm{O_{N1}}$ and x_i is its vertex with list of size four.

Since we already excluded all chords not incident with p_1, at most one of x_1 and x_2 has list of size three or four. By symmetry assume that $|L(x_1)| \in \{1, 5\}$. If $|L(x_2)| \in \{1, 5\}$, then all the possible combinations of such graphs G_1 and G_2 are either L-colorable or equal to $\mathrm{O_{C1}}$. Therefore, $|L(x_2)| \in \{3, 4\}$. Since every chord of F is crossed and incident with an internal vertex of P, we have $x_1 \notin V(F)$, thus G_1 is $\mathrm{O_{N1}}$. Let w be the vertex of G_1 with list of size three, $G' = G - \{w, p_0\} - p_1v$ and L' the list assignment such that $L'(x_1) = \{\varphi_1(x_1)\}$, $L'(v) = \{\varphi_1(v)\}$ and $L'(z) = L(z)$ otherwise. Note that G' is $(p_2p_1x_1v, N \cap V(G'), M \cap E(G'))$-distant and not L'-colorable. If v has degree at least 5 in G, then it has degree at least three in G'. Together with (59), this implies that x_2 is not adjacent to v, hence (T) holds. If v has degree at most four, then $|L(v)| \leq 4$, and by (C), $|L(x_2)| = 4$, and again (T) holds. Therefore, G' violates (O). Since x_1 has degree three in G' and it is adjacent to a vertex with list of size three or four, G' contains (and by (59), is equal to) $\mathrm{O_{P1}}$ or $\mathrm{O_{P2}}$. However, then G is L-colorable. Therefore, we obtain the following conclusion:

(63) *F has no chords.*

An easy corollary is that

(64) *no vertex of P is incident with a crossed edge.*

Indeed, if vp were a crossed edge with $p \in V(P)$, then by (63), we have $v \notin V(F)$. Furthermore, since P is incident with a crossing, we have $\ell(P) \leq 2$. Let L' be the list assignment such that $L'(v) = L(v) \setminus L(p)$ and L' matches L on the rest of the vertices of G. Note that $G - vp$ is not L'-colorable, and by the minimality of G, we conclude that $G - vp$ contains $\mathrm{O_{N1}}$, $\mathrm{O_{N2}}$ or $\mathrm{O_{N3}}$, whose internal vertex with list of size 4 is v. It cannot contain $\mathrm{O_{N1}}$, since v is not adjacent to all vertices of P in $G - vp$. Similarly, it cannot contain $\mathrm{O_{N3}}$, since the edge vp would be crossed twice. If $G - vp$ contains $\mathrm{O_{N2}}$, then G contains $\mathrm{O_{C1}}$. Comparison of bad lists for $\mathrm{O_{N2}}$ and $\mathrm{O_{C1}}$ in Figure 9.4 shows that $\mathrm{O_{C1}}$ is not L-colorable, which is a contradiction to the assumption that (O) holds for G and L.

Consider now a vertex v with three neighbors $p_a, p_b, p_c \in V(P)$, where $a < b < c$. Let K be the cycle $p_ap_{a+1}\ldots p_cv$, and note that K has a chord vp_b. By (64), none of the edges vp_a, vp_b and vp_c is crossed. By (59)(d), K is not separating ((59)(d) allows a vertex of N or a K_4 with a crossed edge in the interior of K; however, this would only be possible if $\ell(P) = 3$, yielding two special subgraphs at distance 1). Suppose that $c - a = \ell(P)$, and let G_2 be the p_avp_c-component of G that does not contain P. Since $v \notin V(F)$, and $v \notin N$ if $\ell(P) = 3$, there is a color in $L(v)$ that does not appear in the lists of vertices in P. Observe that G_2 (with the precolored path p_avp_c) violates either (T) or (O). In the former case, G is either $\mathrm{O_{N1}}$ or $\mathrm{O_{P6}}$. In the latter case, we have $\ell(P) = 2$ by the distance condition, and (58) implies that G_2 is either $\mathrm{O_{M1}}$ or $\mathrm{O_{C1}}$. If G_2 is $\mathrm{O_{M1}}$, then G is $\mathrm{O_{M2}}$, and if G_2 is $\mathrm{O_{C1}}$, then G is L-colorable.

Finally, consider the case that $\ell(P) = 3$ and v is adjacent to say p_0, p_1 and p_2 and is not adjacent to p_3. If $L(p_0) = L(p_2)$, then $G - vp_2$ is a counterexample to Theorem 100 contradicting the minimality of G. Therefore, $L(p_0) \neq L(p_2)$. Since the edges vp_0, vp_1, and vp_2 are not crossed, the degree of p_1 is three. Let $G' = G - p_1 + p_0p_2$, with the list assignment L' such that $L'(v) = L(v) \setminus L(p_1)$ and $L'(z) = L(z)$ otherwise. Note that G' is $(p_0p_2p_3, N \cup \{v\}, M)$-distant, since the rank of the special subgraph p_1p_2 in G is greater than the rank of the special subgraph v (in G'), and any path Q between two special subgraphs S_1 and S_2 that uses the new edge p_0p_2 gives rise to paths between S_1 or S_2 and the middle edge p_1p_2 of P in G, thus implying $\ell(Q) \geq 14 + r(S_1) + r(S_2) + 2r(p_1p_2) - 1 > 7 + r(S_1) + r(S_2)$. We conclude that G' violates (O) and contains O_{N1}, O_{N2} or O_{N3} that is not L'-colorable; however, then G contains a non-L-colorable obstruction O_{P6}, O_{P4} or O_{P5}, respectively. Therefore, we have:

(65) *Every vertex has at most two neighbors in P.*

Suppose now that uv and xy are edges crossing each other and $u, x \in V(F)$. By (64), neither u nor x belongs to P. Let c be the curve formed by the part of the edge uv between u and the crossing and the part of the edge xy between the crossing and x. If c is not part of the boundary of F, then let G_2 be the subgraph of G drawn inside the closed disc bounded by c and the part of the boundary of F between u and x that does not contain P. Note that there are two possible situations, depending on whether G_2 includes the vertices v and y or not. In any case, we can write $G = G_1 \cup G_2$, where $G_1 \cap G_2$ consists only of vertices u and x. Let G_2' be the graph obtained from G_2 by adding a common neighbor w of u and x. No L-coloring of G_1 extends to an L-coloring of G_2' (where w is assigned an arbitrary color different from the colors of u and x). By the minimality of G, we conclude that G_2' (with the precolored path uwx) violates (P), thus $ux \in E(G)$. The conclusion is:

(66) *If u and x are crossing-adjacent and $u, x \in V(F)$, then either $ux \in E(G)$, or the crossing incident with u and x belongs to the boundary of F.*

Similarly, we derive the following property:

(67) *Suppose that $Q = x_1x_2 \ldots x_{t-1}x_t$ is a path in G, where $t \leq 6$ and x_1x_2 crosses $x_{t-1}x_t$. Let c be the closed curve consisting of the path $x_2 \ldots x_{t-1}$ and parts of the edges x_1x_2 and $x_{t-1}x_t$, and let X be the set of vertices of G drawn in the open disc bounded by c. If $x_1 \notin X$, then $X = \emptyset$.*

The proof of (67) proceeds as follows. Observe first that the curve c is not crossed since all its edges are close to a crossing. If the path $x_2 \ldots x_{t-1}$ is induced in G, then the subgraph of G drawn inside the closed disc bounded by c, with the precolored path $x_2x_3 \ldots x_{t-1}$, would be a counterexample to Theorem 100 smaller than G (the distance constraints are satisfied even if $t = 6$, since the middle edge of the path $x_2x_3x_4x_5$ has smaller rank then the crossing, whose distance to x_3x_4

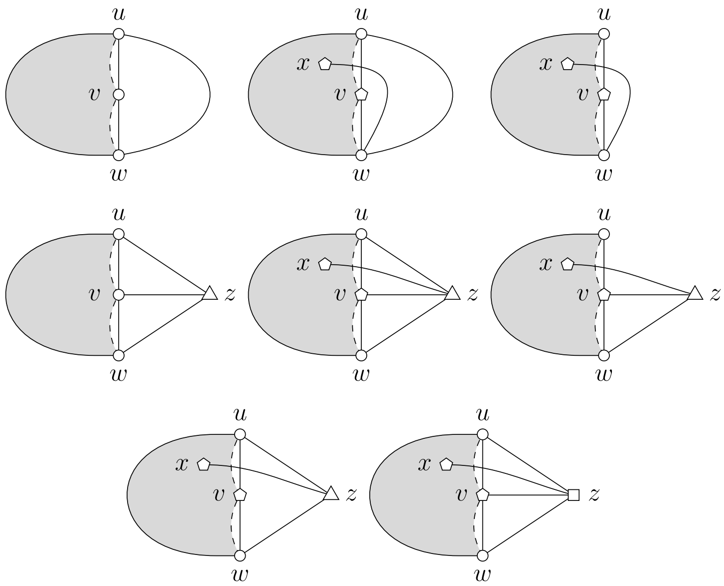


Figure 9.6: Possible cases for G_2 for a 2-chord uvw

in G is one). If $x_2 \ldots x_{t-1}$ contains a chord $x_i x_j$ $(i < j)$, then we first consider $Q' = x_1 x_2 \ldots x_i x_j \ldots x_t$ and then apply (59)(d) to show that no vertices are contained in the interior of the cycle $x_i x_{i+1} \ldots x_j x_i$.

Now, we shall consider the 2-chords of F.

(68) *Let uvw be a 2-chord of F such that vw is not crossed. Let c be the closed curve bounding the outer face of G and q the curve corresponding to the 2-chord uvw. Let c_1 and c_2 be the simple closed curves in $c \cup q$ distinct from c, and let G_1 and G_2 be the subgraphs of G drawn inside c_1 and c_2, respectively, so that $G_1 \cap G_2 = uvw$ and $G_1 \cup G_2$ is equal to G if the edge uv is not crossed, and is equal to $G - xy$ if uv is crossed by the edge xy. If neither u nor w is an internal vertex of P and $P \subseteq G_1$, then one of the following holds (cf. Figure 9.6):*

- *$V(G_2) = \{u, v, w\}$, and either uv is not crossed and $uw \in E(G)$, or uv is crossed by an edge incident with w; in the latter case, uw may or may not be an edge.*
- *$V(G_2) = \{u, v, w, z\}$ for a vertex z with list of size three, and either uv is not crossed and $uz, vz, wz \in E(G)$, or uv is crossed by an edge incident with z, $zw \in E(G)$ and at least one of uz and vz is an edge.*
- *$V(G_2) = \{u, v, w, z\}$ for a vertex z with list of size four adjacent to u, v, w and incident with an edge crossing uv.*

Proof. Let us consider a 2-chord uvw that does not satisfy the conclusion of the claim such that G_2 is maximal. First, suppose that uv is not crossed. An

L-coloring of G_1 does not extend to an L-coloring of G_2, hence G_2 (with the precolored path uvw) violates (P), (T) or (O). If G_2 violates (P) or (T), then by (60) and (63) the outcome of the claim holds. Therefore, we conclude that G_2 violates (O). Since the obstruction in G_2 violating (O) contains a special subgraph with a vertex distinct from v and $v \notin V(F)$, we conclude that $v \notin N$, and hence $|L(v)| = 5$. By (59) and (60) we also conclude that G_2 is the obstruction. Let S be the set of L-colorings of uvw that do not extend to an L-coloring of G_2. The inspection of the non-colorable obstructions with $\ell(P) = 2$ in Figure 9.4 shows that one of the following holds:

(R1) there exists a set A of at most two colors and S contains only colorings ψ such that $\psi(v) \in A$, and furthermore, if $|A| = 2$ then neither u nor w has list of size three; or,

(R2) S contains only colorings ψ such that $\psi(u) = \psi(w)$, and neither u nor w has list of size three.

Indeed, by (58), all obstructions except for O_{M1} and O_{C1} satisfy (R1) with $|A| = 1$. If G_2 is O_{M1} or O_{C1}, then neither u nor w has list of size three, by (M) together with the distance condition and by (C). The inspection of the colorings shows that if G_2 is O_{C1}, then (R1) holds with $|A| = 2$, and if G_2 is O_{M1}, then either (R1) holds with $|A| = 2$, or (R2) holds (the latter is the case when the two lists of size 3 are equal, i.e., $a = c$ in Figure 9.4).

If (R1) holds, then let $G' = G_1$, with the list assignment L' such that $L'(v) = L(v) \setminus A$ and $L'(z) = L(z)$ for $z \neq w$. Note that if $|A| = 2$, then v has no neighbor in G_1 with list of size three by (R1) and by the maximality of G_2. If (R2) holds, let $G' = G_1 + uw$ with the list assignment $L' = L$. In either case, G' is not L'-colorable and it is $(P, N \cap V(G'), M \cap E(G'))$-distant (in the latter case, any path Q between special subgraphs H_1 and H_2 using the added edge uw gives rise to paths from H_1 and H_2 to the special subgraph of G_2, and thus $\ell(Q) \geq 14 + r(H_1) + r(H_2) - 3$). Furthermore, G' satisfies (T) by (63) and (65), and if G' violated (C) or (O), then v or uw would have to belong to a crossing or to an obstruction in G', and the distance between its special subgraph and the special subgraph of G_2 would be at most 4. Note that G' cannot violate (P), as otherwise $u, w \in V(P)$ and G_2 is O_{M1}, and by (59) and (65), v would have degree four and list of size five. Therefore, G' is a counterexample to Theorem 100 smaller than G, which is a contradiction.

Suppose now that uv is crossed by an edge xy, where $x \in V(G_1)$ and $y \in V(G_2)$. If $y = w$, then the conclusion of the claim holds by (66), hence assume that $y \neq w$. Furthermore, $x \neq w$ by (66), and $uw \notin E(G)$ by (60). Let G_1' be the graph obtained from G_1 by adding the edges ux and vx (if they are not present already). Note that this can be done without introducing any new crossings. Since u, v and x are incident with a crossing in G, G_1' is $(P, N \cap V(G_1'), M \cap E(G_1'))$-distant. Furthermore, G_1' does not contain any obstruction, as its special subgraph would be at distance at most 2 from the crossing. By (63), u has at most one neighbor in P within G, hence G'_1 satisfies (T). By (64), u and x cannot belong to P, hence by the minimality of G, there exists an L-coloring φ of G'_1. Let G'_2 be the graph obtained from $G_2 - uv$ by adding the vertex x and edges ux, vx, yx. Consider the list assignment L'_2 for G'_2 such that $L'_2(z) = \{\varphi(z)\}$ for $z \in \{u, v, w, x\}$ and $L'_2(z) = L(z)$ otherwise. Note that G'_2 is not L'_2-colorable and that it is $(uxvw, N \cap V(G'_2), M \cap E(G'_2))$-distant.

Since $y \neq w$ and since $uw \notin E(G'_2)$, the graph G'_2 satisfies (P) for the φ-colored path $uxvw$. If G'_2 violates (T), then by (60) we have that $|L(y)| = 3$ and y is adjacent to at least two of u, v and w. In particular, $y \in V(F)$. Observe that if $vy \in E(G)$, then the yvw-component K that does not contain P can only be a triangle (since otherwise any L-coloring of the other yvw-component K' would extend to K, and K' would contradict the minimality of G). Thus $yw \in E(G)$. By (67) applied to the path $xywvu$, we have $V(G_2) = \{u, v, w, y\}$ and the conclusion of the claim holds. Let us now consider the remaining case that G'_2 satisfies (T), and thus by the minimality of G, it violates (O). By (59) and (60), G'_2 is equal to one of $\mathrm{O}_{\mathrm{P1}}, \ldots, \mathrm{O}_{\mathrm{P6}}$, but not O_{P3} since x has degree 3 in G'_2.

If G'_2 is O_{P1}, then the conclusion of the claim holds. Otherwise, let us define S as the set of colorings ψ of the path $uxvw$ that do not extend to an L-coloring of G'_2 and satisfy $\psi(u) \neq \psi(v)$. The inspection of the obstructions and their problematic list assignments displayed in Figure 9.4 shows that either (R1) or one of the following holds:

(R3) G'_2 is O_{P2} and there exists a color c such that S contains only colorings ψ such that either $\psi(u) = c$ and $\psi(x) = \psi(w)$, or $\psi(x) = c$ and $\psi(u) = \psi(w)$. Moreover, $|L(u)| \neq 3$ and $|L(w)| \neq 3$.

(R4) G'_2 is O_{P4} and there exists a color c such that S contains only colorings ψ satisfying either $\psi(v) = c$ or $\psi(x) = c$. Moreover, $|L(u)| \neq 3$.

Again, the conclusions that the specified vertices do not have lists of size three follow in all applicable cases by noting that otherwise either (C) or the distance condition would be violated. Let us remark that for O_{P2} we have (R1) if the colors a, b, c, d in Figure 9.4 are different; we have (R3) if $b = d$ or $a = d$. To argue for O_{P4}, O_{P5}, O_{P6} we observe that $\psi(x)$ and $\psi(v)$ should be taken from the difference of the lists of the two neighbors of u (so these are colors b, c in Figure 9.4). This yields (R1) with the only exception in the case of O_{P4}, where we cannot argue about $|L(w)| \neq 3$, so we need (R4) in this (and only this) case.

The condition in (R3) that the lists of u and w do not have precisely three elements is argued as follows. Since x has degree 3 in G'_2, the vertex z of O_{P2} with list of size 3 is not the vertex y, and v, w are both adjacent to z. Since $|L(z)| = 3$ and the edge wz is close to a crossing in G, we conclude that $wz \notin M$ and hence $|L(w)| \neq 3$. Since $|L(y)| = 4$, (C) implies that $|L(u)| \neq 3$.

Now, the case when (R1) holds is handled in the same way as the case when uv was not crossed. If (R3) holds, then we let $G_1' = G_1 + uw$ with the list assignment obtained from L by removing c from the list of u (note that $|L(u)| \neq 1$ by (64)); we may need to add an edge incident with u to M in order to satisfy (M). If (R4) holds and $|L(x)| = 5$, then let $G_1' = G_1$ with the list assignment obtained by removing c from the lists of x and v (and adding x to N). In all the cases, G_1' satisfies the assumptions of the theorem. Indeed, (P) is trivial, since $u \notin V(P)$ by (64). Similarly, (T) follows by (63) and (65). Finally, (O) holds since by the distance condition, we could only create $\mathrm{O_{M1}}$, $\mathrm{O_{M2}}$, $\mathrm{O_{N1}}$, $\mathrm{O_{N2}}$ or $\mathrm{O_{N3}}$, and each of them is excluded by (63) or (65). Therefore, G_1' contradicts the minimality of G, since its coloring would extend to an L-coloring of G.

Finally, consider the case that (R4) holds and $|L(x)| \in \{3, 4\}$. By (66), all neighbors of u distinct from x belong to G_2. By (64), we have $u \notin V(P)$, $\deg(u) \geq |L(u)| \geq 4$, and thus u is adjacent to x and $|L(u)| = 4$. Since G_2' is $\mathrm{O_{P4}}$, every coloring of x, v and w extends to an L-coloring of G_2, hence G_1 contradicts the minimality of G. This completes the proof of (68). □

Similarly, one can prove the following:

(69) *Let $u, w \in V(F)$ be distinct vertices, neither of which is an internal vertex of P. Suppose that $v \notin V(F)$ is a vertex adjacent to w and crossing-adjacent to u. Let c be the closed curve not containing P consisting of vw, parts of the crossed edges incident with u and v, and a part of the boundary of F between u and w that does not contain P, and let G_2 be the subgraph of G drawn in the closed disc bounded by c. Then G_2 does not contain the crossing and satisfies one of the following:*

(a) $V(G_2) = \{u, v, w\}$ *and* $uw \in E(G)$*, or*

(b) $V(G_2) = \{u, v, w, z\}$*,* $|L(z)| = 3$ *and* z *is adjacent to* u*,* v *and* w*.*

Proof. By (68), it suffices to consider the case that $uv \notin V(G)$. Let G_1' be the graph obtained from G_1 as follows: If $uw \in E(G)$, then we add the edge uw. If u, v and w have a common neighbor z with list of size three, then we add z and incident edges. If $V(G_1') = V(G)$, then (a) or (b) holds. Otherwise, there exists an L-coloring ψ of G_1' by the minimality of G. Let L' be the list assignment such that $L'(v) = \{\psi(v)\}$, $L'(w) = \{\psi(w)\}$, $L'(u) = \{c\}$ for a new color c, $L'(x) = (L(x) \setminus \{\psi(u)\}) \cup \{c\}$ for each neighbor x of u distinct from v and w and $L'(x) = L(x)$ for all other vertices x. Note that $G_2' = G_2 + uv$ is not L'-colorable, and by the minimality of G, one of the assumptions of the theorem is violated in G_2'. By the construction of G_1' and the choice of ψ, (P) and (T) hold. By the distance condition, the only obstruction that can appear in G_2' is $\mathrm{O_{C1}}$. However, then the 2-chord wvt (where t is the neighbor of u in G_2 with list of size four) contradicts (68). □

Let us now introduce a way of defining list assignments that will be used throughout the rest of the paper. Let φ be any proper partial L-coloring of G such that $\varphi(v) \notin L(p)$ for every pair of adjacent vertices $v \in \mathrm{dom}(\varphi)$ and $p \in V(P)$. For each vertex $z \in V(G)$, let

$$R_z = \bigcup_{p \in V(P) \setminus \mathrm{dom}(\varphi), zp \in E(G)} L(p).$$

We define L_φ to be the list assignment such that

$$L_\varphi(z) = \Big(L(z) \setminus \{\varphi(x) : x \in \mathrm{dom}(\varphi), xz \in E(G)\}\Big) \cup R_z.$$

Let us also define $G_\varphi = G - \mathrm{dom}(\varphi)$. Consider any L_φ-coloring ψ of G_φ. We claim that the combination of φ with ψ is a proper L-coloring of G. Indeed, for any $z \in V(G_\varphi)$, we clearly have $\psi(z) \notin R_z$, and thus $\psi(z) \in L_\varphi(z)$ is different from the colors of the neighbors of z in $\mathrm{dom}(\varphi)$. Since G is not L-colorable, we conclude that G_φ is not L_φ-colorable.

Suppose now that G contains a subgraph H isomorphic to one of the graphs drawn in Figure 9.3 such that the subgraph of H corresponding to full-circle vertices is equal to P, triangle vertices have lists of size *at least* three, square vertices have lists of size *at least* four and pentagonal vertices have lists of size five. Then we say that H is a *near-obstruction.*

(70) *If H is a near-obstruction, then H is isomorphic to one of* $\mathrm{O_{M1}}$*,* $\mathrm{O_{N2}}$*,* $\mathrm{O_{N3}}$ *or* $\mathrm{O_{P3}}$*. Furthermore, $|(V(H) \cap V(F)) \setminus V(P)| \leq 1$, and if $(V(H) \cap V(F)) \setminus V(P) \neq \emptyset$, then H is* $\mathrm{O_{N2}}$ *or* $\mathrm{O_{N3}}$*.*

Proof. By (65), H is isomorphic to one of $\mathrm{O_{M1}}$, $\mathrm{O_{N2}}$, $\mathrm{O_{N3}}$, $\mathrm{O_{C2}}$, $\mathrm{O_{C3}}$, $\mathrm{O_{C4}}$, $\mathrm{O_{C5}}$ or $\mathrm{O_{P3}}$.

By (68), if H is $\mathrm{O_{C5}}$, then $V(G) = V(H) \cup \{z\}$, where z is a vertex of degree three adjacent to p_0, p_2 and the vertex $w \notin V(P)$ in the outer face of H. However, the distance condition implies that $w \notin N$, so that $|L(w)| = 5$. This implies that G is L-colorable, which is a contradiction.

If H is $\mathrm{O_{C2}}$, then let $p_0w_1w_2p_2$ be the path in the outer face of H. If $w_1, w_2 \in V(F)$, then $V(G) = V(H)$ and G is L-colorable by (O). Thus assume that $w_2 \notin V(F)$; hence $|L(w_2)| = 5$. If $w_1 \in V(F)$, then since w_2 has degree at least 5, by (68) we have that $V(G) = V(H) \cup \{z\}$, where z is adjacent to w_1, w_2 and p_2. However, then G is L-colorable. Therefore, $w_1 \notin V(F)$. Let φ be an L-coloring of H and G_2 the $p_0w_1w_2p_2$-component of G that does not contain P. Since φ does not extend to an L-coloring of G_2, it follows that G_2 with the precolored path $p_0w_1w_2p_2$ violates (P), (T) or (O). Since both w_1 and w_2 have degree at least 5, it follows that $p_0w_2 \notin E(G)$ and $w_1p_2 \notin E(G)$, hence (P) holds. Suppose that G_2 violates (T). Then a vertex z with list of size three is adjacent to three vertices among p_0, w_1, w_2 and p_2. If it is adjacent to all four of them, then G contains

O_{C5} which has already been excluded. Otherwise, since w_1 and w_2 have degree at least 5, by (61) z cannot be adjacent to p_0, w_1, and p_2. By symmetry, we may assume that z is adjacent to p_0, w_1, w_2. Then (68) applied to the 2-chord zw_2p_2 shows that there is a vertex z' adjacent to z whose list has size 3, and we see that the special edge zz' gives a contradiction. Finally, if G_2 violates (O), then the obstruction is equal to one of O_{P1}, O_{P2}, O_{P3}, O_{P4}, O_{P5} or O_{P6}, and now it is easy to see (by comparing bad lists for the obstructions) that G is L-colorable.

If H is O_{C3}, then let w_1 be the vertex of H drawn by the triangle and w_2 the vertex of P that is not adjacent to it in G. If H is O_{C4}, then let w_1 and w_2 be the vertices of H drawn by triangles. By symmetry, we can assume that w_1 is the neighbor of p_2. Let $w_1x_1x_2w_2$ be the path in H formed by neighbors of p_1. Note that $|L(w_i)| \in \{1, 5\}$ by (63). Choose an L-coloring φ of the subgraph of G induced by $V(P) \cup \{w_1, w_2\}$ such that $\varphi(w_1) \neq \varphi(w_2)$ and either $|L_\varphi(x_1)| \geq 4$ or $L_\varphi(x_1) \neq L_\varphi(x_2)$. Note that this is possible since $|L(w_1)| = 5$. Let $G' = G - \{p_1, x_1, x_2\} + w_1w_2$ with the list assignment L' such that $L'(z) = \{\varphi(z)\}$ for $z \in \{w_1, w_2\}$ and $L'(z) = L(z)$ otherwise. Observe that G' is not L'-colorable (since every L'-coloring of G' extends to an L-coloring of G) and that it satisfies (P) for the precolored path $w_2w_1p_2$ or $p_0w_2w_1p_2$. By the minimality of G, G' violates (T) or (O). In the former case, by symmetry we can assume that there exists a vertex $z \in V(G)$ such that $|L(z)| = 3$ and z is adjacent to p_2, w_1 and either w_2 or p_0. It follows that G contains a separating 4-cycle formed by non-crossed edges, and by (59) the interior of this 4-cycle contains K_4. By (63) and (68), there are no other vertices in G. Now, it is easy to see that the resulting graph is L-colorable. Therefore, G' violates (O). Since G is (P, N, M)-distant, this is only possible if H is O_{C4}. In this case the obstruction in G' is one of O_{P1}–O_{P6}. Note that the edge w_1w_2 is contained in a triangle; let z be their common neighbor. By (59), the 4-cycle $w_1zw_2p_1$ surrounds K_4 in G. However, the graphs obtained from the obstructions O_{P1}–O_{P6} with the precolored path $p_0w_2w_1p_2$ by adding the vertex p_1 joined to the vertices p_0, w_1, w_2, p_2, deleting the edge w_1w_2, and adding K_4 inside the 4-cycle $w_1zw_2p_1$ are easily seen to be L-colorable.

The remaining obstructions are O_{M1}, O_{N2}, O_{N3}, and O_{P3}. If H is O_{M1} or O_{P3}, then none of the vertices in $V(H) \setminus V(P)$ belongs to F since this would contradict (63). In the other cases, at most one of the vertices of H can belong to F by the same reason. □

Observe that $\ell(P) \geq 2$, since if $\ell(P) < 2$, then we can precolor $2 - \ell(P)$ more vertices of F (by (64), we can extend P in the boundary of F). Let $p_k \ldots p_1p_0v_1v_2 \ldots v_s$ be the vertices contained in the boundary of F in the cyclic order around it. We either have $v_iv_{i+1} \in E(G)$, or v_i and v_{i+1} are crossing-adjacent, for each i. As we already observed, $p_0v_1, p_kv_s \in E(G)$. We also define $v_{s+1} = p_k$, $v_{s+2} = p_{k-1}$, ...

If $s = 0$, then let φ be the L-coloring of p_0. Then G_φ with the list assignment L_φ is a smaller counterexample to Theorem 99. This contradiction shows that

$s \geq 1$.

Suppose that $s = 1$ and let φ be the partial coloring that assigns a color in $L(v_1) \setminus (L(p_0) \cup L(p_k))$ to v_1. Note that if v_1 is adjacent to a vertex x by a crossed edge, then G_φ is $(P, N \cup \{x\}, \emptyset)$-distant, otherwise G_φ is $(P, N, \emptyset)$-distant. By the minimality of G, G_φ with the list assignment L_φ violates (O) and contains an obstruction H, which by (70) is one of $\mathrm{O_{M1}}$, $\mathrm{O_{N2}}$, $\mathrm{O_{N3}}$ or $\mathrm{O_{P3}}$. Note that if $z \in V(H)$ is a vertex with list of size three according to L_φ, then z is adjacent to v_1 and belongs to N; but z is at distance at most one from the special subgraph in the obstruction, contradictory to the assumption that G is (P, N, M)-distant. Therefore, $s \geq 2$.

If v_1 is not adjacent to v_2 (i.e., v_1 and v_2 are crossing-adjacent), then let φ be a coloring that assigns a color from $L(v_1) \setminus L(p_0)$ to v_1 and the color from $L(p_0)$ to p_0. Note that $\ell(P) = 2$ by the distance condition. Let y be the vertex adjacent to v_1 by the crossed edge, and note that G_φ is $(p_1p_2, N \cup \{y\}, M)$-distant. By the minimality of G together with (59), (63) and (65), we conclude that G_φ with the list assignment L_φ violates (N) or (M). In the former case, we have $|L_\varphi(y)| = 3$ and since $|L(y)| = 5$, it follows that y is adjacent to p_0. However, by (60), v_2 would be adjacent to p_0, contrary to (63). In the latter case, p_0 and v_1 have a common neighbor $u \neq y$ adjacent to a vertex w with $|L(w)| = 3$. This contradicts (68). Therefore, $v_1v_2 \in E(G)$, and by symmetry, $v_{s-1}v_s \in E(G)$.

Suppose now that $s = 2$. By symmetry, assume that if v_2 is incident with a crossed edge, then v_1 is incident with a crossed edge as well. If $v_1v_2 \in M$, then let φ be an L-coloring of v_1 and v_2 such that $\varphi(v_1) \notin L(p_0)$ and $\varphi(v_2) \notin L(p_k)$. Otherwise, let φ be a coloring of v_1 by a color in $L(v_1) \setminus L(p_0)$ such that if $|L(v_2)| = 3$, then $\varphi(v_1) \notin L(v_2) \setminus L(p_k)$. Note that this is possible by (59)(f). Let us remark that when $|L(v_2) \setminus \{\varphi(v_1)\}| = 2$, then $L(p_k) = \{\varphi(v_1)\}$ and $L_\varphi(v_2) = L(v_2)$ by the definition of L_φ, and thus we always have $|L_\varphi(v_2)| \geq 3$. If G_φ with the list assignment L_φ violated (C), then v_2 would have to be incident with a crossing, and by the choice of v_1, the vertex v_1 would be incident with the same crossing, which then would not appear in G_φ. Therefore, G_φ satisfies (C). If v_1 is incident with a crossed edge v_1x, then let $N' = N \cup \{x\}$; if v_1 is adjacent to a vertex $y \in N$, then let $N' = N \setminus \{y\}$; otherwise let $N' = N$. If v_1 and v_2 have a common neighbor z belonging to N, then let $M' = M \cup \{v_2z\}$; otherwise let $M' = M \setminus \{v_1v_2\}$. Observe that G_φ is (P, N', M')-distant and that it satisfies (S), (N) and (M). By the minimality of G, G_φ violates (O) and thus G contains a near-obstruction H. By (70), H is $\mathrm{O_{M1}}$, $\mathrm{O_{N2}}$, $\mathrm{O_{N3}}$ or $\mathrm{O_{P3}}$. Observe that $v_1v_2 \notin M$, since otherwise the distance between v_1v_2 and the special subgraph of H (which is also special in G) is at most 3. Every vertex with list of size three according to L_φ either belongs to N or is equal to v_2. If $v_2 \notin V(H)$, then H contains only one vertex with list of size three, hence H is $\mathrm{O_{N2}}$. However, then N contains two adjacent vertices, which is a contradiction. Similarly, we exclude the case that $v_2 \in V(H)$ and H is $\mathrm{O_{N3}}$ or $\mathrm{O_{P3}}$. Therefore $v_2 \in V(H)$ and H is $\mathrm{O_{M1}}$ or $\mathrm{O_{N2}}$. The former is excluded by (63). If H is $\mathrm{O_{N2}}$, then we have $V(G) = V(H) \cup \{v_1\}$

by (60). If v_1 is incident with a crossed edge, then G contains O_{C2}. On the other hand, if v_1 is not incident with a crossed edge, then $|L(v_1)| = 3$, $|L(v_2)| = 4$, $|N| = 1$ and G is L-colorable. This is a contradiction, hence $s \geq 3$.

Next, observe that if v_1 and v_2 are not crossing-adjacent, then $|L(v_1)| = 3$ or $|L(v_2)| = 3$. Otherwise, we could remove a color from the list of v_1. If the edges v_1x and e cross, then $|L(x)| = 5$ by (63), and both vertices incident with e have list of size five by (64) and (66), hence G with the new list assignment satisfies (C). By (70), no obstruction arises (since all vertices with lists of size three or four in the new list assignment are contained in $V(F)$). Thus G with the reduced lists satisfies (O) and contradicts the minimality assumption. Similarly, if v_1 and v_2 are crossing-adjacent and $|L(v_1)| > 3$ and $|L(v_2)| > 3$, then we can assume that $|L(v_1)| = |L(v_2)| = 4$.

If $|L(v_1)| = 3$ or $|L(v_2)| = 3$ and furthermore $v_1v_2, v_2v_3 \notin M$, then let the set $X \subseteq V(F) \setminus V(P)$ and its partial L-coloring φ be defined as in (X1)–(X4) in the proof of Theorem 99. Let us add two more cases for the situation when v_1 and v_2 are crossing-adjacent:

(X5) If $|L(v_1)| = |L(v_2)| = 4$ and $|L(v_3)| \neq 3$, then $X = \{v_1\}$ and $\varphi(v_1) \in L(v_1) \setminus L(p_0)$ is chosen arbitrarily.

(X6) If $|L(v_1)| = |L(v_2)| = 4$ and $|L(v_3)| = 3$, then $X = \{v_2\}$ and $\varphi(v_2) \in L(v_2) \setminus L(v_3)$ is chosen arbitrarily.

Let m be the largest index such that $v_m \in X$. Let us note that $m = 1$ in (X1) and (X5), $m = 3$ in (X4), and $m = 2$ otherwise. Also, $X = \mathrm{dom}(\varphi)$ in all cases except for (X4b), when $X = \{v_1, v_2, v_3\}$ and $\mathrm{dom}(\varphi) = \{v_1, v_3\}$.

(71) *One of the following cases holds:*

(A1) $v_1v_2 \in M$ *or* $v_2v_3 \in M$.

(A2) *Either* v_1 *and* v_2 *or two distinct vertices in dom*(φ) *have a common neighbor in* N.

(A3) *There exists a crossing* q *and two crossing-adjacent vertices* $w_1, w_2 \in V(G_q)$ *such that* $V(G_q) \cap X = \emptyset$, w_1 *has a neighbor in dom*(φ) *and* w_2 *has two neighbors in dom*(φ).

(A4) $v_mv_{m+1} \in E(G)$, *there exists a crossing* q *such that* $V(G_q) \cap X = \emptyset$ *and* $v_{m+1}, v_{m+2} \in V(G_q)$, *and either* $|L(v_{m+1})| = |L(v_{m+2})| = 4$ *or* $|L(v_{m+1})| = 5$ *and* $|L(v_{m+2})| = 3$.

(A5) $v_mv_{m+1} \in E(G)$, $|L(v_{m+1})| \in \{3, 4\}$ *and there exists a crossing* q *such that* $V(G_q) \cap X = \emptyset$, $v_{m+1} \in V(G_q)$ *and a neighbor* $w \notin V(F)$ *of* v_m *is crossing-adjacent to* v_{m+1}.

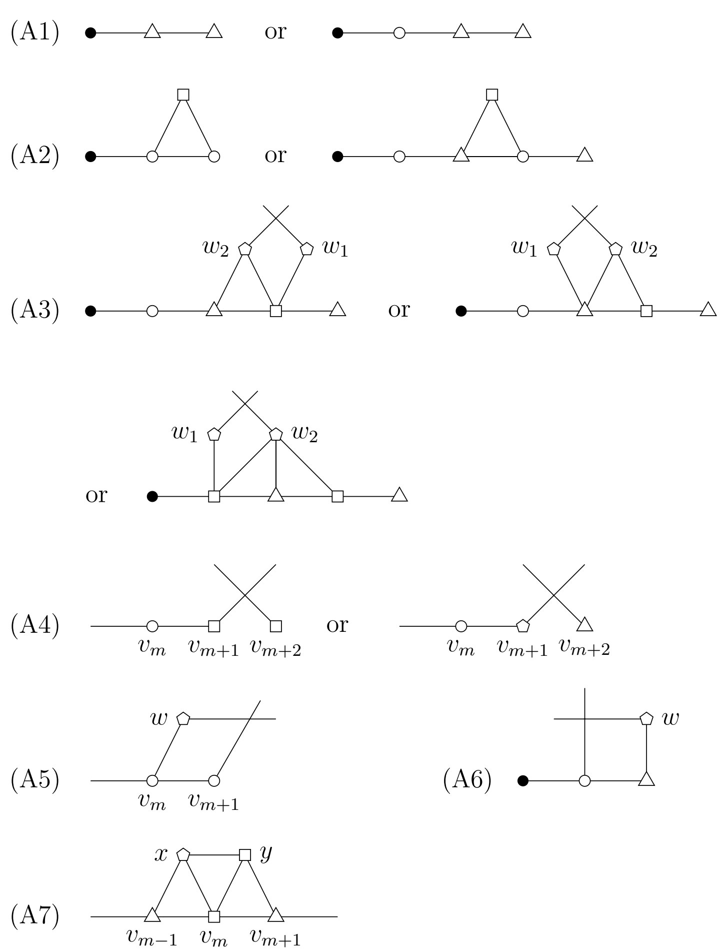


Figure 9.7: Possible outcomes of (71)

(A6) *$v_1 \notin X$ and there exists a crossing q such that $V(G_q) \cap X = \emptyset$, $v_1 \in V(G_q)$ and a neighbor $w \notin V(F)$ of v_2 is crossing-adjacent to v_1.*

(A7) *$|X| \geq 2$ and there exists a path $v_{m-1}xyv_{m+1}$, where x and y are neighbors of v_m and $y \in N$.*

Proof. See Figure 9.7 for the illustration of the possibilities. Note that if (A1) does not hold, then X and φ are defined.

Suppose now for a contradiction that none of (A1)–(A7) holds. Let us consider the graph $G'_0 = G - \mathrm{dom}(\varphi)$ with the list assignment L_φ, and let G' be the graph obtained from G'_0 by repeatedly removing vertices whose list is larger than their degree. If $\mathrm{dom}(\varphi) \neq X$, then we have case (X4b) and $X \setminus \mathrm{dom}(\varphi) = \{v_2\}$. If v_2 is not incident with a crossing, then its degree in G'_0 is 1, and since $|L_\varphi(v_2)| \geq 2$, it is not present in G'. On the other hand, if v_2 were incident with a crossing, then the fact that $|L(v_1)| = |L(v_3)| = 4$ would contradict (C). This shows that $G' \subseteq G - X$. Observe also that G' is not L_φ-colorable.

Next, we argue that G' satisfies the assumptions of Theorem 100 (with the sets $N' \subseteq V(G')$ and $M' \subseteq E(G')$ defined as the minimal sets such that (S), (N) and (M) hold), thus contradicting the minimality of G. The property (P) holds trivially, (T) holds by (65). To verify (S), note first that the only vertices not in the outer face of G' with list of size less than five are those belonging to N, or incident with a crossed edge joining them in G to a vertex in $\mathrm{dom}(\varphi)$; and the vertices of the latter kind have list of size four. Thus, they have been added into N' without violating the distance condition since the rank of special vertices in N' is smaller than the rank of the crossing. Next, suppose that a vertex $v \in V(G') \setminus V(P)$ satisfies $|L_\varphi(v)| \leq 2$. Note that $v \notin V(F)$ by the choice of X and φ and by (63). It follows that $v \in N$ and v has two colored neighbors in X, thus (A2) holds. This confirms that (S) is satisfied.

Now, let us consider property (C). Let q be a crossing in G' and suppose that (C) is violated at q, i.e., there exist distinct $u, v \in V(G_q)$ such that $|L_\varphi(u)| = 3$ and $|L_\varphi(v)| \in \{3, 4\}$. If both u and v belong to F, then by (63) and (66) we have that u and v are crossing-adjacent, $\{u, v\} = \{v_{m+1}, v_{m+2}\}$ and $L(v_{m+2}) = L_\varphi(v_{m+2})$. It follows that $|L(v_{m+1})| \neq 3$ and that (A4) holds. If $u \in V(F)$ and $v \notin V(F)$ and u and v are not crossing-adjacent, then since $V(G_q) \cap X = \emptyset$, (68) implies that (A4) holds. If $u \in V(F)$ and $v \notin V(F)$ and u and v are crossing-adjacent, then we apply (69). The outcome (a) of (69) gives (A5) or (A6). The outcome (b) gives a vertex $w \in X$ that is adjacent to v and a vertex z with $|L(z)| = 3$ that is adjacent to u, v and w. Therefore, $|L(u)| \neq 3$, so u has a neighbor in X. This is only possible in the subcase (X4a) of the definition of X, where $z = v_2$, $w = v_3$ and $u = v_1$, thus obtaining (A6). If $u \notin V(F)$ and $v \in V(F)$, then u has two neighbors in $\mathrm{dom}(\varphi)$. Since $V(G_q) \cap X = \emptyset$, (68) implies that u and v are crossing-adjacent. By (69), one of the neighbors of u

in X is also adjacent to v and has list of size three, and by the choice of X, we conclude that (A6) holds. Finally, if $u, v \notin V(F)$, then they are crossing-adjacent by (60) and the fact that $V(G_q) \cap X = \emptyset$, and (A3) holds.

Therefore, we can assume that G' satisfies (C). Let us now consider the newly created special subgraphs in G'. If $v \in N' \setminus N$, then v is adjacent to a vertex of X by an edge containing a crossing q, and no other vertex of G_q belongs to X. Therefore, there is at most one such vertex. Consider now an edge $xy \in M' \setminus M$; we will show that either there exists a crossing q such that $\{x, y\} = V(G_q) \setminus X$, or at least one of x and y belongs to N. Note that xy has been added to M' because $|L_\varphi(x)| = |L_\varphi(y)| = 3$. Suppose that $x, y \notin N$. If $x, y \notin V(F)$, then both x and y have two neighbors in $\mathrm{dom}(\varphi)$. It is easy to see using (60) and (61) that this implies that x and y are crossing-adjacent in G via the edges joining x, y with X. If $x, y \in V(F)$, then by (63) we can assume that $x = v_{m+1}$ and $y = v_{m+2}$; but then $|L_\varphi(x)| \neq 3$ or $|L_\varphi(y)| \neq 3$ by the choice of X, which is a contradiction. Finally, suppose that say $x \in V(F)$ and $y \notin V(F)$; then y has two neighbors in $\mathrm{dom}(\varphi)$ and, in particular, we have cases (X2) or (X4). By (68), we have $x \in \{v_1, v_{m+1}\}$. If $x = v_1$, then y would be a common neighbor of v_1, v_2 and v_3, contradicting the choice of X (assumptions of (X4b) are satisfied, hence we would have $v_1 \in X$). If $x = v_{m+1}$, then $|L(v_m)| = 4$ and therefore one of the edges $v_{m-1}y$ and $v_{m+1}y$ is crossed since $\deg(v_m) \geq 4$. However, by the choice of X we have $|L(v_{m-1})| = |L(v_{m+1})| = 3$, contradicting (C).

It follows that $d(S_1, S_2) \geq 7 + r(S_1) + r(S_2)$ whenever S_1 is a special subgraph of G that is also special in G' and S_2 is any special subgraph of G'. Suppose now that S_1 and S_2 are both distinct newly created special subgraphs in G'. Note that $|N' \setminus N| \leq 1$ and if $N' \setminus N \neq \emptyset$, then $M' \setminus M = \emptyset$. It follows that $S_1, S_2 \in M' \setminus M$. As proved in the previous paragraph, each edge in $M' \setminus M$ is incident with a special subgraph in G that is adjacent to X. By the distance condition, we conclude that there exists a path xyz in G' such that $|L_\varphi(x)| = |L_\varphi(y)| = |L_\varphi(z)| = 3$ and $y \in N$. Note that at most one of x and z can have two neighbors in $\mathrm{dom}(\varphi)$, as otherwise G would contain a crossing at distance at most one from y; thus we may assume that $x \in V(F)$. By (68), $x \in \{v_1, v_{m+1}, v_{m+2}\}$. If $x = v_{m+2}$, then we would have $|L(v_{m+1})| = |L(x)| = 3$ and $v_{m+1}x \in M$ would be at distance one from $y \in N$, which is a contradiction; therefore, $x \neq v_{m+2}$. If $x = v_1$, then (A2) holds, hence $x = v_{m+1}$ and $z \notin V(F)$ has two neighbors in $\mathrm{dom}(\varphi)$. However, then $|L(v_m)| = 4$, hence $\deg(v_m) \geq 4$ and v_m is adjacent to y and (A7) holds. We conclude that G' is (P, N', M')-distant.

Finally, suppose that G' violates condition (O), and thus G contains a near-obstruction H. By (70), H is one of $\mathrm{O_{M1}}$, $\mathrm{O_{N2}}$, $\mathrm{O_{N3}}$ or $\mathrm{O_{P3}}$.

- If H is $\mathrm{O_{M1}}$, then let xy be the edge of H that belongs to M', where x is adjacent to p_2. Note that $x, y \notin V(F)$ by (63) and $xy \notin M$. If $x \notin N$, then x has two neighbors v_i and v_j in $\mathrm{dom}(\varphi)$, where $i < j$. By (68) applied to p_2xv_i, we have $j = i + 1$ and by the choice of X, $|L(v_j)| = 4$; hence v_j is incident with a crossing and thus $y \notin N$. Consequently, y is also adjacent to v_i and v_j. However, note that $|L(v_i)| = 3$, contradicting (C) for G. Therefore, $x \in N$ is adjacent to v_j, and y is adjacent to both v_i and v_j. By (68) applied to p_0yv_j, we have $i = 1$, $j = 2$ and $|L(v_1)| = 3$, and by (68) applied to p_2xv_2, we have that $s = 3$ and $|L(v_3)| = 3$. However, then G is L-colorable.

- Next, suppose that H is $\mathrm{O_{N2}}$ and let x and y be the vertices in the outer face of H such that $|L_\varphi(x)| = 3$ and $|L_\varphi(y)| = 4$. By (63), $y \notin V(F)$. If $x \in V(F)$, then by (68) we have $s \leq 2$, which is a contradiction, hence $x \notin V(F)$. Thus x has two neighbors in $\mathrm{dom}(\varphi)$ and y has one, and by (68) we conclude that $s = 3$ and $|L(v_1)| = |L(v_3)| = 3$. It follows that $X = \{v_1, v_2\}$, x is adjacent to v_1 and v_2, and y is adjacent to v_2. There are two cases, either v_2 is incident with a crossed edge or $|N| = 1$; in both of them, G is L-colorable.

- If H is $\mathrm{O_{N3}}$, then let xyz be the path in the outer face of H such that $|L_\varphi(x)| = |L_\varphi(z)| = 3$, $|L_\varphi(y)| = 4$ and z is adjacent to p_1. By (63), $z \notin V(F)$, thus z has two neighbors $w_1, w_2 \in \mathrm{dom}(\varphi)$, and by (68), we can assume that the neighbors of w_1 are w_2, z and an endvertex of P, and that $|L(w_1)| = 3$. Since $y \notin V(F)$, y is adjacent to w_2. Since x cannot have more than one neighbor in $\mathrm{dom}(\varphi)$, we have $x \in V(F)$. If $xw_2 \notin E(G)$, then (68) implies that x is adjacent to a vertex with list of size three, and thus $|L_\varphi(x)| = |L(x)| > 3$. This is a contradiction, hence $xw_2 \in E(G)$. By the choice of X, $|L(x)| = 3$. Again, we distinguish two cases depending on whether w_2 is incident with a crossed edge (in this case $|L(w_2)| = 5$ by (C)) or $|N| = 1$. In both cases, G is L-colorable.

- Therefore, H is $\mathrm{O_{P3}}$. But then two of the vertices of H have two neighbors in $\mathrm{dom}(\varphi)$, hence G contains a crossing at distance at most one from P, contradictory to the assumption that G is (P, N, M)-distant.

We have shown that G' satisfies all conditions of Theorem 100 for the list assignment L_φ. This gives a contradiction to the minimality of G and proves claim (71). □

Each case among (A1)–(A7) in (71) contains a special subgraph. Thus, G contains a special subgraph S whose distance from p_0 is at most $2 + r(S)$. Consequently, $\ell(P) = 2$. Next, we consider the set $X' \subseteq \{v_s, v_{s-1}, v_{s-2}\}$ defined symmetrically to X and conclude that there exists a special subgraph S' (satisfying one of (A1)–(A7) with v_i replaced by v_{s+1-i}) whose distance to p_2 is at most $2 + r(S')$. It follows that $d(S, S') \leq 6 + r(S) + r(S')$, and since G is (P, N, M)-distant, we have $S = S'$.

Next, we show that

(72) *S consists of two edges incident with a crossing.*

Proof. If not, then either $S \in M$ or $S \in N$. Suppose first that $S \in M$. Then (A1) holds and $s \leq 4$. Since $s \geq 3$, we can by symmetry assume that $S = v_2v_3$. If v_2, v_3 and v_i have no common neighbor for $i \in \{1, 4\}$ ($i = 1$ if $s = 3$), then let φ be an arbitrary L-coloring of S (such that $\varphi(v_3) \notin L(p_2)$ if $s = 3$). Observe that G_φ cannot contain an obstruction since its special subgraph would be a special subgraph in G, too close to the special edge v_2v_3. Now it is easy to check using previously proved properties of G that G_φ satisfies all conditions of Theorem 100. (The same reasoning will be applied in the sequel without repeating it.) Therefore, G_φ with the list coloring L_φ is a counterexample to Theorem 100, contradicting the minimality of G. Thus, by symmetry, we may assume that v_1, v_2 and v_3 have a common neighbor w. In that case, w is not adjacent to v_4 by (68). Let φ be an L-coloring of v_1 and v_3 such that $\varphi(v_1) \notin L(p_0)$, $\varphi(v_3) \notin L(p_2)$ and $|L_\varphi(v_2)| \geq 2$. Then $G' = G - \{v_1, v_2, v_3\}$ with the list assignment L_φ contradicts the minimality of G since any L_φ-coloring of G' can be extended to v_2 by using a color in $L_\varphi(v_2)$, and can henceforth be extended to G.

Let us now consider the case that $S \in N$, hence (A2) or (A7) holds. Let i and j be the smallest and the largest integer, respectively, such that S is adjacent to v_i and v_j. By (68) we have $j \in \{i+1, i+2\}$. We consider the two possible values of j separately:

- Suppose first that $j = i+1$. If $|X| \geq 2$, then $|L(v_m)| \geq 4$ and $|L(v_{m+1})| = 3$, hence (A7) cannot hold for both X and X'. If both X and X' satisfy (A2), then since $s \geq 3$, we can assume that $v_2, v_3 \in X$ have a common neighbor in N. By the choice of X, we have $|L(v_4)| = 3$, hence $s = 4$ and $v_2, v_3 \in X'$. However, then $|L(v_1)| \geq 4$ by the choice of X and $|L(v_1)| = 3$ by the choice of X', which is a contradiction.

  Hence, we can assume that (A7) holds for X and (A2) for X'; then we either have $s = m + 1$, or we have $s = m + 2$ and $X' = \{v_m, v_{m+1}\}$. If there exists an L-coloring φ of v_{m-1} and v_{m+1} such that their colors are distinct from the colors of their neighbors in P and $|L_\varphi(v_m)| \geq 3$, then $G' = G - \{v_{m-1}, v_m, v_{m+1}\}$ with the list assignment L_φ contradicts the minimality of G. (Observe that G' satisfies (O), since no special subgraph of G is at distance at most two from S. A new special subgraph would appear in G' only if S would be adjacent to v_{m+2}, which is not the case since $j = i + 1$.)

  We conclude that no such coloring exists, hence both v_{m-1} and v_{m+1} have a neighbor in P and $s = 3$. Furthermore, $|L(v_1)| = 3$ and $L(v_1) \setminus L(p_0) \subset L(v_2)$. Let w' be the common neighbor of S and v_1. Suppose that there exists a color $c \in L(w')$ different from the colors of the neighbors of w' in P such that either $c \notin L(v_2)$, or v_1 has degree three and $c \notin L(v_1) \setminus L(p_0)$.

In this case, we let φ be the partial coloring such that $\varphi(w') = c$ and let $G' = G - \{w', v_2\}$ if $c \notin L(v_2)$ and $G' = G - \{w', v_1, v_2\}$ if $c \in L(v_2)$. Observe that G' is not L_φ-colorable. Furthermore, it satisfies the assumptions of Theorem 100, with the edge Sv_3 belonging to M (the condition (O) holds by (70), the distance condition and (63)). This contradicts the minimality of G, and thus no such color c exists. Since $|L(w')| > |L(v_2)|$, it follows that w' has a neighbor in P. By (68), w' is not adjacent to p_2, hence it is adjacent to p_0 or p_1. However, then (61) and (63) imply that v_1 has degree three, and since $|L(v_1) \setminus L(p_0)| = 2$ and w' has at most two neighbors in P, the color c exists. This is a contradiction.

- It remains to consider the case when $j = i + 2$. In this case S is adjacent to v_i and v_{i+2}, and by (68) we conclude that v_{i+1} is a vertex of degree 3 with neighbors v_i, v_{i+2}, and S. Thus, $|L(v_{i+1})| = 3$. Suppose first that both X and X' satisfy (A7). If there exists a coloring φ of S by a color different from the colors of its neighbors in P such that $\varphi(S) \notin L(v_i) \cap L(v_{i+1}) \cap L(v_{i+2})$, then $G' = G - \{S, v_i, v_{i+1}, v_{i+2}\}$ with the list assignment L_φ is a counterexample contradicting the minimality of G (since in this case any L_φ-coloring of G' extends to an L-coloring of G). Otherwise, note that S is not adjacent to p_0 or p_2 by (68), hence S is adjacent to p_1 and $L(S) \setminus L(p_1) = L(v_{i+1}) \subseteq L(v_i) \cap L(v_{i+2})$. However, in this case we let φ be the L-coloring of X as chosen in the proof of Theorem 99, and note that $\varphi(v_i) \notin L(v_{i+1}) = L(S) \setminus L(p_1)$. Then $G - X$ with the list assignment L_φ for any other vertex z is a counterexample contradicting the minimality of G.

  Hence, we can assume that say X' satisfies (A2). Let us first consider the case that X satisfies (A2) as well. Note that $v_{i+2} \notin X$, as otherwise $|L(v_{i+3})| = 3$ by the choice of X, and thus $v_{i+1} \notin X'$, contradictory to the assumption that X' satisfies (A2). Symmetrically, $v_i \notin X'$. Since $|L(v_{i+1})| = 3$, we cannot have $\{v_i, v_{i+1}\} \subseteq X$, thus $i = 1$, and by symmetry, $s = 3$. Observe that we cannot color S by a color $\varphi(S) \notin L(v_{i+1})$, as otherwise $G - \{S, v_{i+1}\}$ with the list assignment L_φ would contradict the minimality of G. Therefore, S has a neighbor in P, and by (68), this neighbor is p_1. By (61), the 4-cycle $p_1p_0v_1S$ is not separating, and by (63), v_1 has degree three. This is a contradiction, since $|L(v_1)| > 3$.

  Therefore, X satisfies (A7). Note that v_{i+1} cannot be the element of X' with the smallest index, thus $i + 2 = s$. As before, we exclude the case that S can be colored by a color not belonging to $L(v_i) \cap L(v_{i+1})$, hence S has a neighbor in P. By (68), S is not adjacent to p_0 or p_2, hence S is adjacent to p_1. However, by (61), the 4-cycle $p_1Sv_{i+2}p_2$ is not separating, and by (63), v_{i+2} is not adjacent to p_1. Thus, v_{i+2} has degree three and list of size at least four, which is a contradiction.

This completes the proof of the case when $S \in N$. $\square$

Therefore, (72) holds and S consists of two edges incident with a crossing q. We conclude that each of X and X' satisfies one of (A3), (A4), (A5) or (A6). If one of them satisfies (A4), then $|V(G_q) \cap V(F)| = 2$ by (63). If it satisfies (A6), then by (63), (64) and (66) we have $|V(G_q) \cap V(F)| = 1$. If it satisfies (A3), then similarly $|V(G_q) \cap V(F)| \leq 1$, and if it satisfies (A5) then $1 \leq |V(G_q) \cap V(F)| \leq 2$.

(73) *Neither X nor X' satisfies* (A3).

Proof. Suppose for a contradiction that X satisfies (A3). Let w_1 and w_2 be as in the description of (A3). Note that w_2 is adjacent to v_{m-1} and v_m (even if $v_{m-1} \notin \mathrm{dom}(\varphi)$, in the case (X4b)) and that $|L(v_{m-1})| = |L(v_{m+1})| = 3$.

Let us first consider the case that $|V(G_q) \cap V(F)| = \emptyset$. In this case X' satisfies (A3) as well, i.e., there exists $w_2' \in V(G_q)$ adjacent to v_b and v_{b+1}, where b is the smallest index of an element of X', and another vertex w_1' of G_q that has one neighbor in X'. Since $|L(v_b)| \neq 3$, we have $b \notin \{m-1, m+1\}$. Consequently, $|X \cap X'| \leq 1$, and $w_2' \neq w_2$ by (68).

We now distinguish two cases regarding whether w_2 is adjacent or crossing-adjacent to w_2' in G_q.

- Suppose that $w_2 w_2'$ is a crossed edge. Then $b \neq m$ by (60) and the assumption that G_q is disjoint with F; thus $b \geq m+2$. Let G_1 and G_2 be the subgraphs of G intersecting in $v_m w_2 w_2' v_b$, such that $G_1 \cup G_2 = G - e$, where e is the edge crossing $w_2 w_2'$, and $P \subset G_1$. By (68), we have that $w_1 \neq w_2'$, $w_1' \neq w_2$ and that if $w_1 = w_1'$, then w_1 belongs to G_2. By symmetry, assume that w_1 belongs to G_2. If w_1 is adjacent to v_b, then $b = m+2$ by (68). Let $T = \{v_m, v_{m+1}, v_{m+2}, w_1\}$. By using (67) it is easy to see that $|L(t)| = \deg(t)$ for each $t \in T \setminus \{w_1\}$ and that $\deg(w_1) \leq 6$. By the minimality of G, there exists an L-coloring φ of $G - T$. Consider the subgraph G' of G induced by T with the list assignment L_φ. We have $|L_\varphi(v_{m+1})| \geq 3$ and $|L_\varphi(z)| \geq 2$ for $z \in T \setminus \{v_{m+1}\}$. If $L_\varphi(w_1) \neq L_\varphi(v_m)$, then we color w_1 by a color in $L_\varphi(w_1) \setminus L_\varphi(v_m)$ and extend this coloring to the rest of G'. Similarly, G' is L_φ-colorable if $L_\varphi(w_1) \neq L_\varphi(v_{m+2})$. If $L_\varphi(v_m) = L_\varphi(w_1) = L_\varphi(v_{m+2})$, then we color v_{m+1} by a color in $L_\varphi(v_{m+1}) \setminus L_\varphi(w_1)$ and again we can extend this to an L_φ-coloring of G'. It follows that G is L-colorable, which is a contradiction.

  Therefore, w_1 is not adjacent to v_b, and in particular $w_1 \neq w_1'$ and $w_1' \in V(G_1)$. Let φ be an L-coloring of G_1, which exists by the minimality of G. Since w_1 is not adjacent to v_b, note that w_1 has at most three neighbors in G_1 different from w_2'. Hence, we can additionally choose a color $\varphi(w_1)$ for w_1 different from the colors of its neighbors in G_1 so that $\varphi(w_1) \neq \varphi(w_2')$. Let $G_2' = G_2 - w_2 + w_1 w_2'$. Note that G_2' gives an instance of Theorem 100 with the precolored path $P' = v_m w_1 w_2' v_b$, since the added edge $w_1 w_2'$ can

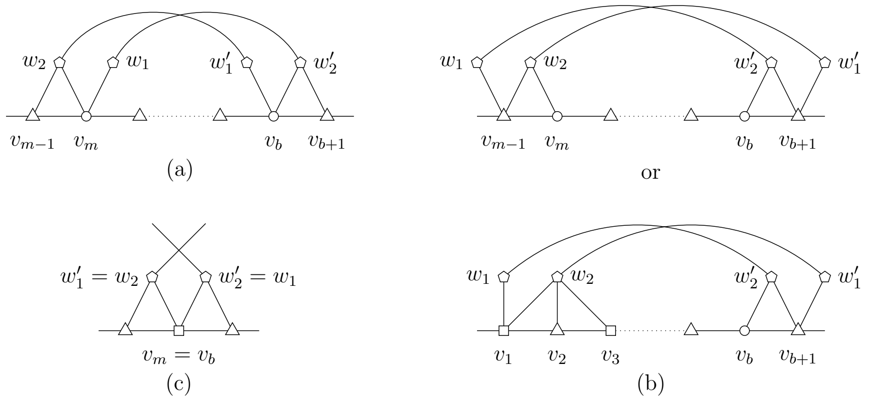


Figure 9.8: Subcases when w_2 and w_2' are crossing-adjacent

be drawn without crossings following the crossed edges of G that are no longer in G_2'. It is clear that G_2' satisfies validity and distance constraints. Note that φ does not extend to an L-coloring of G_2'. Thus G_2' violates (T) or (O). In the former case, the vertex violating (T) must be v_{m+1} and we would have $b = m + 2$. Consequently, v_b would have degree at most three, which is a contradiction. In the latter case, since $|L(v_{m+1})| = |L(v_{b-1})| = 3$ and v_b has degree at least three in G_2', we have that G_2' is equal to $\mathrm{O_{P5}}$ or $\mathrm{O_{P6}}$. In both cases, any L-coloring of $G_1 - \{v_m, v_b\}$ would extend to an L-coloring of G, a contradiction.

- Suppose now that w_2 is crossing-adjacent to w_2'. Let G_1 and G_2 be the subgraphs of G intersecting in $\{v_b, w_2', w_2, v_m\}$, where $P \subset G_1$ and $G_1 \cup G_2$ is equal to G without the crossed edges. We have two subcases: either $b > m$ or $b = m$.

  – If $b > m$, then (68) implies that w_2' has no neighbor in X, and thus $w_1 \neq w_2'$. Symmetrically, $w_1' \neq w_2$. Considering the drawing of G in the plane, we conclude that the edges of G_q are w_1w_2' and $w_1'w_2$.

    If $w_1, w_1' \notin V(G_1)$ (see Figure 9.8(a)), then $w_1v_m, w_1'v_b \in E(G)$. Let φ be an L-coloring of $G_1 + \{w_1w_2', w_1'w_2, w_1w_1'\}$ which exists by the minimality of G, and note that φ does not extend to an L-coloring of $G_2' = G_2 + w_1w_1'$. Observe that G_2' provides an instance for Theorem 100 with the precolored path $v_mw_1w_1'v_b$. It is easy to see that we can choose the colors of w_1, w_1', v_m and v_b so that G_2' satisfies the assumptions of the theorem (once the coloring of $G_1 - \{v_m, v_b\}$ is fixed, we still have two possible choices for the colors of v_m and v_b and three possible choices for the colors of w_1 and w_1'). This is a contradiction. The case that $w_1, w_1' \in V(G_1)$ (see Figure 9.8(b)) is excluded similarly.

– If $b = m$, then let w_2z and w'_2z' be the edges of G_q (note that we have $w_1 = w'_2$ and $w'_1 = w_2$). Suppose that $z, z' \in V(G_2)$. Note that $V(G_2) \neq \{z, z', w_2, w'_2, v_m\}$, since otherwise z would have degree at most four and $|L(z)| = 5$. Therefore, the subgraph of G induced by $V(G_1) \cup \{z, z'\}$ has an L-coloring ψ by the minimality of G. Let L' be the list assignment for $G'_2 = G_2 - \{z, z'\}$ obtained from L by removing the colors of z and z' according to ψ from the lists of their neighbors and by setting $L'(w_2) = \{\psi(w_2)\}$, $L'(v_m) = \{\psi(v_m)\}$ and $L'(w'_2) = \{\psi(w'_2)\}$. Note that G'_2 satisfies (O) by the distance condition and (P) by the choice of ψ, and since G is not L-colorable, we conclude that G'_2 violates (T). Therefore, G_2 contains a vertex adjacent to w_2, w'_2, v_m, z and z', and by (60), z and z' have degree at most four. This is a contradiction.

  Therefore, we have $z, z' \in V(G_1)$ (see Figure 9.8(c)), and by (67), $\deg(v_m) = 4$. Let $S_1 = L(v_2)$ if $m = 3$ and $S_1 = L(v_1) \setminus L(p_0)$ if $m = 2$. Note that $S_1 \subset L(v_m)$, as otherwise we consider the partial coloring φ with $\varphi(v_{m-1}) \in S_1 \setminus L(v_m)$ and conclude that G_φ with the list assignment L_φ contradicts the minimality of G. Suppose that there exists a color $c \in L(w_2) \setminus L(v_m)$, or that $\deg(v_{m-1}) = 3$ and there exists a color $c \in L(w_2) \setminus S_1$, such that this color c is distinct from the colors of the neighbors of w_2 in P. Let $G' = G - \{w_2, v_m\}$ if $\deg(v_{m-1}) > 3$ and $G' = G - \{w_2, v_m, v_{m-1}\}$ if $\deg(v_{m-1}) = 3$, with the list assignment L' obtained from L by removing c from the lists of neighbors of w_2 and setting $L'(v_{m-1}) = L(v_{m-1})$ if v_{m-1} belongs to $V(G')$ (observe that $c \notin S_1$ and that in any L'-coloring of G', the color of v_{m-1} must belong to S_1). Note that L' is $(P, N \cup \{z\}, M)$-valid. Every L'-coloring of G' would extend to an L-coloring of G, thus G' is not L'-colorable. By the minimality of G, we conclude that G' violates (O), and by (70) and the distance condition, G' contains $\mathrm{O_{N2}}$ or $\mathrm{O_{N3}}$. However, then z is adjacent to two vertices of P and to z' and w'_2, and at least one of z' and w'_2 has a list of size three according to L', which is a contradiction since $|L(z')| = |L(w'_2)| = 5$.

  We conclude that there exists no such color c. Since $|L(v_m)| = 4$ and $|L(w_2)| = 5$, we conclude that w_2 has a neighbor in P. By (68), w_2 is not adjacent to p_2, and if it were adjacent to p_0, then we would have $m = 2$, $\deg(v_1) = 3$ and there would exist a color $c \in L(w_2) \setminus (S_1 \cup L(p_0) \cup L(p_1))$. Therefore, w_2 is adjacent to p_1. By symmetry, w'_2 is adjacent to p_1 as well. However, the edges w_2p_1 and w'_2p_1 are not crossed by (64), and thus the crossing is contained inside the 4-cycle $v_mw_2p_1w'_2$, contrary to (61).

We conclude that $V(G_q) \cap V(F) \neq \emptyset$. By (63), $w_2 \notin V(F)$. Let w be the vertex joined to w_2 by a crossed edge, and let w_1w' be the other crossing edge.

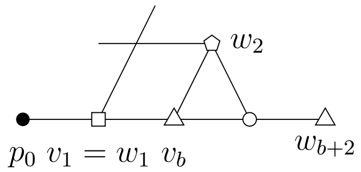


Figure 9.9: Subcase combining (A3) and (A5).

Since $V(G_q) \cap X = \emptyset$, by (68) we have $w \notin V(F)$. Since v_m has degree at least four, we cannot have $w_1 = v_{m+1}$, thus by (63) and (64), we have $w_1 \notin V(F) \setminus \{v_1\}$. If $w_1 \notin V(F)$ and $x \in X$ is a neighbor of w_1, then the 2-chord xw_1w' separates P from either w_2 or w, and neither w_2 nor w belongs to F, contrary to (68). We conclude that $w_1 = v_1$ and $V(G_q) \cap V(F) = \{v_1\}$, hence $v_1 \notin X$ and X was chosen according to (X4a).

Since $|V(G_q) \cap V(F)| = 1$, X' must satisfy (A3), (A5) or (A6). If X' satisfied (A3), the conclusions of the preceding paragraph would apply symmetrically and we would have $v_1 = v_b$, which is a contradiction. Similarly, X' cannot satisfy (A6). The remaining possibility is that (A5) holds for X'. Then $v_1 = v_{b-1}$ and $v_b = v_2$. The situation is shown in Figure 9.9. Since X was chosen according to (X4a), we have $|L(v_b)| = |L(v_{b+2})| = 3$; in particular, $s \geq 4$ and $b \leq s - 2$. This is only possible if X' has been chosen according to (X4), but then $|L(v_b)| > 3$. This is a contradiction, showing that (A3) does not occur. $\square$

Next, we claim that

(74) $|V(G_q) \cap V(F)| = 1$.

Proof. Since X does not satisfy (A3), if $|V(G_q) \cap V(F)| \neq 1$ then $|V(G_q) \cap V(F)| = 2$ and each of X and X' satisfies (A4) or (A5). By (63) and (66), $V(G_q) \cap V(F) = \{v_{m+1}, v_{m+2}\}$ and v_{m+1} is crossing-adjacent to v_{m+2}. Let $v_{m+1}w$ and $v_{m+2}w'$ be the crossed edges. By symmetry, we can assume that $|L(v_{m+1})| \geq |L(v_{m+2})|$. By (C), either $|L(v_{m+1})| \geq |L(v_{m+2})| \geq 4$ or $|L(v_{m+1})| = 5$ and $|L(v_{m+2})| = 3$. Therefore, X was chosen according to the rules (X1) or (X3) and $|L(v_m)| = 3$.

If $L(v_{m+2}) \neq L(v_{m+1})$, then let c be a color in $L(v_{m+1}) \setminus L(v_{m+2})$. If v_{m+1} is not adjacent to v_{m+2}, then let c be an arbitrary color in $L(v_{m+1})$. In both cases, let φ be an L-coloring of v_m and v_{m+1} such that $\varphi(v_{m+1}) = c$ and if $m = 1$, then $\varphi(v_1) \notin L(p_0)$. It is easy to see that L_φ is a $(P, N \cup \{w\}, M)$-valid list assignment for G_φ. Therefore, G_φ violates (O). By (70), G_φ contains O_{N2} or O_{N3}. It follows that w is adjacent to p_1 and to p_0 or p_2. However, if w is adjacent to p_0, then by (68), v_{m+2} is incident with a chord of F, contradicting (63). If w is adjacent to p_2, then v_{m+2} has degree at most three in G_φ by (68), and since $|L_\varphi(v_{m+2})| \geq 4$, G_φ contains neither O_{N2} nor O_{N3}. This is a contradiction, implying that $L(v_{m+1}) = L(v_{m+2})$ (and in particular, $|L(v_{m+1})| = |L(v_{m+2})| = 4$), and $v_{m+1}v_{m+2} \in E(G)$. By the choice of X', we have $|L(v_{m+3})| = 3$.

Suppose now that $w'v_m \in E(G)$. Note that v_{m+1} has degree at least four, so it is adjacent to w'. Let $S_1 = L(v_m)$ if $m \neq 1$ and $S_1 = L(v_m) \setminus L(p_0)$ if $m = 1$. Note that $S_1 \subseteq L(v_{m+1})$, as otherwise we can choose an L-coloring φ of v_m such that $\varphi(v_m) \in S_1 \setminus L(v_{m+1})$, and $G_1 = G - \{v_m, v_{m+1}\}$ with the list assignment L_φ is a counterexample contradicting the minimality of G (note that G_1 cannot contain an obstruction since no internal vertex in G_1 has its list decreased and q is not a crossing in G_1). Since $L(v_{m+1}) = L(v_{m+2})$, we conclude that $S_1 \subseteq L(v_{m+2})$. Let G' be the graph obtained from $G - v_{m+1}$ by identifying v_m with v_{m+2}, and give the resulting vertex z the list of v_m. Note that G' satisfies the validity and the distance conditions of Theorem 100 (with the edge zv_{m+3} added to M). Since every coloring of G' gives rise to an L-coloring of G, condition (O) is violated in G' by the minimality of G. However, G' contains neither $\mathrm{O_{M1}}$ nor $\mathrm{O_{M2}}$ (and the exclusion of other obstructions is obvious). Therefore, $w'v_m \notin E(G)$, and by symmetry, $wv_{m+3} \notin E(G)$.

Let $S_2 = L(v_{m+3})$ if $m+3 \neq s$ and $S_2 = L(v_{m+3}) \setminus L(p_2)$ if $m+3 = s$. Suppose now that there exists an L-coloring φ of v_{m+1} and v_{m+2} such that $\varphi(v_{m+1}) \notin S_1$ and $\varphi(v_{m+2}) \notin S_2$. Then L_φ is a $(P, N, \{ww'\})$-valid list assignment for G_φ, and by the minimality of G, G_φ violates (O). By (70), G_φ contains $\mathrm{O_{M1}}$ (the other cases are easily excluded: $\mathrm{O_{N2}}$ and $\mathrm{O_{N3}}$ since no internal vertex gets a reduced list and $\mathrm{O_{P3}}$ since $\ell(P) = 2$). But then w' is adjacent to p_0, and the 2-chord $p_0w'v_{m+2}$ contradicts (68). Therefore, no such coloring φ exists. It follows that $|S_1| = |S_2| = 3$ and $S_1 \subseteq L(v_{m+1})$. Since $L(v_{m+1}) = L(v_{m+2})$, we also have that $S_1 = S_2$. Since $|S_1| = |S_2| = 3$, claim (59)(f) implies that $m = 2$ and $s = 6$. Similarly, we conclude that $L(v_1) = L(p_0) \cup L(v_2)$ and $L(v_6) = L(p_2) \cup L(v_5)$, as otherwise we can color and remove v_1 or v_6.

Let us now consider the case that v_2, v_3 and w' have no common neighbor. If v_1, v_2 and v_3 have no common neighbor, then let φ be an L-coloring of v_2, v_3 and v_4 such that $\varphi(v_4) \notin L(v_5)$. Otherwise, let φ be an L-coloring of v_1, v_3 and v_4 such that $\varphi(v_4) \notin L(v_5)$ and $\varphi(v_1) = \varphi(v_3)$. In the former case, let $G' = G_\varphi$, in the latter case let $G' = G_\varphi - v_2$. Observe that L_φ is a valid list assignment for G' (after possibly adding the edge ww' into M) and that any L_φ-coloring of G' extends to G. Furthermore, G' satisfies (O) by (70), since w' cannot be adjacent to p_0. Hence, G_φ contradicts the minimality of G. Therefore, v_2, v_3 and w' have a common neighbor x', and by symmetry, v_4, v_5 and w have a common neighbor x (see Figure 9.10).

By (68), we have $x \neq x'$ and x is adjacent neither to p_0 nor to p_2. Furthermore, if $xp_1 \in E(G)$, then consider the cycle $K = p_1p_2v_6v_5x$. Since v_6 has degree at least four, we conclude by (62) that K has two chords incident with v_6. However, that contradicts (63). Therefore, x (and symmetrically x') has no neighbor in P. By (68), neither w nor w' is adjacent to p_0 or p_2. Claims (60) and (61) imply that $x'w, xw', xx' \notin E(G)$. Since both w and w' have degree at least 5,

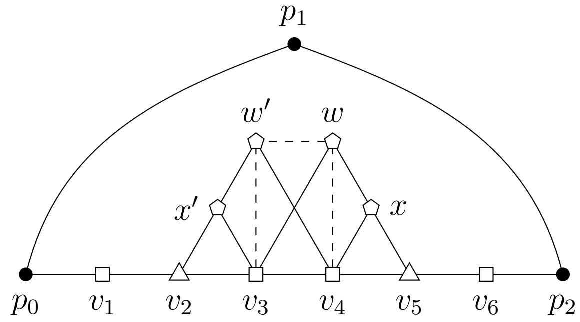


Figure 9.10: A subcase in the proof when X and X' satisfy (A4) or (A5). The dotted edges may or may not be present.

we conclude that each of them has a neighbor that is different from all vertices shown in Figure 9.10. Suppose that $w'p_1 \notin E(G)$. Then let φ be an L-coloring of x and w' such that $\varphi(x), \varphi(w') \notin L(v_4)$ (note that these colors do not belong to the lists of v_2, v_3 and v_5, as well as to $L(v_1) \setminus L(p_0)$ and $L(v_6) \setminus L(p_2)$). Let $G' = G - \{x, w', v_3, v_4\}$ if $\deg(w) > 5$ and $G' = G - \{x, w', v_3, v_4, w\}$ if $\deg(w) = 5$. Note that G' is not L_φ-colorable since any L_φ-coloring of G' extends to G. Furthermore, the only possible vertices with list of size three in G' are v_2, v_5, w and a common neighbor u of x and w' distinct from w and v_4, if such a vertex exists. By (61), if u exists, then $\deg(w) = 5$ and $w \notin V(G')$. Furthermore, by (61), u and w are not adjacent to v_2 and v_5. Therefore, L_φ is a valid list assignment, the distance condition implies that G' satisfies (O), and thus G' contradicts the minimality of G.

We conclude that $w'p_1 \in E(G)$. Let G_1 and G_2 be the $p_1w'v_4$-components of G, where G_1 contains p_0. Consider an L-coloring of G_2. Note that v_3 has only two neighbors in $G_2 - w'$, thus the coloring of G_2 can be extended to v_3 in such a way that its color is different from the color of w'. Then $G_1 - v_4 + w'v_3$ (with the precolored path $p_0p_1w'v_3$) violates (O). Observe that only v_1 and v_2 have list of size at most four and that x' is a common neighbor of v_3 and w'. Therefore, x' is a vertex in the corresponding obstruction K, and v_2 is a vertex in K with list of size 3. It follows that K is equal to O_{P4}. However, then $v_1p_1 \in E(G)$, contradicting (63). $\square$

Therefore, $|V(G_q) \cap V(F)| = 1$, and thus each of X and X' satisfies (A5) or (A6). Since $s \geq 3$, we can assume that X' satisfies (A5). Suppose first that X satisfies (A6), and thus $b = 2$. Since $v_1 \notin X$, the inspection of possible cases for X and X' shows that we have $|L(v_2)| = 3$, $X' = \{v_2\}$, and $s = 3$. If v_1, v_2 and v_3 have no common neighbor, then consider any L-coloring φ of v_1 and v_2 such that $\varphi(v_1) \notin L(p_0)$, and observe that G_φ with the list assignment L_φ is a counterexample contradicting the minimality of G: since v_1, v_2, v_3 do not have a common neighbor, we do not get adjacent vertices with lists of size 3; but we may need to add the neighbor z of v_1 along the crossed edge into the set N. The resulting graph satisfies (O), since z is not adjacent to p_0 and p_2 by (68) and (64). Hence, we can assume that v_1, v_2 and v_3 have a common neighbor w, and thus $\deg(v_2) = 3$. Similarly, we conclude that $L(v_1) = L(p_0) \cup L(v_2)$ (if not, we color v_1 with a color in $L(v_1) \setminus (L(p_0) \cup L(v_2))$ and then consider $G' = G - \{v_1, v_2\}$) and that $L(v_3) = L(p_2) \cup L(v_2)$ (if not, we can color v_3 by a color in $L(v_3) \setminus (L(p_2) \cup L(v_2))$ and then consider $G' = G - \{v_2, v_3\}$). By (61), (64) and (68), w has no neighbor in P. Let u be the vertex adjacent to w by the crossed edge, let φ be an L-coloring of w such that $\varphi(w) \notin L(v_2)$ and let $G' = G - \{v_2, w\}$ Note that L_φ is a $(P, N \cup \{u\}, \emptyset)$-valid list assignment for G' and that G' satisfies (O), since no vertex has list of size three. Thus, G' is a counterexample to Theorem 100 contradicting the minimality of G.

Therefore, both X and X' satisfy (A5) and $b = m + 2$. Moreover, (61) implies that the neighbor w' of v_b in $V(G_q) \setminus \{v_{m+1}\}$ is different from w (the neighbor of v_m). Let y be the vertex joined to v_{m+1} by a crossed edge. If $|L(v_{m+1})| \neq 3$, then both X and X' are chosen by cases (X1) or (X3) and $|L(v_m)| = |L(v_{m+2})| = 3$. The condition (A5) implies $|L(v_{m+1})| = 4$. However, in that case we have $|L(v_{m+2})| \neq 3$ both in (X1) and (X3), which is a contradiction. Therefore, $|L(v_{m+1})| = 3$. Consequently, X and X' were chosen by (X2) or (X4) and we have $|L(v_m)|, |L(v_{m+2})| \geq 4$ and $|L(v_{m-1})| = |L(v_{m+3})| = 3$. Since $\deg(v_m) \geq 4$, (68) implies that w has no neighbor in F other than p_1, v_m and v_{m+1}, and by symmetry, the only possible neighbors of w' in F are p_1, v_{m+1} and v_{m+2}.

Let $S_1 = L(v_{m-1})$ if $m = 3$ and $S_1 = L(v_{m-1}) \setminus L(p_0)$ if $m = 2$. Let $S_2 = L(v_{b+1})$ if $b = s - 2$ and $S_2 = L(v_{b+1}) \setminus L(p_2)$ if $b = s - 1$. By symmetry, we can assume that if $m = 2$, then $b = s - 1$. Let S be the set of colors $c \in L(v_{m+1})$ such that either

(a) $L(v_{m+2}) = S_2 \cup \{c\}$, or

(b) $|L(v_m)| = 4$, $c \notin S_1$ and $S_1 \cup \{c\} \subseteq L(v_m)$.

If $m = 2$, then we have $b = s - 1$, $|S_1| = |S_2| = 2$, there are at most two colors with the property (b) and no colors with the property (a). If $m = 3$, then $|S_1| = 3$ and $|S_2| \leq 3$, there is at most one color with the property (b) and at most one color with the property (a). It follows that $|S| \leq 2$. Let φ be an L-coloring of v_{m-1}, v_{m+1} and v_{m+2} chosen so that $\varphi(v_{m+2}) \notin S_2$, $\varphi(v_{m+1}) \notin S$, $\varphi(v_{m-1}) \in S_1$ and $|L(v_m) \setminus \{\varphi(v_{m-1}), \varphi(v_{m+1})\}| \geq 3$. Note that the choices for $\varphi(v_{m+2})$ and $\varphi(v_{m-1})$ are possible, since $\varphi(v_{m+1})$ does not satisfy (a) and (b), respectively. Consider $G' = G - \{v_{m-1}, v_{m+1}, v_{m+2}\}$ with the list assignment L_φ. By (68), v_{m-1} has no common neighbor with v_{m+1} and none with v_{m+2}, and the only common neighbor of v_{m+1} and v_{m+2} is w'. Therefore, the only vertices with list of size three are v_1 if $m = 3$, v_m, v_{m+3} and w'. Since w' is not adjacent to v_{m+3}, L_φ is $(P, N \cup \{y\}, \emptyset)$-valid. Furthermore, y is adjacent neither to p_0 nor to

p_2 by (68), hence G' satisfies (O) by (70) and contradicts the minimality of G. This completes the proof of Theorem 100. □

Proof of Theorem 98. Let G be a graph with crossing number at most two. We may assume that G is nonplanar. Consider a drawing of G in the plane with one or two crossings and let L be a list assignment such that each vertex has five admissible colors. Let xy and uv be two edges crossing each other at the crossing q. Suppose first that the edges xy and uv do not participate in another crossing. Now remove the two edges and add the edges xu, uy, yv, and vx (if they are not already present). This gives rise to a graph G' with at most one crossing, and we can redraw it so that the cycle $xuyv$ bounds the outer face. Now we φ-precolor the path xuy such that $\varphi(x) \neq \varphi(y)$, and give v the list $L(v) \setminus \{\varphi(u)\}$. Theorem 100 now implies that G' has a list coloring which in turn shows that G is L-colorable.

If the edge uv participates in another crossing, then xy does not participate in another one. Suppose that the segment of uv from u to the crossing q does not contain the other crossing. Then we proceed similarly as above: we remove the edges xy and uv and add edges xu and uy. The resulting graph is planar and the path $P = xuy$ is part of a facial walk. Thus we may φ-precolor the path so that $\varphi(x) \neq \varphi(y)$ and then remove $\varphi(u)$ from the list of v. Now, we apply Theorem 100 with $N = \{v\}$ to obtain a coloring that again confirms that G is L-colorable. □

Chapter 10

3-list-colorability of planar graphs with (≤ 4)-cycles far apart[1]

Motivated by Grötzsch's theorem, Havel asked whether there exists a constant d such that if the distance between each pair of triangles in a planar graph is at least d, then the graph is 3-colorable. This question was open for many years, finally being answered in affirmative by Dvořák et al. [24] (although the bound on d is impractically large). Due to the result of Voigt [76], an analogous question for 3-list-colorability needs also to restrict 4-cycles: does there exist a constant d such that if the distance between each pair of (≤ 4)-cycles in a planar graph is at least d, then the graph is 3-list-colorable? We give a positive answer to this question:

Theorem 101. *If G is a planar graph such that the distance between each pair of (≤ 4)-cycles is at least 26, then G is 3-list-colorable.*

This bound is quite reasonable compared to one given for Havel's problem [24]. However, it is far from the best known lower bound of 4, given by Aksionov and Mel'nikov [3].

10.1 Proof of Theorem 101

For a subgraph H of a graph G, let $d(H) = \min_F d(H, F)$, where the minimum goes over all (≤ 4)-cycles F of G distinct from H. Let $t(G) = \min_H d(H)$, where the minimum goes over all (≤ 4)-cycles H of G. A *path of length k* (or a *k-path*) is a path with k edges and $k+1$ vertices. For a path or a cycle X, let $\ell(X)$ denote its length. Let r be the function defined by $r(0) = 0$, $r(1) = 2$, $r(2) = 4$, $r(3) = 9$, $r(4) = 13$ and $r(5) = 16$. For a path P, let $r(P) = r(\ell(P))$. Let $B = 26$. Using the proof technique of precoloring extension developed by Thomassen [69], we show the following generalization of Theorem 101:

[1]The results of this chapter are based on Dvořák [21].

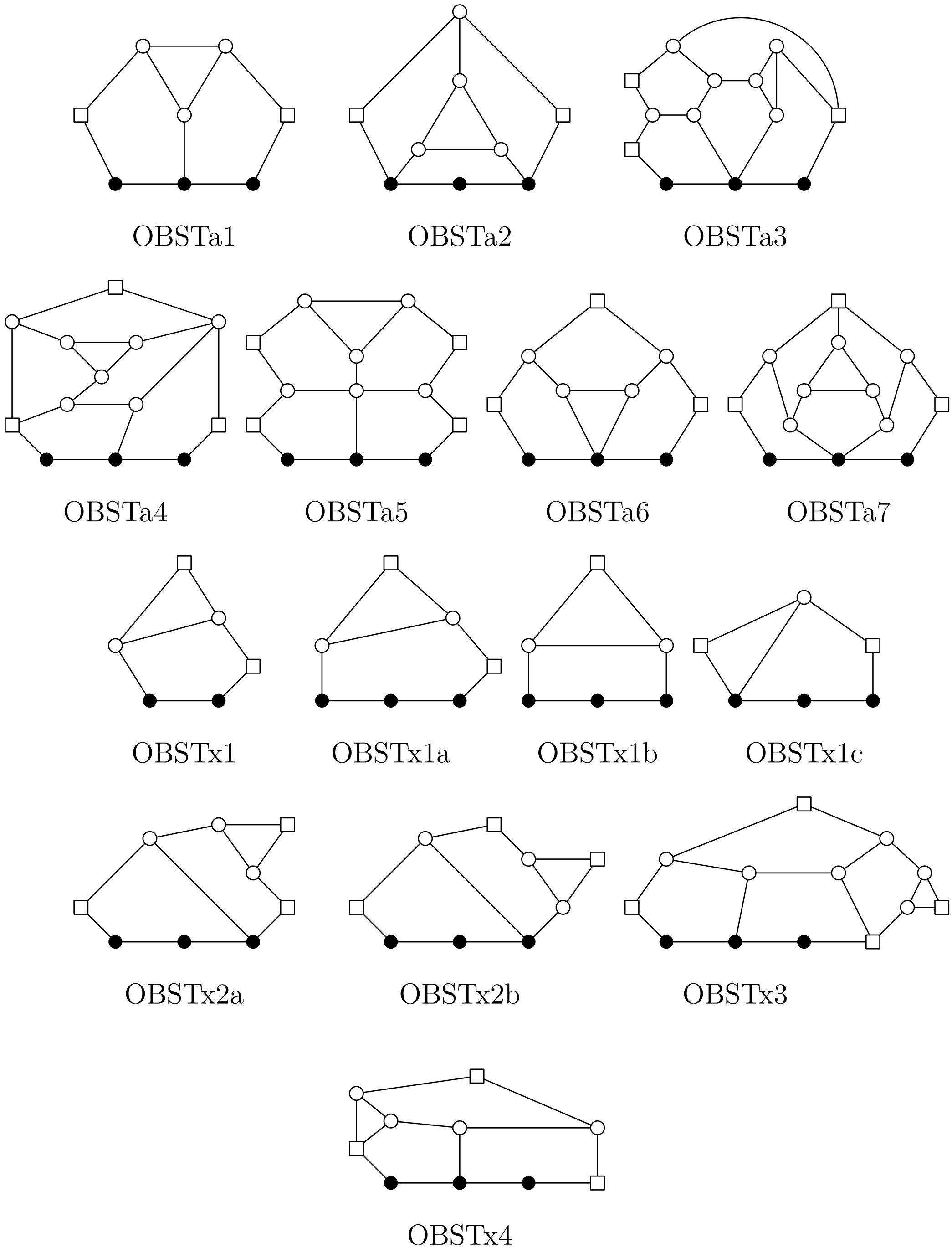


Figure 10.1: Forbidden configurations of Theorem 102, $\ell(P) \leq 2$

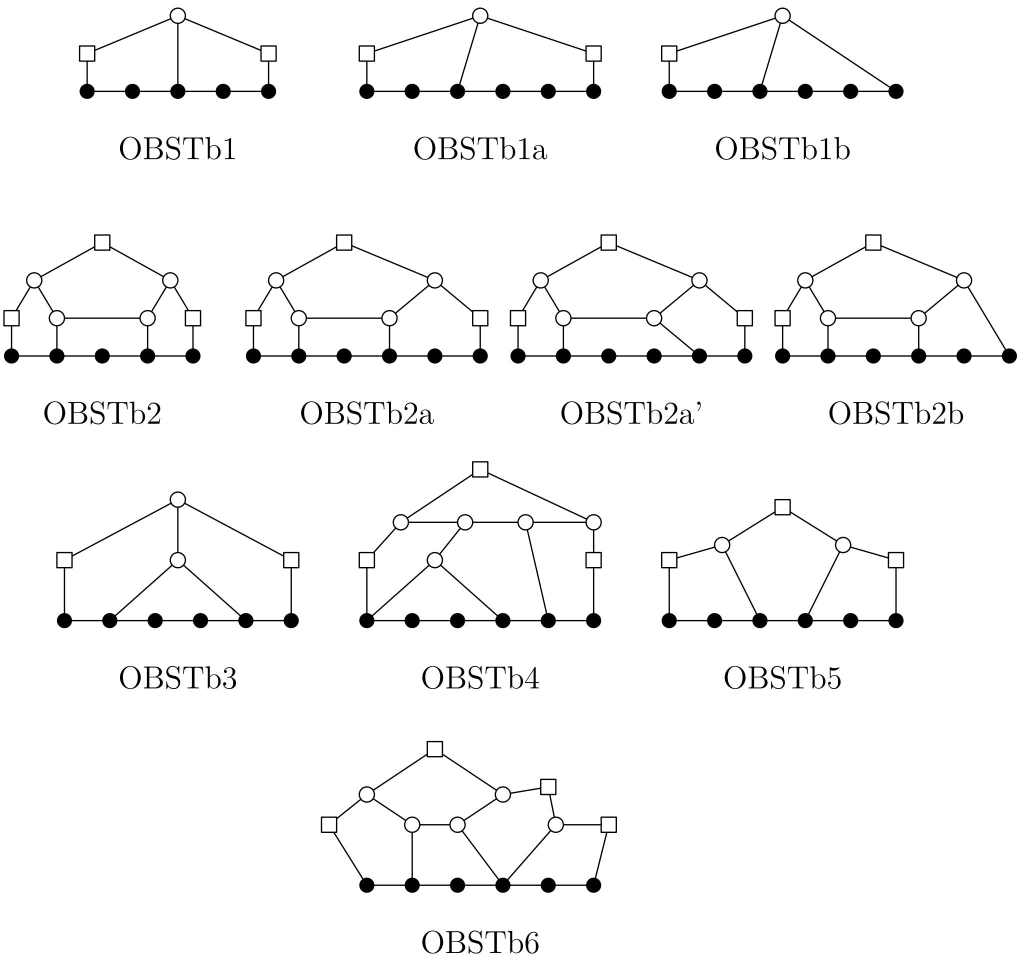


Figure 10.2: Forbidden configurations of Theorem 102, $\ell(P) \leq 5$

Theorem 102. *Let G be a planar graph with the outer face C such that $t(G) \geq B$, and P a path such that $V(P) \subseteq V(C)$. Let L be a list assignment such that*

(S1) $|L(v)| = 3$ *for all* $v \in V(G) \setminus V(C)$;

(S2) $2 \leq |L(v)| \leq 3$ *for all* $v \in V(C) \setminus V(P)$;

(S3) $|L(v)| = 1$ *for all* $v \in V(P)$, *and the colors in the lists give a proper coloring of the subgraph of G induced by $V(P)$;*

(I) the vertices with lists of size two form an independent set;

(T) if uvw is a triangle, $|L(u)| = 2$ and v has a neighbor with list of size two distinct from u, then w has no neighbor with list of size two distinct from u; and

(Q) if a vertex v with list of size two has two neighbors w_1 and w_2 in P, then $L(v) \neq L(w_1) \cup L(w_2)$.

In this situation, if $\ell(P) \leq 2$ and

(OBSTa) every subgraph $H \subseteq G$ isomorphic to one of the graphs drawn in Figure 10.1 is L-colorable,

then G is L-colorable. Furthermore, if $\ell(P) \leq 5$, $d(P) \geq r(P)$ and

(OBSTb) every subgraph $H \subseteq G$ isomorphic to one of the graphs drawn in Figure 10.2 is L-colorable,

then G is L-colorable.

Note that we view the single-element lists as a precoloring of the vertices of P. Also, P does not have to be a part of the facial walk of C, as we only require $V(P) \subseteq V(C)$. The notation used in Figures 10.1 and 10.2 is the following: We mark the vertices of P (precolored vertices) by full circles, the vertices with list of size three by empty circles, and the vertices with list of size two by empty squares. In the conditions (OBSTa) and (OBSTb), we require the lists of the vertices of H according to L to match the sizes prescribed by Figures 10.1 and 10.2.

Let us remark that the assumption (T) is necessary—Figure 10.3 shows a non-L-colorable graph G_1 with only one precolored vertex x_1 satisfying all other assumptions of Theorem 102. By repeating the left part of this graph, x_1 can be made arbitrarily far apart from the triangle. Let G_2 and G_3 with precolored vertices x_1 and x_2 be the copies of G_1 with the color A replaced by colors A' and A'', respectively, in the lists of all vertices. Let G be the graph obtained from G_1, G_2 and G_3 by identifying the vertices x_1, x_2 and x_3 to a single vertex whose list is $\{A, A', A''\}$. Note that G is a counterexample to Theorem 102 without the

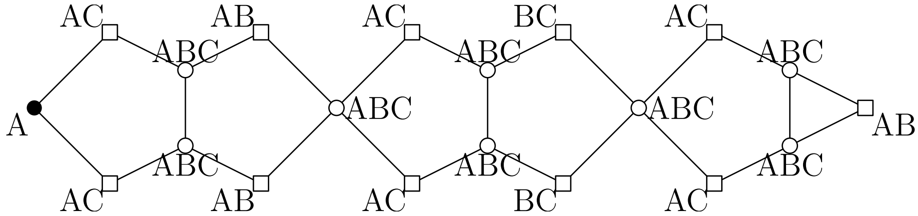
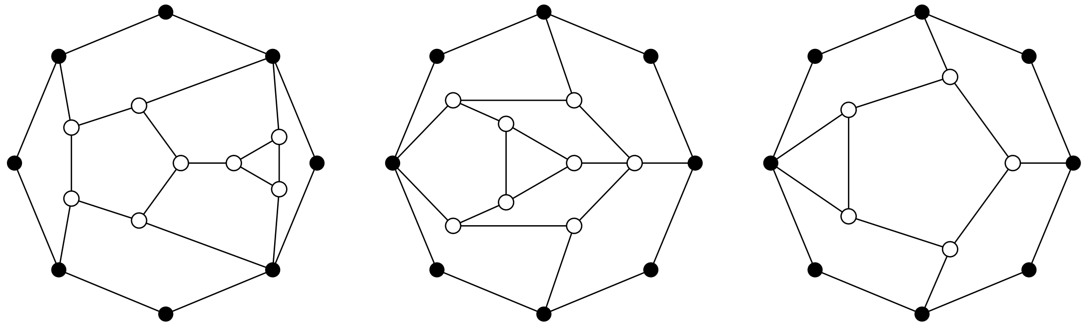


Figure 10.3: Assumption (T) is necessary



(a) (b) (c)

Figure 10.4: C-obstacles

assumption (T) and that G has no precolored vertices and $t(G)$ can be arbitrarily large.

In his paper showing that every planar graph with at most three triangles is 3-colorable, Aksionov [2] also proved that if G is a plane graph with exactly one (≤ 4)-cycle, then any precoloring of a 5-face of G extends to a 3-coloring of G. Thomassen [69] showed that in a planar graph of girth 5, any precoloring of an induced cycle C of length at most 9 extends to a 3-coloring, unless a vertex has three neighbors in C. Walls [77] extended this characterization for cycles of length at most 11 (giving more subgraphs that prevent the coloring from being extended), Thomassen [71] generalized it for list-coloring, and Dvořák and Kawarabayashi [22] extended both of these results for the cycles of length 12. Similarly, Theorem 102 implies a result regarding extension of a precoloring of a (≤ 8)-cycle, assuming that (≤ 4)-cycles are far apart.

Let C be a (≤ 8)-cycle. We call a plane graph F a *C-obstacle* if $C \subseteq F$ bounds the outer face of F, F contains exactly one (≤ 4)-cycle, and

O1: $F - V(C)$ is a tree (with at most $\ell(C) - 6$ vertices), or

O2: $F - V(C)$ is a graph (with at most $\ell(C) - 3$ vertices) whose only cycle is a triangle, or

O3: F is one of the graphs drawn in Figure 10.4.

Corollary 103. *Let G be a plane graph with the outer face bounded by an induced (≤ 8)-cycle C, such that $t(G) \geq B$. Furthermore, assume that G does not contain a C-obstacle as a subgraph. Let L be an assignment of lists of size 1 to the vertices of C and lists of size 3 to the other vertices of G. If L prescribes a proper coloring of C, then G is L-colorable.*

Let us give a proof of this result in a slightly more general setting, which we are going to use in the inductive proof of Theorem 102. A graph G_1 is *smaller* than G_2 if

- G_1 has smaller number of (≤ 4)-cycles than G_2, or
- G_1 and G_2 have the same number of (≤ 4)-cycles and satisfy $|V(G_1)| < |V(G_2)|$, or
- G_1 and G_2 have the same number of (≤ 4)-cycles, $|V(G_1)| = |V(G_2)|$ and $|E(G_1)| < |E(G_2)|$.

Lemma 104. *Let G be a plane graph satisfying the assumptions of Corollary 103. If Theorem 102 holds for all graphs smaller than G, then G is L-colorable.*

Proof. Suppose for a contradiction that G is a non-L-colorable graph satisfying the assumptions, such that Lemma 104 holds for all graphs smaller than G. Let $K \neq C$ be a (≤ 8)-cycle in G, and H the subgraph of G drawn in the closed disk bounded by K. If $H \neq K$, then, by the minimality of G, $G - (E(H) \setminus E(K))$ has an L-coloring φ, and since G is not L-colorable, the precoloring of K given by φ does not extend to an L-coloring of H. By the minimality of G, we conclude that either K is not an induced cycle in H or H contains a K-obstacle F. Assume the latter. Note that each internal face K' of F has length at most 7, and let H' be the subgraph of G drawn in the closed disk bounded by K'. Since F contains a (≤ 4)-cycle and $t(G) \geq B$, K' is an induced cycle in H' and H' does not contain any K'-obstacle. It follows that $H' = K'$ for every internal face K' of F, and thus $H = F$. We conclude that

(75) *every (≤ 8)-cycle $K \neq C$ in G either bounds a face, has a chord drawn inside the disk bounded by K, or the subgraph drawn inside K is a K-obstacle.*

In particular, every (≤ 5)-cycle bounds a face.

Consider a vertex $v \in V(G) \setminus V(C)$, and assume that v has more than one neighbor in C. If v has at least three neighbors in C, then G contains the C-obstacle consisting of v, C and three edges incident with v (satisfying the condition O1). Thus, suppose that v has exactly two neighbors $w_1, w_2 \in V(C)$. Furthermore, suppose that $\ell(C) \leq 7$ or that w_1 and w_2 are non-adjacent. Let K_1 and K_2 be the two cycles formed by $w_1 v w_2$ and the two paths between w_1 and w_2 in C, and note that $\ell(K_1), \ell(K_2) \leq 8$ and both K_1 and K_2 are induced cycles. By (75) and the assumption that $t(G) \geq B$, we conclude that at least one of K_1 and K_2 (say K_1) bounds a face. By the minimality of G, v has degree at least three, thus K_2 does not bound a face. Again, since $t(G) \geq B$, this implies that $\ell(K_1) \geq 5$ and $6 \leq \ell(K_2) \leq 7$. Thus, the subgraph F_2 drawn inside K_2 is a K_2-obstacle satisfying condition O1 or O2, and $F_2 \cup K_1$ is a C-obstacle in G. It follows that

(76) *no vertex $v \in V(G) \setminus V(C)$ has more than one neighbor in C, unless $\ell(C) = 8$ and the neighbors of v in C are adjacent.*

Also, observe that

(77) *if $\ell(C) = 8$ and v has two adjacent neighbors w_1 and w_2 in C, then no neighbor x of v distinct from w_1 and w_2 is adjacent to a vertex in C,*

as otherwise (75) together with $t(G) \geq B$ implies that x has two (non-adjacent) neighbors in C.

Suppose now that two adjacent vertices $v_1, v_2 \in V(G) \setminus V(C)$ both have a neighbor in C. By (76) and (77), each of them has exactly one such neighbor; let $w_i \in V(C)$ be the neighbor of v_i, for $i \in \{1, 2\}$. Furthermore, suppose that both (induced) cycles K_1 and K_2 consisting of $w_1v_1v_2w_2$ together with a path joining w_1 with w_2 in C have length at least 6. Note that $\ell(K_1) + \ell(K_2) = \ell(C) + 6$, thus $\ell(K_1), \ell(K_2) \leq \ell(C)$ and $\ell(C) \geq 6$. Since $t(G) \geq B$, (75) implies that say K_1 bounds a face and the subgraph of G in K_2 is a K_2-obstacle. Consider the graph G' obtained from G by contracting an edge e of the path $K_1 - \{w_1, v_1, v_2, w_2\}$ and giving the resulting vertex a color different from the color of its neighbors. By (75), e does not belong to a (≤ 5)-cycle in G, thus the contraction does not create any (≤ 4)-cycle. Also, as G contains only one cycle of length at most 4 (drawn inside K_2), the restriction on the distance between (≤ 4)-cycles in G' is vacuously true. The graph G' is not L-colorable, and by the minimality of G, it contains an obstacle satisfying O1 or O2. However, this gives a corresponding C-obstacle in G. Therefore,

(78) *if each of two adjacent vertices $v_1, v_2 \in V(G) \setminus V(C)$ has a neighbor in C, then they together with a path in C bound a face of length at most* 5.

If $3 \leq \ell(C) \leq 4$, then consider the graph G' obtained from G by subdividing an edge of C by $5 - \ell(C)$ new vertices, and giving these vertices distinct colors that do not appear in any of the lists of G. Note that G' is smaller than G, since it contains fewer (≤ 4)-cycles, and by the minimality of G, we conclude that G' is L-colorable. However, that gives an L-coloring of G, thus we may assume that $\ell(C) \geq 5$.

Let us now show that there exists a set $X \subseteq V(C)$ of $\max(1, \ell(C) - 5)$ consecutive vertices of C such that

- every path of length at most 3 whose endvertices belong to X is contained in the subgraph of G induced by X, and

- no vertex of X has a neighbor in a triangle.

If $\ell(C) \leq 7$, then by (76), at most three vertices of C are incident with or have a neighbor in a triangle, and at most two vertices are incident with a 4-cycle. Since $t(G) \geq B$, these cases are mutually exclusive, thus we can choose X as a subset of the remaining (at least $\ell(C) - 3$) vertices. Hence, suppose that $\ell(C) = 8$ and $C = v_1 v_2 \ldots v_8$. If say $v_2 v_3$ is an edge of a triangle, then none of $v_5, \ldots, v_8$ has a neighbor in a triangle. If $v_5 v_6 v_7$ is not a part of the boundary walk of a 5-face, then set $X = \{v_5, v_6, v_7\}$; otherwise, $v_6 v_7 v_8$ is not a part of the boundary walk of a 5-face by (76), and we set $X = \{v_6, v_7, v_8\}$. We choose the set X in the same way in case that a triangle shares a single vertex v_2 with C, or a 4-cycle shares at most two vertices v_2 and v_3 with C, or no (≤ 4)-cycle intersects C and at least 4 consecutive vertices v_5, v_6, v_7 and v_8 have no neighbor in a triangle. It remains to consider the case that no (≤ 4)-cycle intersects C and among each 4 consecutive vertices, at least one has a neighbor in a triangle. If three vertices of C had a neighbor in a triangle, then (75) would imply that $G - V(C)$ is a triangle, giving a C-obstacle satisfying O2. Therefore, two opposite vertices of C, say v_1 and v_5, have a neighbor in a triangle. However, this contradicts (76) or (78).

Let $C - X = v_1 v_2 \ldots v_k$, where $k = \ell(C) - |X| \leq 5$. Let $G' = G - X$, with the list assignment L' obtained from L by removing from the list of each vertex the color of its neighbor (if any) in X. Furthermore, we set $L'(v_1) = L(v_1) \cup L(v_2)$ and $L'(v_k) = L(v_k) \cup L(v_{k-1})$. By the choice of X, G' with the list assignment L' satisfies the assumptions of Theorem 102, and every vertex incident with a triangle that does not belong to $V(C)$ has list of size three. An L'-coloring of G would correspond to an L-coloring of G, thus we conclude that $k = 5$ (and hence $\ell(C) \geq 6$) and G' contains a subgraph H isomorphic to one of the graphs OBSTa1 – OBSTa7 drawn in Figure 10.1 (with matching lengths of lists according to L'). However, a case analysis shows that

- if H is OBSTa1 or OBSTa2, then G contains a C-obstacle satisfying (O2),
- if H is OBSTa3, then G contains the C-obstacle drawn in Figure 10.4(a).
- if H is OBSTa4, OBSTa5 or OBSTa7, then G contains the C-obstacle drawn in Figure 10.4(b).
- if H is OBSTa6, then G contains the C-obstacle drawn in Figure 10.4(c).

□

Let us now give a short outline of the proof of Theorem 102. We basically follow the proof of Grötzsch's theorem by Thomassen [69], which the reader should be familiar with. We consider the hypothetical smallest counterexample. First, we give constraints on short paths Q whose endvertices belong to $V(C)$ and internal vertices do not belong to $V(C)$ (claims (80), (81) and (83) in the proof), by splitting the graph along Q, coloring one part and extending the coloring to the second one, with Q playing the role of the precolored path in the second part. However, due to the existence of counterexamples to the statement "every precoloring of a path of length two can be extended" (depicted in Figure 10.1), we cannot exclude such paths entirely. However, using the ability to color vertices of a path of length up to 5 if we can in the process ensure that there are no (≤ 4)-cycles nearby, we can strengthen these constraints sufficiently if the vertices of Q are close to P (claims (89) and (92)). Then, as in the Thomassen's proof, we try to color up to five appropriately chosen vertices of G near to P and remove their colors from the lists of their neighbors, so that the resulting graph G' satisfies the assumptions of Theorem 102. This may only fail if a (≤ 4)-cycle T appears near to the colored vertices, making (I) or (T) false (claims (93) and (95)). Note that this implies that $\ell(P) \leq 2$. Many of these problematic configurations (those where T is a 4-cycle, or where (T) is false in G') can be reduced by precoloring up to three more vertices near to T, extending the precolored path and at the same time removing some vertices so that T disappears. Still, some cases (e.g., when T contains a vertex in C whose distance from P is at most four) remain. However, then we observe that we can apply the symmetric argument on the other side of P, and if that fails as well, a (≤ 4)-cycle T' must be close to the vertices that we try to color there as well. Since the distance between any two (≤ 4)-cycles in G is at least B, it follows that $T' = T$, which implies that G contains a short path Q with endvertices in C. Using the constraints on such paths, we can find a suitable set of vertices to color and remove in this case as well, finally finishing the proof.

Let us now provide the details of this argument, which unfortunately turns out to be rather lengthy and technical.

Proof of Theorem 102. Suppose that G together with lists L is a smallest counterexample, i.e., Theorem 102 holds for every graph smaller than G and G satisfies the assumptions of Theorem 102, but G is not L-colorable. Let C be the outer face of G and P a path with $V(P) \subseteq V(C)$ as in the statement of the theorem. We first derive several properties of this counterexample. Note that each vertex v of G has degree at least $\max(2, |L(v)|)$, and if two vertices u and v are adjacent, then $L(u) \cap L(v) \neq \emptyset$, unless uv is an edge of P. In particular, if $v \notin V(P)$ is adjacent to a vertex $p \in V(P)$, then $L(p) \subset L(v)$.

Lemma 104 implies that

(79) *every (≤ 8)-cycle K in G either bounds a face, has a chord drawn inside the disk bounded by K, or the subgraph drawn inside K is a K-obstacle.*

In particular, every (≤ 5)-cycle in G bounds a face. Furthermore,

(80) *The graph G is 2-connected.*

Proof. Clearly, G is connected. Suppose that G is not 2-connected, and let $G = G_1 \cup G_2$, where $V(G_1) \cap V(G_2) = \{v\}$ and $|V(G_1)|, |V(G_2)| \geq 2$. If say $P \subseteq G_1$, then by the minimality of G, an L-coloring φ_1 of G_1 exists. Let L_2 be the list assignment such that $L_2(x) = L(x)$ for $x \neq v$ and $L(v) = \{\varphi_1(v)\}$. By the minimality of G, we have that G_2 is L_2-colorable. However, this gives an L-coloring of G. Similarly, in case that the cut-vertex v is an internal vertex of P, the minimality of G implies that both G_1 and G_2 are L-colorable, giving an L-coloring of G. This is a contradiction. $\square$

A *chord* of a cycle K is an edge $e \notin E(K)$ joining two vertices of K. A vertex of a path is *internal* if its degree in the path is two, and an *endvertex* otherwise.

(81) *Every chord of C joins two vertices u and v with list of size three, such that either u and v have a common neighbor with list of size two, or there exists a triangle $w_1w_2w_3$ with $|L(w_2)| = 2$, a neighbor $z \notin \{w_2, w_3\}$ of w_1 with $|L(z)| = 2$, and $uz, vw_3 \in E(G)$ or $uw_3, vz \in E(G)$.*

Proof. Let uv be a chord of C. Let $G = G_1 \cup G_2$, where $V(G_1) \cap V(G_2) = \{u, v\}$ and $|V(G_1)|, |V(G_2)| \geq 3$. By symmetry, we may assume that $|V(G_1) \cap V(P)| \geq |V(G_2) \cap V(P)|$. If $u, v \in V(P)$, then by the minimality of G, both G_1 and G_2 are L-colorable, and their colorings combine to an L-coloring of G. This is a contradiction, thus we may assume that $v \notin V(P)$. Let $P_i = (P \cap G_i) \cup \{uv\}$ for $i \in \{1, 2\}$.

By the minimality of G, there exists an L-coloring φ of G_1. Let L' be the list assignment such that $L'(x) = L(x)$ for $x \notin \{u, v\}$ and $L'(x) = \{\varphi(x)\}$ for $x \in \{u, v\}$. Since G is not L-colorable, G_2 is not L'-colorable, thus it violates (Q), (OBSTa) or (OBSTb).

Suppose first that u is not an internal vertex of P. Then only two vertices are precolored in G_2, and thus G_2 contains either a vertex with list of size two adjacent to u and v or OBSTx1. By (I) and (T), neither u nor v have a list of size two. Furthermore, note that u cannot be an endvertex of P: Otherwise, we have $d(P) \leq 2$, thus $\ell(P) \leq 2$. Let $c \neq \varphi(v)$ be a color in $L(v) \setminus L(u)$ and L_2 the list assignment such that $L_2(v) = \{c\}$ and $L_2(x) = L(x)$ for $x \neq v$. Note that G_2 with list assignment L_2 satisfies (Q) and (OBSTa), and by the minimality of G, G_2 is L_2-colorable. It follows that G_1 cannot be L_2-colorable. However, we have $d(P_1) \geq B - 4 \geq r(P_1)$ in G_1. Since G_1 is not L_2-colorable, it follows that G_1 violates (Q). However, that implies that G contains a non-L-colorable OBSTx1c, OBSTx2a or OBSTx2b, which is a contradiction. Therefore, the chord uv satisfies the conclusion of (81) in this case.

Let us now consider the case that u is an internal vertex of P. By the choice of G_1 and G_2, we have $2\ell(P_2) \leq \ell(P) + 2$. Suppose first that $\ell(P_2) = 2$. By the minimality of G, we conclude that (S3), (Q) or (OBSTa) fails for G_2 with the list assignment L'. This implies that $d(P) \leq 3$, and since G satisfies the assumptions of Theorem 102, we have $\ell(P) = 2$. However, by symmetry G_1 with the precolored path P_1 also fails (S3), (Q) or (OBSTa), implying that $t(G) \le 6$. This is a contradiction.

Therefore, we may assume that $\ell(P_2) = 3$, and thus $\ell(P) \ge 4$ and $d(P) \ge r(P)$. Note that $d(P_2) \ge d(P) - 1$, and thus $d(P_2) \ge r(P_2)$. By the minimality of G, we have that G_2 fails (Q), and G_2 contains a vertex w with $|L(w)| = 2$ adjacent both to v and to an endvertex of P. Analogously, G_1 (with the precolored path P_1) also fails (Q), or $\ell(P) = 5$ and G_1 fails either (S3) or (OBSTb) due to a subgraph isomorphic to OBSTb1 or OBSTb2. The obstruction in G_1 together with the 5-cycle G_2 form one of the subgraphs H described in (OBSTb), namely OBSTb1, OBSTb1a, OBSTb1b, OBSTb5 or OBSTb6; and by (79), this subgraph H is unique. By (OBSTb), H has an L-coloring ψ. However, by the minimality of G, this implies that the precoloring that assigns v the color $\psi(v)$ extends both to G_1 and G_2, contradicting the assumption that G is not L-colorable. □

Let us note that (81) implies that P is a subpath of C. Furthermore, observe that there exists an L-coloring of the subgraph of G induced by $V(C)$, unless G contains a non-L-colorable OBSTx1, OBSTx1a or OBSTx1b. Lemma 104 then implies that

(82) $\ell(C) \ge 9$.

Proof. If $\ell(C) \le 8$, then G would contain a C-obstacle H, and by (79), it would actually be equal to this C-obstacle. Since each C-obstacle contains a (≤ 4)-cycle whose distance from any vertex of C is at most 4, this is only possible if $\ell(P) \le 2$. However, a straightforward case analysis shows that either G is L-colorable or violates (OBSTa). More precisely,

- If H satisfies (O1) and $|V(H) \setminus V(C)| = 1$, then G contains OBSTa1 or is L-colorable.
- If H satisfies (O1) and $|V(H) \setminus V(C)| = 2$, then G contains OBSTa6 or OBSTx4, or is L-colorable.
- If H satisfies (O2) and $|V(H) \setminus V(C)| = 3$, then G contains OBSTa2 or is L-colorable.
- If H satisfies (O2) and $|V(H) \setminus V(C)| = 4$, then G is L-colorable.
- If H satisfies (O2) and $|V(H) \setminus V(C)| = 5$, then G contains OBSTa3, OBSTa4 or OBSTa7, or is L-colorable.
- If H satisfies (O3), then G is L-colorable.

□

For $k \geq 2$, a *k-chord* of a cycle K is a path $Q = q_0 q_1 \ldots q_k$ of length k joining two distinct vertices of K, such that $V(K) \cap V(Q) = \{q_0, q_k\}$. We consider a chord to be a *1-chord*. Suppose that neither q_0 nor q_k is an internal vertex of P. Let G_1 and G_2 be the maximal connected subgraphs of G intersecting in Q, such that $P \subseteq G_1$. We say that Q *splits off a face* if G_2 is a cycle. For one of the obstructions O drawn in Figures 10.1 and 10.2, the k-chord Q *splits off* O if G_2 is isomorphic to O and

- the vertices drawn in the Figures by full circles coincide with the (not necessarily proper) subpath of Q consisting of the vertices $x \in V(Q)$ such that $|L(x)| \in \{1, 3\}$, and
- the sizes of the lists of all other vertices of G_2 are equal to those given by Figure 10.1 or 10.2.

(83) *Let $Q = q_0 q_1 \ldots q_k$ be a k-chord of C such that no endvertex of Q is an internal vertex of P and Q does not split off a face. If $k \leq 2$, or if $k = 3$ and q_3 has list of size two, then Q splits off one of the obstructions drawn in Figure 10.1.*

Proof. Suppose for a contradiction that there exists a k-chord Q violating (83). Let G_1 and G_2 be the maximal connected subgraphs of G intersecting in Q, such that $P \subseteq G_1$. Let us choose Q among all (≤ 3)-chords of C that violate (83) so that $|V(G_2)|$ is minimal.

By the minimality of G, there exists an L-coloring φ of G_1. Let L' be the list assignment such that $L'(x) = L(x)$ if $x \notin V(Q)$, $L'(q_3) = \{\varphi(q_2), \varphi(q_3)\}$ if $k = 3$ and $L'(q_i) = \{\varphi(q_i)\}$ for $0 \leq i \leq 2$. Observe that G_2 is not L'-colorable, thus it violates (Q) or (OBSTa). Let H be the minimal subgraph of G_2 that contains Q and violates (Q) or (OBSTa). Note that H contains a (≤ 4)-cycle T whose distance to any vertex of H is at most four. By (79), each face of H except for the outer one is also a face of G.

We claim that $G_2 = H$, that is, Q splits off H. Otherwise, consider a k'-chord $Q' \neq Q$ of G_2 that is a subpath of the union of Q and of the outer face of H. If Q' satisfies the assumptions of (83), then by the choice of Q, we have that that Q' splits off a subgraph H' that is either a face or an obstruction drawn in Figure 10.1. However, H' contains a (≤ 4)-face T', whose distance to Q' is at most three. It follows that $d(T, T') \leq 7 < B$, which is a contradiction. Therefore, Q' does not satisfy the assumptions of (83). Since every vertex with list of size two in H belongs to the outer face of G, the inspection of the graphs in Figure 10.1 shows that this is only possible if $k = 3$, H is OBSTx1 and $Q' = q_3 q_2 q_1 uv$ for vertices $u, v \in V(H) \setminus V(Q)$ such that $|L(u)| = 3$ and $|L(v)| = 2$. However, in this case let G_1' and G_2' be the subgraphs of G that intersect in Q', let φ' be an L-coloring of G_1' and let L_2 be the list assignment such that $L_2(x) = \{\varphi'(x)\}$ for $x \in \{v, q_1, q_2\}$, $L_2(q_3) = \{\varphi'(q_2), \varphi'(q_2)\}$, $L_2(v) = \{\varphi'(u), \varphi'(v)\}$ and $L_2(x) = L(x)$

for other vertices $x \in V(G_2')$. Since $t(G) \geq B$ and H contains T, we conclude that G_2' satisfies the assumptions of Theorem 102, hence G_2' is L_2-colorable. This gives an L-coloring of G, which is a contradiction. □

(79) and (83) imply that G does not contain a subgraph isomorphic to ones described in (OBSTa) or (OBSTb), such that the sizes of the lists match those prescribed by Figures 10.1 and 10.2: If G contained such a subgraph H, we would conclude that $G = H$ as in the proof of (83), and by the assumptions, G would be L-colorable.

(84) *If $Q = q_0q_1q_2$ is a 2-chord of C in G, then at most one endvertex of Q belongs to P.*

Proof. Suppose that both q_0 and q_2 belong to P. Then Q together with a subpath of P forms a cycle K of length at most $\ell(P) + 2$, and by (79) together with the assumption that $d(P) \geq r(P)$ if $\ell(P) > 2$, this cycle bounds a face. Observe that q_1 cannot have a neighbor in P distinct from q_0 and q_2. Let L' be the list assignment such that $L'(q_1) \subseteq L(q_1)\setminus(L(q_0)\cup L(q_2))$ has size one and $L'(x) = L(x)$ for $x \neq q_1$. Let $G' = G - q_0q_2$ if K is a triangle and $G' = G - (V(K) \setminus V(Q))$ otherwise. Note that the vertices with list of size one form an induced path P' in G', and the length of P' is at most $\ell(P) - 1$ if K has length at least 5 and at most $\ell(P) + 1$ otherwise. In the former case, if $d(P) \geq r(P)$, then $d(P') \geq r(P')$, since $d(P') \geq d(P) - 1$. In the latter case, we have $\ell(P) \leq 2$ and $d(P') \geq r(P')$, since $d(K) \geq B$. Since G' is smaller than G and is not L'-colorable, we conclude that it violates (Q) or (OBSTb). However, in these cases, G itself would violate (OBSTb): If G' violates (Q), then G contains OBSTb1b; if G' contains OBSTb1, then G contains OBSTb3; and if G' contains OBSTb2, then G' contains OBSTb4. □

(85) *Suppose that C has either a 3-chord $Q = q_0q_1q_2q_3$, or a 4-chord $Q = q_0q_1q_2q_3q_4$ such that $|L(q_4)| = 2$, where no endvertex of Q is an internal vertex of P. Let G_1 and G_2 be the maximal connected subgraphs of G that intersect in Q, such that $P \subseteq G_1$. Assume that either*

- *$\ell(P) \geq 4$ and $d(P, q_i) \leq r(4) - r(3) = 4$ for $0 \leq i \leq 3$, or*
- *G_1 contains a ($\leq$4)-cycle T such that $d(P, q_i) \leq B - r(3)$ for $0 \leq i \leq 3$.*

Then G_2 is a 5-cycle, and hence q_0 and q_3 have a common neighbor with list of size two (equal to q_4 if Q is a 4-chord).

Proof. Let φ be an L-coloring of G_1 that exists by the minimality of G. Let L_2 be the list assignment such that $L_2(q_i) = \{\varphi(q_i)\}$ for $0 \leq i \leq 3$, if Q is a 4-chord, then $L_2(q_4) = \{\varphi(q_3), \varphi(q_4)\}$, and $L_2(x) = L(x)$ for $x \notin V(Q)$. The graph G_2 is not L_2-colorable. Furthermore, we have $d(q_0q_1q_2q_3) \geq r(q_0q_1q_2q_3)$, since either $\ell(P) \geq 4$

and $d(q_0q_1q_2q_3) + (r(4) - r(3)) \geq d(P) \geq r(P)$, or $d(q_0q_1q_2q_3) + (B - r(3)) \geq B$. By the minimality of G, we conclude that G_2 violates (Q), hence a vertex x with a list of size two is adjacent to both q_0 and q_3. Furthermore, by (79) and (83), G_2 is equal to the 5-face $q_0q_1q_2q_3x$. □

We may assume that $\ell(P) \geq 2$; otherwise, we can color $2 - \ell(P)$ vertices adjacent to P in C so that the resulting list assignment L' either still satisfies the assumptions of Theorem 102 or violates (OBSTa). But, in the latter case, (79) and (83) would imply that G with the list assignment L' is equal to one of the obstructions in Figure 10.1. However, then it is easy to see that G either is L-colorable or contains OBSTx1. Let $P = p_0p_1 \ldots p_m$, where $m = \ell(P)$.

A subgraph H of G is a *near-obstruction* if it is isomorphic to one of the graphs in Figure 10.1 or 10.2, where the vertices drawn by full circles coincide with the vertices of H belonging to P and the sizes of lists of other vertices of H are greater or equal to the sizes prescribed by the Figure. A near-obstruction H is *tame* when for every vertex v of H that is depicted in Figure 10.1 or 10.2 by a square, if v is adjacent to a vertex in P, then $v \in V(C)$.

(86) *The graph G contains no tame near-obstruction.*

Proof. Suppose that H is a tame near-obstruction in G, and let K be the cycle bounding the outer face of H. Let $Q_0 = q_0q_1 \ldots q_k$ be the subpath of K vertex-disjoint with P such that $V(K) \subseteq V(Q_0) \cup V(P)$. Suppose first that both q_0 and q_k are adjacent to an endvertex of P, say q_0 to p_0 and q_k to p_m; by the assumption that $d(P) \geq r(P)$ and that H is tame and by (81), this is the case unless H is OBSTx1 and $\ell(P) = 2$. Let Q be the path consisting of Q_0 and those of the edges q_0p_0 and q_kp_m that do not belong to C.

Note that $|V(H)| < |V(G)|$, since otherwise either G violates (OBSTa) or (OBSTb), or is L-colorable. Let $G' = G - (V(H) \setminus V(Q))$. By the minimality of G, the graph H is L-colorable. Let φ be an L-coloring of H, and let L' be the list assignment such that $L'(x) = \{\varphi(x)\}$ if $x \in V(Q)$ and $L'(x) = L(x)$ otherwise. Note that G' is not L'-colorable, and by the minimality of G, it cannot satisfy the assumptions of Theorem 102. But, clearly G' satisfies (I) and (T). Let us now discuss several cases; we always assume that the precolored vertices of the drawing of H in Figure 10.1 or 10.2 are labeled from left to right, i.e., p_0 is the the leftmost precolored vertex in the drawing.

- *H* **is OBSTx2a or OBSTx2b:** Since q_1p_2 is not a chord by (81), we have $q_1 \notin V(C)$. By (83), the 2-chord $q_0q_1p_2$ splits off a subgraph H' which is isomorphic to one of the graphs drawn in Figure 10.1. Since $V(H) \neq V(G)$, H' is not OBSTx1. Since $H \subseteq G$, we have that q_1 has degree at least three in H' and that q_1, p_2 and two vertices of a triangle are incident with a common 5-face in H'. This implies that H' is OBSTa1, OBSTa3 or OBSTx4. However, then q_0 is adjacent to a vertex with list of size two in

H', and thus $|L(q_0)| = 3$. It follows that the 5-cycle $p_0p_1p_2q_1q_0$ has at least two L-colorings, and at least one of them extends to H'. Therefore, G is L-colorable, which is a contradiction.

- $\ell(\mathbf{Q}) \le \mathbf{5}$**:** Since $t(G) \ge B$ or $d(P) \ge r(P)$, no vertex of Q is contained in a (≤ 4)-cycle. The inspection of the graphs depicted in Figures 10.1 and 10.2 shows that among any three consecutive internal vertices of Q, at least one has degree two in H. This implies that Q is an induced path in G, since otherwise by (79), G would contain a vertex of degree two with list of size three. Similarly, we conclude that in G, no vertex with list of size two has two neighbors in Q, unless H is OBSTa1 (or OBSTx2a, but that was already excluded). However, if H is OBSTa1 and q_0 and q_3 have a common neighbor x with list of size two, then (79) and (83) imply that $V(G) = V(H) \cup \{x\}$, and it is easy to see that G is L-colorable. We conclude that G' satisfies (S3) and (Q).

  Let us discuss several subcases regarding m:

  - $m = 2$: That is, H is one of the obstructions drawn in Figure 10.1, except for OBSTa5, OBSTx1, OBSTx2b or OBSTx3 (or OBSTx2a, which was already excluded). Note that in all these cases, $\ell(Q) \le 4$. Also, H contains a triangle whose distance from any vertex of Q is at most three, and thus G' satisfies $d(Q) \ge r(Q)$. It follows that G' violates (OBSTb), i.e., $\ell(Q) = 4$, H is OBSTa3, OBSTa4, OBSTa6, OBSTa7, OBSTx1a, OBSTx1b or OBSTx4 and G' is OBSTb1 or OBSTb2. Since G does not contain a vertex of degree two with list of size three, if G' is OBSTb2, then H is OBSTa7. The case analysis of the possible combinations of H and G' shows that G is L-colorable, which is a contradiction.
  - $m = 4$: The case that H is OBSTb1 is excluded by (83), since $d(P) \ge d(T)$, thus H is OBSTb2. (83) furthermore implies that $|L(q_2)| = 3$, and thus we may choose the L-coloring φ so that $\varphi(q_1) \notin L(q_0) \setminus L(p_0)$. Let L'' be the list assignment defined by $L''(q_0) = (L(q_0) \setminus L(p_0)) \cup \{\varphi(q_1)\}$ and $L''(x) = L'(x)$ otherwise. Note that only a path $q_1q_2q_3q_4$ of length three is precolored in G' according to this list assignment and $d(q_1q_2q_3q_4) \ge d(P) - 3 \ge r(P) - 3 \ge r(q_1q_2q_3q_4)$ and thus G' is L''-colorable. This gives an L-coloring of G, which is a contradiction.
  - $m = 5$: By (84), H cannot be OBSTb3 or OBSTb4. Thus, H is OBSTb1a, OBSTb1b, OBSTb2a, OBSTb2a', OBSTb2b or OBSTb5, and $\ell(Q) \le 4$. We conclude that G' is OBSTb1 or OBSTb2 and $\ell(Q) = 4$ (excluding the cases that H is OBSTb1a or OBSTb1b). Note that q_2 has degree two in H, and since it has degree at least three in G, we conclude that G' is OBSTb1. The case analysis of the

possible combinations of H and G' shows that G is L-colorable, which is a contradiction.

- $\ell(\mathbf{Q}) > \mathbf{5}$**:** Thus, H is OBSTa5, OBSTx3 or OBSTb6. Let us discuss these cases separately:

  – H is OBSTa5: Let w be the common neighbor of q_1 and q_6, and w' the common neighbor of w, q_3 and q_4. If there exist colors $c_1 \in L(q_1) \setminus (L(q_0) \setminus L(p_0))$ and $c_2 \in L(q_6) \setminus (L(q_7) \setminus L(p_2))$ so that $L(w) \neq L(p_1) \cup \{c_1, c_2\}$, then consider the graph $G_1 = G - V(P)$ with the list assignment L_1 such that $L_1(q_1) = \{c_1\}$, $L_1(q_6) = \{c_2\}$, $L_1(w)$ chosen as an arbitrary one-element subset of $L(w) \setminus (L(p_1) \cup \{c_1, c_2\})$, $L_1(q_0) = (L(q_0) \setminus L(p_0)) \cup \{c_1\}$, $L_1(q_7) = (L(q_7) \setminus L(p_2)) \cup \{c_2\}$ and $L_1(x) = L(x)$ otherwise. The graph G_1 cannot be L_1-colorable, thus it violates (OBSTa). This is only possible if G_1 is OBSTa1, but then $V(G) = V(H)$ and thus G is L-colorable.

    So, we have $|L(q_0)| = |L(q_7)| = 3$, $L(q_1) = (L(q_0) \setminus L(p_0)) \cup \{c_1\}$, $L(q_6) = (L(q_7) \setminus L(p_2)) \cup \{c_2\}$ and $L(w) = L(p_1) \cup \{c_1, c_2\}$. Let ψ be an L-coloring of $q_1q_0p_0p_1p_2q_7q_6$ such that $\psi(q_1), \psi(q_6) \notin L(w) \setminus L(p_1)$. Let $G_2 = G - (V(P) \cup \{w'\})$, with the list assignment L_2 such that $L_2(x) = \{\psi(x)\}$ for $x \in \{q_0, q_1, q_6, q_7\}$, $L_2(w)$ is an arbitrary singleton list disjoint with $L_2(q_1)$ and $L_2(q_6)$ and $L_2(x) = L(x)$ otherwise. Since an L_2-coloring of G_2 corresponds to an L-coloring of G (choosing the color of w' different from the colors of q_3 and q_4, and the color of w different from the color of p_1 and w_2), we have that G_2 is not L_2-colorable. By (79), G_2 satisfies (S3) and (Q), and the internal face of G_2 incident with w has length at least six, thus G_2 satisfies (OBSTb). Furthermore, since $d(q_3q_4w') \geq B$ in G, we have $d(q_0q_1wq_6q_7) \geq B - 3 \geq r(q_0q_1wq_6q_7)$. Therefore, G_2 is a counterexample to Theorem 102 smaller than G, which is a contradiction.

  – H is OBSTx3: Let $q_1w_1w_2q_3$ be the path in H such that $w_1, w_2 \neq q_2$. If $|L(q_0)| = 2$, then consider an L-coloring ψ of the subgraph of G induced by $\{q_0, q_1, w_1, w_2, p_0, p_1\}$ such that $\psi(w_2) \notin L(q_7) \setminus L(p_2)$. Let L' be the list assignment defined by $L'(q_0) = \{\psi(q_0), \psi(q_1)\}$, $L'(x) = \{\psi(x)\}$ for $x \in \{q_1, w_1, w_2\}$, $L'(q_7) = (L(q_7) \setminus L(p_2)) \cup \{\psi(w_2)\}$ and $L'(x) = L(x)$ otherwise. We conclude that $G - V(P)$ is not L'-colorable, thus it violates (OBSTa). Note that w_1 has degree two in $G - V(P)$ and the face with that it is incident does not share any vertex with the triangle, and that q_7 is not incident with the triangle, thus $G - V(P)$ contains OBSTx2a. By (79) and (83), $G - V(P)$ is equal to OBSTx2a. However, then q_2, q_5 and q_7 have list of size two and G contains OBSTx3, which is a contradiction.

So, we have $|L(q_0)| = 3$. Then, there exist $c_1 \in L(q_1) \setminus (L(w_1) \setminus L(p_1))$ and $c_0 \in L(q_0) \setminus L(p_0)$ such that $c_0 \neq c_1$. Let G_1 be the graph obtained from $G - \{p_0, p_1, w_1, w_2\}$ by adding the edge q_1p_2. Let c be a color that does not appear in any of the lists of G. Let L_1 be the list assignment such that $L_1(q_0) = \{c_0\}$, $L_1(q_1) = \{c_1\}$, $L_1(p_2) = \{c\}$, $L_1(q_7) = (L(q_7) \setminus L(p_2)) \cup \{c\}$ and $L_1(x) = L(x)$ for all other vertices of G_1. Observe that G_1 is not L_1-colorable. Furthermore, the distance of q_1 from the triangle $q_4q_5q_6$ is three both in G and G_1, and the distance of q_1 and q_7 to any other (≤ 4)-cycle is at least $B - 3$, thus $t(G_1) \geq B$. The internal face F of G_1 incident with q_1p_2 has length at least six, as otherwise the cycle $F - q_1p_2 + q_1w_1p_1p_2$ has length at most seven and contradicts (79). Furthermore, observe that neither q_0 nor q_1 is adjacent to a vertex of the triangle $q_4q_5q_6$, thus G_1 contains neither OBSTx1 nor OBSTx1a. It follows that G_1 satisfies (OBSTa), and thus it is a counterexample to Theorem 102 smaller than G. This is a contradiction.

– H is OBSTb6: Let $q_1w_1w_2p_3$ be the path in H with w_1 adjacent to p_1. If $|L(q_6)| = 2$, then let c' be the unique color in $L(q_6) \setminus L(p_5)$, and note that there exists $c \in L(q_5) \setminus (L(p_3) \cup \{c'\})$. Let $G_1 = G - \{p_4, p_5\}$ and let L_1 be the list assignment such that $L_1(q_5) = \{c\}$, $L_1(q_6) = \{c, c'\}$ and $L_1(x) = L(x)$ for $x \notin \{q_5, q_6\}$. Note that G_1 is not L_1-colorable, and since a path of length 4 is precolored in G_1 and H is a subgraph of G, we conclude that G_1 contains OBSTb2. However, this implies that G contains OBSTb6, which is a contradiction.

  Therefore, $|L(q_6)| = 3$. Then, there exists an L-coloring ψ of the subgraph of G induced by $\{q_3, q_4, q_5, q_6, p_3, p_5\}$ such that $\psi(q_3) \notin L(w_2) \setminus L(p_3)$. Let G_2 be the graph obtained from $G - (V(P) \cup \{w_1, w_2\})$ by adding a vertex w adjacent to q_0 and q_3. Let c be a new color that does not appear in $L(q_0) \cup L(q_3)$. Let L_2 be the list assignment such that $L_2(x) = \psi(x)$ for $x \in \{q_3, q_4, q_5, q_6\}$, $L_2(w) = \{c\}$, $L_2(q_0) = (L(q_0) \setminus L(p_0)) \cup \{c\}$ and $L_2(x) = L(x)$ otherwise. Observe that an L_2-coloring of G_2 corresponds to an L-coloring of G, thus G_2 is not L_2-colorable. Furthermore, a path $P_2 = wq_3q_4q_5q_6$ of length 4 is precolored in G_2. Let us remark that the newly added vertex w is not incident with a (≤ 4)-cycle, as otherwise either $t(P) < r(P)$ in G, or (79) implies that q_2 is a vertex of degree two with list of size three. Furthermore, $t(G_2) \geq B$, since only the added path q_0wq_3 could result in shortening the distance between (≤ 4)-cycles, in G we have $d(q_0) \geq d(P) - 1 \geq r(P) - 1$ and $d(q_3) \geq d(P) - 2 \geq r(P) - 2$, and $2r(P) - 1 > B$. Also, $d(P_2) \geq d(P) - 2 \geq r(P_2)$.

  Note that G_2 satisfies (S3), since w is not adjacent to q_6 and $d(P) \geq r(P)$. Similarly, G_2 satisfies (Q), since otherwise (79) would imply that

q_4 is a vertex of degree two with list of size three. Hence, G_2 violates (OBSTb). Since q_4 has degree at least three, G_2 contains OBSTb1. But then q_4 and q_0 have a common neighbor x, and the existence of q_2 together with $d(P_2) \geq r(P_2)$ contradicts (79) applied to the 7-cycle $q_0q_1w_1w_2q_3q_4x$.

Finally, let us consider the case that say q_0 is not adjacent to an endvertex of P, that is, $\ell(P) = 2$, H is OBSTx1, q_0 is adjacent to p_1 and q_3 is adjacent to p_2. An L-coloring of H does not extend to an L-coloring of the subgraph G' that is split off by the path $p_0p_1q_0q_1q_2q_3$. If p_0 and q_1 have a common neighbor with list of size two, then either G is L-colorable or contains OBSTa1. Otherwise, G' satisfies (S3) and (Q), as q_1 cannot be a vertex of degree two with list of size three. Therefore, G' violates (OBSTb). If $|L(q_3)| = 2$, then G' may only be OBSTb1, OBSTb1b, OBSTb2 or OBSTb2b. OBSTb1 and OBSTb1b are excluded, since q_1 must have degree at least three; if G' is OBSTb2, then G is L-colorable, and if G' is OBSTb2b, then G contains OBSTa3. If $|L(q_3)| = 3$, then there exist L-colorings ψ_1 and ψ_2 of H such that $\psi_1(q_0) = \psi_2(q_0)$, $\psi_1(q_1) \neq \psi_2(q_1)$, $\psi_1(q_2) \neq \psi_2(q_2)$ and $\psi_1(q_3) \neq \psi_2(q_3)$. The inspection of the graphs in Figure 10.2 shows that at least one of ψ_1 and ψ_2 extends to an L-coloring of G', unless G' contains a subgraph H' isomorphic to OBSTb1, OBSTb1a, OBSTb1b, OBSTb3 or OBSTb5. By (79) and (83) we conclude that $G' = H$ and $G = H \cup H'$. However, all possible combinations of H and H' result in an L-colorable graph, which is a contradiction. □

Let $v_1, v_2, \ldots, v_s$ be the vertices of $C - V(P)$ labeled so that $C = p_0 \ldots p_m v_1 v_2 \ldots v_s$, where $s = \ell(C) - m - 1$. Let us also define $v_0 = p_m$.

(87) *For* $1 \leq i \leq 4$*, if the edge* $v_{i-1}v_i$ *is not contained in a cycle of length at most* 4 *and a vertex* $v \in V(G)$ *is adjacent to both* v_i *and an endvertex* p *of* P*, then* $v \in V(C)$*.*

Proof. Suppose for a contradiction that $v \notin V(C)$. Let G_2 be the subgraph of G that is split off by the 2-chord v_ivp according to (83), and $G_1 = G - (V(G_2) \setminus \{v_i, v, p\})$. If $p = p_m$, then $i \in \{3, 4\}$, since $v_{i-1}v_i$ does not belong to a (≤ 4)-cycle. By (79) and the fact that every vertex of degree two has list of size two, we have that $i = 4$ and G_2 contains a triangle. It follows that $m \leq 2$. Consider an L-coloring ψ of G_2, and let L_1 be the list assignment such that $L_1(v) = \{\psi(v)\}$, $L_1(v_4) = \{\psi(v_i)\}$ and $L_1(x) = L(x)$ for $x \notin \{v, v_4\}$. Note that G_1 is not L_1-colorable. By (81), (83), (82) and the assumption that $v \notin V(C)$, we conclude that G_1 satisfies (S3) and (Q). Therefore, using (79) and (83) we conclude that G_1 is equal to (OBSTb1) or (OBSTb2). However, all combinations of (OBSTb1) or (OBSTb2) with a $p_mv_1v_2v_3v_4v$-obstacle are L-colorable.

Let us now consider the case that $p = p_0$. Since a (≤ 4)-cycle in G_2 is in distance at most 4 from P, we have $\ell(P) \leq 2$. Let K be the cycle of length at most 8 formed by the 2-chord v_ivp_0, the path P, and the vertices v_1, v_2, …, v_i. Since $t(G) \geq B$, G_1 cannot be a K-obstacle, and if K is not a face, then $\ell(K) = 8$ and K has a chord splitting K to two 5-faces. If K is not a face, then since each vertex with list of size three has degree at least three, we conclude that $|L(v_1)| = |L(v_3)| = 2$, $|L(v_2)| = 3$ and the chord of K is v_2p_0. However, this contradicts (81). Therefore, K is a face. Since v has degree at least three, G_2 is not a face. Furthermore, G_2 is not (OBSTx1b), thus $|L(v_i)| = 3$. Hence, there exist L-colorings ψ_1 and ψ_2 of K such that $\psi_1(v) \neq \psi_2(v)$ and $\psi_1(v_i) \neq \psi_2(v_i)$. The inspection of the graphs in Figure 10.1 shows that at least one of ψ_1 and ψ_2 extends to an L-coloring of G_2, giving an L-coloring of G. This is a contradiction. □

(88) *Suppose that $m = 5$. For $1 \leq i \leq 4$, if a vertex $v \in V(G)$ is adjacent to both v_i and to $p \in \{p_1, p_4\}$, then $v \in V(C)$, unless $p = p_4$ and $i = 2$, or $p = p_1$ and $i = s - 1$.*

Proof. Suppose that $v \notin V(C)$ is adjacent to p_4 and v_i. Since $d(P) \geq r(P)$ and every vertex with list of size three has degree at least three, (79) implies that $i = 2$.

Hence, assume that $v \notin V(C)$ is adjacent to p_1 and v_i. Let $Q = p_0p_1vv_i$, let G_1 be the subgraph of G drawn in the cycle bounded by $vp_1 \ldots p_5v_1 \ldots v_i$ and $G_2 = G - (V(G_1) \setminus V(Q))$. By the minimality of G, there exists an L-coloring φ of G_1. Let L_2 be the list assignment such that $L_2(x) = \varphi(x)$ for $x \in \{v, v_i\}$ and $L_2(x) = L(x)$ otherwise; the graph G_2 cannot be L_2-colorable. Since only an induced path Q of length three is precolored in G_2 (and $d(Q) \geq d(P) - 2 \geq r(P) - 2 \geq r(Q)$), we conclude that G_2 violates (Q), thus there exists a vertex w with list of size two adjacent to p_0 and v_i. By (81), we have $C = p_0p_1 \ldots p_5v_1 \ldots v_iw$, and thus $i = s - 1$. □

(89) *If v_i has degree two and is incident with a triangle, then $i \geq 4$. Furthermore, if $4 \leq i \leq 6$, v_i has degree two and is incident with a triangle, then $|L(v_{i+2})| \neq 2$.*

Proof. Suppose first that v_i is such a vertex, with $1 \leq i \leq 3$. Clearly, this is only possible if $\ell(P) \leq 2$. By the minimality of G, the subgraph G_0 of G induced by $V(P) \cup \{v_1, \ldots, v_{i+1}\}$ has an L-coloring ψ. Let L' be the list assignment such that $L'(x) = \{\psi(x)\}$ for $x \in \{v_1, \ldots, v_{i+1}\}$ and $L'(x) = L(x)$ otherwise, and let $Q = p_0p_1p_2v_1 \ldots v_{i-1}v_{i+1}$. Let $G' = G - v_i$. Then, G' is not L'-colorable. Furthermore, by (81) and (82), G' satisfies (Q). Since $d(Q) \geq d(v_{i-1}v_iv_{i+1}) - 4 \geq B - 4 \geq r(Q)$, G' violates (OBSTb), and by (79) and (83), G' is equal to one of the graphs drawn in Figure 10.2. If $i = 2$, then either G' is OBSTb1 and thus G contains OBSTx2b, or G' is OBSTb2 and G is L-colorable. Therefore, $i = 3$. If $|L(v_1)| = 3$, then we can assume that $\psi(v_2) \notin L(v_1) \setminus L(p_2)$, thus there exist two L-colorings of the subgraph of G_0 that differ only in the color of v_1. Furthermore, the degree of v_1 in G' is at least three. The inspection of the graphs drawn in Figure 10.2 shows that at least one of these colorings extends to G', which is a contradiction. If $|L(v_1)| = 2$, then by (T) we have that either G' is OBSTb1b and G contains OBSTx2a, or G' is OBSTb2b and G contains OBSTx3.

Suppose now that $4 \le i \le 6$ and $|L(v_{i+2})| = 2$. Again, $m = 2$. By (T), $|L(v_{i-2}| = 3$, and by (81), $p_0p_1p_2v_1 \dots v_{i-1}v_{i+1}$ is an induced path. Thus, there exists its L-coloring ψ such that $L(v_i) \ne \{\psi(v_{i-1}), \psi(v_{i+1})\}$ and $\psi(v_{i+1}) \notin L(v_{i+2})$. Let $G' = G - \{v_{i-1}, v_i, v_{i+1}\}$ with the list assignment L' such that $L'(v_j) = \{\psi(v_j)\}$ for $1 \le j \le i-3$, $L'(v_{i-2}) = \{\psi(v_{i-3}), \psi(v_{i-2})\}$, $L'(x) = L(x) \setminus \{\psi(y)\}$ for a vertex $x \in V(G')$ with a neighbor $y \in \{v_{i-1}, v_{i+1}\}$ and $L'(x) = L(x)$ otherwise. The graph G' is not L'-colorable. Furthermore, by (81), (S2) holds, and by (83), (I) is satisfied as well. Let w be a common neighbor of two vertices of the path $Q = p_0p_1p_2v_1 \dots v_{i-3}$ in G'. By (81), we have $w \ne v_{i-2}$ and $|L(w)| = 3$. Furthermore, $|L'(w)| = 3$, since otherwise w would be adjacent to v_{i-1} or v_{i+1} as well, and (79) would imply that v_{i-2} has degree two in G. This shows that (Q) is true. Note that $d(Q) \ge B - 7 > r(P)$. Therefore, G' violates (OBSTb). This implies that $i \ge 5$; observe that there exist L-colorings ψ_1 and ψ_2 of Q such that $\psi_1(v_{i-1}) = \psi_2(v_{i-1}) = \psi(v_{i-1})$, $\psi_1(v_{i+1}) = \psi_2(v_{i+1}) = \psi(v_{i+1})$, $\psi_1(v_{i-2}) \ne \psi_2(v_{i-2})$, $\psi_1(v_{i-3}) \ne \psi_2(v_{i-3})$ and if $i = 6$, then $\psi_1(v_1) = \psi_2(v_1)$. Note that v_{i-4} is not adjacent to a vertex x with $|L'(x)| = 2$ and that v_{i-2} is the only such vertex adjacent to v_{i-3}, by (81), (79) and the fact that v_{i-2} has degree at least three in G. Since neither ψ_1 nor ψ_2 extends to an L'-coloring of G', the inspection of the graphs depicted in Figure 10.2 shows that $i = 6$ and G' contains OBSTb3. If v_8 is adjacent to p_0, then G contains OBSTx3. Otherwise, (81) and (83) imply that the edge of OBSTb3 incident with v_{i-2} (distinct from $v_{i-3}v_{i-2}$) is a chord of C that splits off OBSTx1 in G; however, the resulting graph is L-colorable. □

(90) *We have $|L(v_1)| = 2$ or $|L(v_2)| = 2$.*

Proof. Suppose that $|L(v_1)| = |L(v_2)| = 3$. Let L' be the list assignment such that $L'(v_1) = L(v_1) \setminus L(p_m)$ and $L'(x) = L(x)$ otherwise. Let $G' = G - p_mv_1$. By (81), G' with the list assignment L' satisfies (I). Suppose that (T) is violated. Then there exists a triangle $w_1w_2w_3$ such that either $v_1 = w_2$ and both w_1 and w_3 have a neighbor with list of size two, or $|L(w_2)| = 2$, w_1 is adjacent to v_1 and w_3 has a neighbor w distinct from w_1 with list of size two. By (83), the former is not possible, and in the latter case, we have $w_1 = v_2$, $w_2 = v_3$ and $w_3 = v_4$. However, that contradicts (89). Therefore, (T) holds. Furthermore, by (81), v_1 is not adjacent to any vertex of P other than p_m, and thus (Q) is satisfied. Since an L'-coloring of G' would give an L-coloring of G, it follows that G' with the assignment L' violates (OBSTa) or (OBSTb). However, this implies that G with the list assignment L contains a tame near-obstruction H, contradicting (86). □

(91) *If $\ell(P) = 5$, then $\ell(C) \ge 10$.*

Proof. By (82), we have $\ell(C) \geq 9$. Suppose that $\ell(C) = 9$. By (90), either $|L(v_1)| = 2$ or $|L(v_2)| = 2$. Applying (90) symmetrically on the other end of P, we also have that $|L(v_2)| = 2$ or $|L(v_3)| = 2$. Therefore, either $|L(v_1)| = |L(v_3)| = 2$ and $|L(v_2)| = 3$, or $|L(v_1)| = |L(v_3)| = 3$ and $|L(v_2)| = 2$. In the former case, L-color the path $v_1v_2v_3$ so that v_1 gets a color different from the color of p_5 and v_3 a color different from the color of p_0. Let $G' = G - \{v_1, v_2, v_3\}$, with the list assignment L' obtained from L by removing the colors of the vertices v_1, v_2 and v_3 from the lists of their neighbors. Note that G' satisfies (I), since otherwise $v_1v_2v_3$ would be a part of a 5-cycle, and by (79), v_2 would have degree two. Furthermore, (T) is satisfied since $d(P) \geq r(P)$ and (Q) is satisfied by (84). Note also that no vertex adjacent to p_0 or p_5 has list of size 2, thus G' satisfies (OBSTb). This is a contradiction, since an L'-coloring of G' corresponds to an L-coloring of G.

In the latter case, let G' be the graph with list assignment L' obtained from G by coloring v_2 from its list arbitrarily, removing v_2 and removing its color from the lists of its neighbors. Again, (I), (T) and (Q) are obviously satisfied by G'. Furthermore, since $d(P) \geq r(P)$, the distance between any pair of vertices of G' with list of size two is at least three. This implies that G' satisfies (OBSTb), unless it contains OBSTb1b. However, that is excluded by (84). □

Let X be the set of vertices defined as follows: If $|L(v_1)| = 3$ (and thus $|L(v_2)| = 2$ by (90) and $|L(v_3)| = 3$) and $|L(v_4)| = 3$, then $X = \{v_2\}$. If $|L(v_1)| = 3$ and $|L(v_4)| = 2$, then $X = \{v_2, v_3\}$. If $|L(v_1)| = 2$ (and thus $|L(v_2)| = 3$) and $|L(v_3)| = 3$, then $X = \{v_1\}$. If $|L(v_1)| = |L(v_3)| = 2$ (and thus $|L(v_4)| = 3$) and $v_5 = p_0$ or $|L(v_5)| = 3$, then $X = \{v_2, v_3\}$. Otherwise, $X = \{v_2, v_3, v_4\}$.

(92) *Let $Q = q_0q_1 \ldots q_k$ be a k-chord of C such that no endvertex of Q is an internal vertex of P and Q does not split off a face. If $k \leq 2$, or if $k = 3$ and q_3 has list of size two, then $q_0 \notin X$.*

Proof. Let G_2 be the subgraph of G that is split off by Q and $G_1 = G - (V(G_2) \setminus V(Q))$. Let Q be chosen so that G_2 is as large as possible. Let i be the index such that $v_i = q_0$. By (83) we can assume that $\ell(P) = 2$, since otherwise G_2 contains a triangle whose distance from q_0 is at most four, hence its distance from P is at most 8, contradicting $d(P) \geq r(P)$.

By (81) and (89), the path consisting of P and $v_1v_2v_3v_4$ is induced. Suppose now that $q_k \in \{v_1, v_2, v_3, v_4\}$, and let K be the cycle bounded by Q and a subpath of $v_1v_2v_3v_4$. Since Q does not split off a face, (79) implies that $\ell(K) \geq 6$, thus $k = 3$ and $\{q_0, q_k\} = \{v_1, v_4\}$. If $q_0 = v_1 \in X$, then $|L(v_1)| = 2$ and $|L(v_2)| = |L(v_3)| = 3$. However, (79) implies that v_2 or v_3 has degree two, which is a contradiction.

If $q_0 = v_4 \in X$, then (79), (89) and the choice of X imply that either $v_2q_2 \in E(G)$, or v_2, q_2 and q_0 are adjacent to vertices of a triangle T. In the former case, let ψ_1 and ψ_2 be L-colorings of the subgraph of G induced by $V(P) \cup \{v_1, v_2, q_2\}$

such that $\psi_1(v_1) = \psi_2(v_1)$, $\psi_1(v_2) \neq \psi_2(v_2)$ and $\psi_1(q_2) \neq \psi_2(q_2)$, let $G' = G - v_1v_2$ and let L_1 and L_2 be the list assignments such that $L_j(x) = \{\psi_j(x)\}$ for $x \in \{v_1, v_2, q_2\}$ and $L_j(x) = L(x)$ otherwise. Note that G' satisfies (Q) by (83) and that G' is not L_j-colorable for $j \in \{1, 2\}$, thus G' with both of these assignments violates (OBSTb). This is only possible if G' contains OBSTb3, but then G contains OBSTx4. In the latter case, let t_1 and t_2 be the vertices of T adjacent to v_2 and v_4, respectively, let ψ be an L-coloring of $p_mv_1v_2v_3v_4$ such that either $\psi(v_2) \notin L(t_1)$ or $L(t_1) \setminus \{\psi(v_2)\} \neq L(t_2) \setminus \{\psi(v_4)\}$, and let G' be the graph obtained from $G - V(T)$ by identifying v_2 with v_3 to a new vertex z. Note that z is not contained in a (≤ 4)-cycle by (79), and observe that $t(G') \geq B$. let L' be the list assignment defined in the following way: $L'(v_i) = \{\psi(v_i)\}$ for $i \in \{1, 4\}$, $L'(z) = \{c\}$ for a new color c that does not appear in any of the lists, and $L'(x) = L(x)$ for any other vertex $x \in V(G')$. Observe that G' is not L'-colorable and satisfies (Q) by (81) and (82), hence G' contains a subgraph H violating (OBSTb). Since q_1 has degree at least three, (79) implies that $v_1zv_4q_1q_2$ is the only cycle of length at most 5 in G' containing z, and that every cycle of length 6 containing z also contains q_1. It follows that $q_1 \in V(H)$. Unless H is OBSTb1b or OBSTb2b, $|L'(q_1)| = 3$ implies that $v_5 \in V(H)$, thus v_4 has degree at least three in H. Note that H is neither OBSTb1b nor OBSTb2b, since then we would have $v_5 \notin V(H)$ and a (≤ 3)-chord contained in the outer face of H incident with v_4 would contradict (83). The only obstruction in that the endvertex of the precolored path has degree greater than two is OBSTb4, however H is not OBSTb4 since q_1 is not adjacent to p_m.

Therefore, $q_k \notin \{v_1, v_2, v_3, v_4\}$. By (83), G_2 is one of the graphs depicted in Figure 10.1. Observe that there exists a color $c \in L(q_0)$ such that every L-coloring of Q that assigns c to q_0 extends to an L-coloring of G_2. Suppose first that there exists an L-coloring ψ of the path $P' = p_0p_1p_2v_1 \ldots v_i$ such that $\psi(q_0) = c$. Let L_1 be the list assignment such that $L_1(x) = \{\psi(x)\}$ for $x \in \{v_1, \ldots, v_{i-1}\}$, $L_1(v_i) = \{\psi(v_i), \psi(v_{i-1})\}$ and $L_1(x) = L(x)$ otherwise. Note that the path $P_1 = P' - v_i$ that is precolored in G_1 has length at most 5. Furthermore, G_2 contains a triangle whose distance from v_i is at most 4, thus $d(P_1) \geq B - 10 \geq r(P_1)$, and since G is not L-colorable, G_1 is not L_1-colorable. By (81), G_1 satisfies (I) and (Q). Note that the distance in G_1 from v_i to any triangle is at least $B - 4 > 1$, thus G_1 satisfies (T). We conclude that G_1 violates (OBSTb), and thus $i \in \{3, 4\}$. The choice of Q implies that if $Q' \neq Q$ is a path in G_1 of length at most three from a vertex v_j with $j \leq i$ to a vertex with list of size two, then the endvertex of Q' is q_0 and Q' bounds a face. The inspection of the graphs in Figure 10.2 shows that G_1 can only satisfy this condition if it contains OBSTb1, OBSTb1a or OBSTb1b. However, if G_1 contains one of these graphs, then (79) and (81) imply that both v_1 and v_2 have degree two, which is a contradiction.

Let us now consider the case that there is no L-coloring of the path P' assigning the color c to v_i. Since the path P' is induced, this is only possible if $i = 1$, or if $i = 2$ and $|L(v_1)| = 2$. If $|L(v_i)| = 2$, then $i = 1$ and (83) implies that $k = 2$

and G_2 is OBSTx1b. However, that is excluded by (89). Therefore, $|L(v_i)| = 3$. There exist two L-colorings ψ_1 and ψ_2 of P' such that $\psi_1(v_i) \neq \psi_2(v_i)$, and by the minimality of G, both of them extend to L-colorings φ_1 and φ_2 of G_1. Furthermore, neither φ_1 nor φ_2 extends to an L-coloring of G_2. The inspection of the graphs in Figure 10.1 shows that this is only possible if G_2 is OBSTa1 or OBSTx1c, or if $k = 3$ and G_2 is OBSTa2 or OBSTx2a. The case that G_2 is OBSTx2a is excluded by (89). Let us discuss the rest of the cases separately:

- If G_2 is OBSTa1, then there exists a color $c_1 \in L(q_1) \setminus \{\psi_1(q_0)\}$ such that every coloring of Q that assigns $\psi_1(q_0)$ to q_0 and c_1 to q_1 extends to an L-coloring of G_2. By (83), no neighbor of q_1 has list of size two. Let L' be the list assignment such that $L'(v_j) = \{\psi_1(v_j)\}$ for $1 \leq j \leq i$, $L'(q_1) = \{\psi_1(q_0), c_1\}$ and $L'(x) = L(x)$ otherwise. Note that G_1 is not L'-colorable, thus it violates (Q) or (OBSTb). If (OBSTb) is violated, i.e., G_1 contains OBSTb1 or OBSTb2, then G contains a ($\leq$3)-chord contradicting the choice of Q, thus suppose that (Q) is false. Then, (83) implies that $i = 2$ and q_1 is adjacent to p_1. However, then consider the path $Q' = p_0p_1q_1q_2$ (or $Q' = p_0p_1q_1q_2q_3$ if $k = 3$). Similarly to (85), we conclude that p_0 and q_2 have a common neighbor with list of size two, and since q_2 has degree at least three, this common neighbor is not equal to q_3. However, then G contains OBSTa5.

- If G_2 is OBSTx1c, then by (89), q_0 has degree two in G_2. Since neither φ_1 nor φ_2 extends to an L-coloring of G_2, this implies that Q is a 3-chord. Note that there exists an L-coloring φ of the path $p_mv_1 \ldots v_{i+2}$ such that $\varphi(v_{i+2}) \notin L(q_3)$. Let L' be the list assignment such that $L'(v_j) = \{\varphi(v_j)\}$ for $1 \leq j \leq i+1$, $L'(v_{i+2}) = \{\varphi(v_{i+1}), \varphi(v_{i+2})\}$ and $L'(x) = L(x)$ otherwise. The graph $G' = G - v_{i+2}q_3$ is not L'-colorable, thus it contains a subgraph H violating (OBSTb). By (83), if $i = 2$ then G' does not contain OBSTb1 or OBSTb2, hence $v_{i+1}, v_{i+2} \in V(H)$. By (79), we conclude that v_i has degree at least three in H, and by the choice of Q, we have $q_3 \in V(H)$. By (79) and (83), we have $G' = H$. If H is OBSTb3, then G is OBSTx4. Otherwise, G contains a subgraph H' depicted in Figure 10.5. Observe that every L-coloring of $G - V(H')$ extends to an L-coloring of G, contradicting the minimality of G.

- If G_2 is OBSTa2, then let w_1 and w_2 be the neighbors of v_i and v_{i+2}, respectively, that are incident with the triangle T of the configuration. Since neither φ_1 nor φ_2 extends to an L-coloring of G_2, we have $L(w_1) = L(w_2)$. Let φ be a coloring of the path $p_mv_1 \ldots v_{i+2}$ such that $\varphi(v_i) \neq \varphi(v_{i+2})$. Let G' be the graph obtained from $G-(V(T) \cup \{v_{i+1}\})$ by adding the edge v_iv_{i+2}, and L' the list assignment such that $L'(v_j) = \{\varphi(v_j)\}$ for $1 \leq j \leq i+2$ and $L'(x) = L(x)$ otherwise. Note that G' is not L'-colorable. By (79), no (≤ 4)-cycle in G' contains the edge v_iv_{i+2}, thus the minimality of G

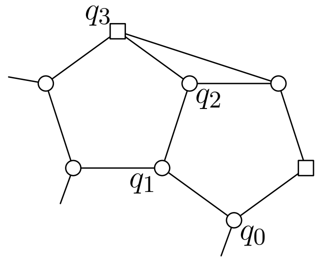


Figure 10.5: A configuration from claim (92).

implies that G' violates (Q) or (OBSTb). If G' violates (Q), then q_3 is adjacent to p_0, and since q_1 has degree at least three, (79) applied to the cycle $p_0p_1 \ldots q_0q_1q_2q_3$ shows that $i = 2$ and q_1 is adjacent to p_1. It follows that G contains OBSTa4. Suppose now that G' contains a subgraph H violating (OBSTb). Observe that v_{i+3} belongs to H; and, the inspection of the graphs in Figure 10.2 shows that v_{i+3} has degree two in H. However, since Q is a 3-chord, $v_{i+3} = q_3$ has degree at least three in G, contradicting (79) or (83).

□

Let k be the index such that $v_k \in X$ and $v_{k+1} \notin X$. We now show that G contains one of several subgraphs near to P; see Figure 10.6 for cases (A4) and (A5).

(93) *One of the following holds:*

(A1) $|X| = 3$ *and* $v_2v_3v_4$ *is a part of the boundary walk of a* 5*-face, or*

(A2) a vertex of X *is incident with a triangle, or*

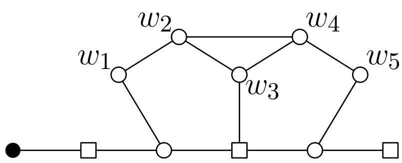
(A3) an edge of the path $p_mv_1v_2 \ldots v_k$ *is incident with a* 4*-face, or*

(A4) $|X| = 3$ *and there exists a path* $w_1w_2w_3w_4w_5$ *in* $G - (X \cup V(P))$ *such that* $w_2w_4, v_2w_1, v_3w_3, v_4w_5 \in E(G)$*, or*

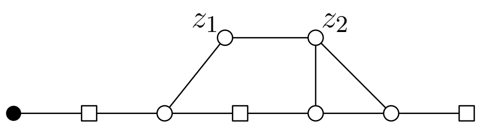
(A5) $|L(v_1)| = |L(v_3)| = |L(v_6)| = 2$ *and there exist adjacent vertices* $z_1, z_2 \in V(G) \setminus (X \cup V(P))$ *such that* $z_1v_2, z_2v_4, z_2v_5 \in E(G)$*.*

Proof. Assume for a contradiction that X satisfies none of these conditions. Since no vertex of X is incident with a triangle, (81) implies that the subgraph R induced by $V(P) \cup \{v_1, \ldots, v_k\}$ is either a path or equal to the cycle C. Observe that there exists an L-coloring ψ of R such that

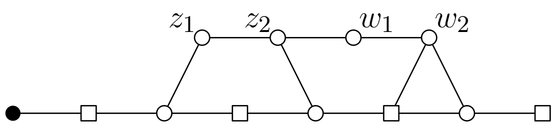
- if $v_1 \in X$, then $\psi(v_1) \notin L(p_m)$, and



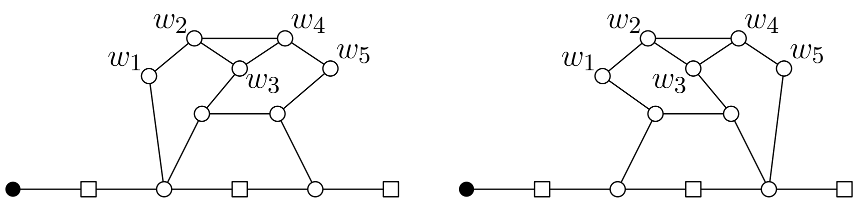
(A4)



(A5)



(B3)



(B4)

Figure 10.6: Configurations from claims (93) and (95)

- if $v_1 \notin X$ and $|L(v_1)| = 2$, then $\psi(v_2)$ is different from the unique color in $L(v_1) \setminus L(p_m)$, and

- if $|L(v_{k+1})| = 2$, then $\psi(v_k) \in L(v_k) \setminus L(v_{k+1})$.

Let $G' = G - X$ and let L' be the list assignment obtained from L by removing the colors of vertices of X from the lists of their neighbors, with the following exception: if $v_1 \notin X$ and $|L(v_1)| = 2$, then $L'(v_1) = L(v_1)$ (note that still, an L'-coloring of G' corresponds to an L-coloring of G, since $\psi(v_2)$ does not belong to $L(v_1) \setminus L(p_m)$). By (81), no neighbor of a vertex of X other than v_1 and v_{k+1} has list of size less than three in L; furthermore, since (A2) and (A3) are false, no vertex of G' has two neighbors in X. It follows that G' satisfies (S2). By (81) and (84), no vertex of $V(G) \setminus V(P)$ has two neighbors in P, thus (Q) holds. Let us now show that (I) holds: otherwise, there would exist adjacent vertices $w_1, w_2 \in V(G')$ such that $|L'(w_1)| = |L'(w_2)| = 2$. We may assume that $|L(w_1)| = 3$, and thus w_1 has a neighbor in X. If $|L(w_2)| = 3$, then w_2 has a neighbor in X as well, and by (79), it follows that (A1), (A2) or (A3) holds. If $|L(w_2)| = 2$ and $w_1 \notin V(C)$, then (92) is contradicted, unless (A2) holds. If $w_1 \in V(C)$, then since (A2) is false, (92) implies that $w_1 \in \{v_1, v_{k+1}\}$. If $w_1 = v_1$, then the chord w_1w_2 contradicts (81), hence $w_1 = v_{k+1}$ and $w_2 = v_{k+2}$. However, the set X was chosen so that if $|L(v_{k+1})| = 3$, then $|L(v_{k+2})| = 3$, which is a contradiction.

Suppose now that (T) is violated, that is, there exists a path $w_1w_2w_3w_4w_5$ in G' such that $|L'(w_1)| = |L'(w_3)| = |L'(w_5)| = 2$ and $w_2w_4 \in E(G)$. If $|L(w_3)| = 2$, then by (T) and symmetry, we may assume that $|L(w_1)| = 3$, and hence w_1 has a neighbor $x \in X$. If $w_1 \notin \{v_1, v_{k+1}\}$, then (92) implies that a subpath of $xw_1w_2w_3$ splits off a face F, and since $|L(w_3)| = 2$, we have $\ell(F) \leq 4$. However, $d(F, w_2w_3w_4) < B$, which is a contradiction. If $w_1 = v_1$, then by (83), a subpath of $w_1w_2w_3$ splits off a triangle T or OBSTx1. However, then (A2) holds. It follows that $w_1 = v_{k+1}$. If $|L(w_5)| = 3$, by symmetry we have $w_5 = v_{k+1} = w_1$, which is a contradiction. Therefore, $|L(w_5)| = 2$ and by (83), $w_3w_4w_5$ is a subpath of C. Since the triangle $w_2w_3w_4$ is outside of the subgraph split off by $w_1w_2w_3$, we also conclude that $w_1w_2w_3 \subset C$, thus $w_j = v_{k+j}$ for $1 \leq j \leq 5$. However, then $k \leq 3$, since both v_{k+1} and v_{k+2} have a list of size three, and $|L(v_{k+5})| = 2$ and v_{k+3} is a vertex of degree two incident with a triangle, contradicting (89). Thus, $|L(w_3)| = 3$ and w_3 has a neighbor $y \in X$. If $|L(w_1)| = |L(w_5)| = 3$, then each of them has a neighbor in X, and thus (A4) holds. Therefore, assume that say $|L(w_1)| = 2$. If $w_3 \notin \{v_1, v_{k+1}\}$, then by (92) a subpath of $yw_3w_2w_1$ splits off a face of length at most four whose distance from $w_2w_3w_4$ is less than B, which is a contradiction. Similarly, (83) shows that $w_1w_2w_3 \subset C$, hence $w_j = v_{k+4-j}$ for $1 \leq j \leq 3$. If $|L(w_5)| = 2$, a symmetrical argument would show that $w_5 = v_{k+3} = w_1$, thus we have $|L(w_5)| = 3$ and w_5 has a neighbor in X. By the choice of X, it follows that (A5) holds.

Therefore, G' satisfies (S1), (S2), (S3), (I), (Q) and (T), and by the minimality of G, we conclude that G' violates (OBSTa) or (OBSTb). Thus G contains a near-obstruction H, and by (86), there exists a vertex $v \in V(H) \setminus V(C)$ such that $|L'(v)| = 2$. By (87), v is not adjacent to an endvertex of P, hence either $m = 2$ and H is OBSTx1, or $m = 5$ and H is OBSTb1 or OBSTb2, with a vertex $p \in \{p_0, p_m\}$ not contained in H. Let v_t be the neighbor of v in X.

Suppose first that $m = 2$. Let $q_0q_1q_2q_3$ be the subpath of the outer face of H, where $q_0q_2 \in E(G)$ and $q_3 = v$. If $p = p_0$, then H is drawn inside the closed disk bounded by $K = p_2p_1vv_tv_{t-1}\ldots v_1$. Then, (79) implies that $t \geq 3$. Since at most one of v_2 and v_3 has degree two, only the vertices q_0, q_1 and q_2 are contained in the open disk bounded by K. Since at most one of v_1 and v_2 has degree two, v_2 is adjacent to a vertex of the triangle $q_0q_1q_2$. Considering the path $Q = p_0p_1vv_t$, as in (85) we conclude that Q splits off a face and p_0 and v_t have a common neighbor with list of size two. However, such a graph G is L-colorable. Hence, suppose that $p = p_2$ and observe that $t = 2$ and v_2 has list of size three. Therefore, v_2 has degree at least three, and $q_1, q_2, q_3 \not\in V(C)$ by (85). It follows that $|L(q_1)| = 3$ and q_1 is adjacent to a vertex $x \in X$. Note that x and p_0 have a common neighbor with list of size two by (85) applied to $xq_1q_0p_0$. But, such a graph G is L-colorable.

Let us now consider the case that $m = 5$. By (88) and symmetry (we will no longer use any properties of the set X), we may assume that $p = p_5$ and v is adjacent to v_2 and p_4. Let K be the cycle bounding the outer face of H and $Q = K - (V(P) \cup \{v_1\}) = q_0q_1\ldots$, where q_0 is adjacent to p_0. By (87), we have $q_0 \in V(C)$. Let $G_1 = G - (V(H) \setminus V(Q))$.

If H is OBSTb1, then note that v_2 has degree at least three, thus by (85) q_0 and v_2 have a common neighbor with list of size two. However, then G contains OBSTb2b. Therefore, H is isomorphic to OBSTb2. There exists an L-coloring φ of H such that $\varphi(q_1) \not\in L(q_0) \setminus L(p_0)$. Let L_1 be the list assignment defined by $L_1(x) = \varphi(x)$ for $x \in V(Q) \setminus \{q_0\}$, $L_1(q_0) = (L(q_0) \setminus L(p_0)) \cup \{\varphi(q_1)\}$ and $L_1(x) = L(x)$ otherwise; G_1 cannot be L_1-colorable. Since a path $Q - q_0$ of length 4 is precolored in G_1 and $d(Q - q_0) \geq d(P) - 3 \geq r(P) - 3 = r(Q - q_0)$, the minimality of G implies that G_1 violates (Q) or (OBSTb). In the former case, as q_2 cannot be a vertex of degree two with a list of size three, (83) implies that G consists of H and a vertex with list of size two adjacent to q_2 and v_2, and it is L-colorable. Similarly, in the latter case, G_1 must be OBSTb2 and G is L-colorable. This is a contradiction. $\square$

Let H be one of the obstructions from Figure 10.1 or 10.2. A set $U \subseteq V(H)$ *has lists determined by the rest of* H if whenever L_1 and L_2 are two list assignments to H such that

- the size of the list of each vertex is given by Figure 10.1 or 10.2,
- $L_1(x) = L_2(x)$ for each $x \not\in U$,

- vertices with list of size one give a proper coloring of the path induced by them, and
- H is neither L_1-colorable nor L_2-colorable,

then $L_1 = L_2$. That is, the list assignment that does not extend to H is uniquely determined once it is known on all the vertices except for those in U. We call H *k-determined* if every subset U of vertices of H of size at most k consisting only of vertices with list of size two has lists determined by the rest of H. A straightforward case analysis shows the following.

(94) *All graphs in Figures 10.1 and 10.2 are* 1*-determined. All except OBSTa2, OBSTx1c, OBSTx2b, OBSTb1, OBSTb1a, OBSTb3, OBSTb5 and OBSTb6 are* 2*-determined.*

Let us now further discuss the subcase (A1) of (93); see Figure 10.6 for cases (B3) and (B4).

(95) *If $|X| = 3$ and $v_2v_3v_4z_2z_1$ is a* 5*-face, then there exists*

(B1) a triangle incident with v_2, v_4, z_1 or z_2, or

(B2) a 4*-face incident with z_1 or z_2, or*

(B3) adjacent vertices w_1, $w_2 \in G - (X \cup \{z_1, z_2\})$ such that $w_1z_2, w_2v_5, w_2v_6 \in E(G)$, and furthermore, $|L(v_7)| = 2$, or

(B4) a path $w_1w_2w_3w_4w_5$ in $G - (X \cup \{z_1, z_2\})$ such that $w_2w_4 \in E(G)$, and either $v_2w_1, z_1w_3, z_2w_5 \in E(G)$ or $z_1w_1, z_2w_3, v_4w_5 \in E(G)$ (possibly with $w_1 = v_1$ in the former case or $w_5 = z_5$ in the latter case).

Proof. Suppose that none of these conditions is satisfied. Since v_2 and v_4 have list of size three, they must have degree at least three in G, and thus (92) implies that $z_1, z_2 \notin V(C)$, unless (B1) holds. Let φ be the coloring of X, $G' = G - X$ and L' the list assignment to G' as chosen in the proof of (93). Note that $|L'(z_1)|, |L'(z_2)| \geq 2$. As in the proof of (93), we conclude that $G' - \{z_1, z_2\}$ is L'-colorable.

There exist at least two L'-colorings φ_1 and φ_2 of the path z_1z_2 such that $\varphi_1(z_1) \neq \varphi_2(z_1)$ and $\varphi_1(z_2) \neq \varphi_2(z_2)$. For $i \in \{1, 2\}$, let L_i be the list assignment obtained from L' by removing the colors of z_1 and z_2 according to φ_i from the lists of their neighbors. Then (92) implies that L_i satisfies (S2), and by (84), (Q) holds as well.

Let G'' be the graph obtained from $G' - \{z_1, z_2\}$ by repeatedly removing the vertices whose degree is less than the size of their list both in L_1 and in L_2. Note that G'' is L_i-colorable if and only if G is L-colorable, for $i \in \{1, 2\}$. Let us argue that (I) is satisfied in G''. Unless (B1) or (B2) holds, (92) implies that no neighbor of z_1 and z_2 other than v_2 and v_4 lies in C, and furthermore, there exists no path wxy, where $w \in \{z_1, z_2\}$, $x \notin \{v_2, v_4, z_1, z_2\}$ and $|L(y)| = 2$. Thus, (I) holds unless there exists a path $wxyv$ with $w \in \{z_1, z_2\}$, $v \in \{v_2, v_4, z_1, z_2\}$ and $x, y \in V(G) \setminus (V(C) \cup \{z_1, z_2\}$. Since (B1) and (B2) are false, we have $w = z_1$ and $v = v_4$ or $w = z_2$ and $v = v_2$. However, then (79) implies that z_1 or z_2 has degree two, which is a contradiction.

Let us now consider the condition (T) for G''. Suppose that there exists a path $w_1w_2w_3w_4w_5$ with $w_2w_4 \in E(G)$ and $|L_i(w_1)| = |L_i(w_3)| = |L_i(w_5)| = 2$ for some $i \in \{1, 2\}$. If $|L(w_3)| = 2$, then by (T) and symmetry, we may assume that $|L(w_1)| = 3$, and thus $w_1 \notin \{v_1, v_5\}$ and by (92), $w_1 \notin V(C)$. Consider the (≤ 5)-chord Q contained in $X \cup \{z_1, z_2, w_1, w_2, w_3, w_4\}$ such that the subgraph F of G that is split off by Q contains neither P nor the triangle $w_2w_3w_4$. We have $d(Q) \geq B - 3 \geq r(Q)$ in F, since the triangle $w_2w_3w_4$ intersects Q. By the minimality of G and the choice of Q, we conclude that F violates (S3), (Q) or (OBSTb) (with the list assignment matching L on $V(F) \setminus V(Q)$ and an L-coloring of the rest of the graph on Q). If F violates (OBSTb), then by (79) and (83), F is isomorphic to one of the graphs in Figure 10.2. Since $|L(w_3)| = 2$, this is only possible if $\ell(Q) = 5$ and $w_5 \in V(F) \setminus V(Q)$. However, note that v_5 has degree two in F and thus it has degree one in $G - X$. It follows that $v_5 \notin V(G'')$, and similarly we conclude that $(V(F) \setminus V(Q)) \cap V(G'') = \emptyset$. This implies that $w_5 \notin V(G'')$, which is a contradiction. If F violates (S3) or (Q), then (79) and (83) imply that Q splits off a face. In particular, we have $v_4 \in V(Q)$. If (S3) fails, then we have that $v_5 = w_3$ and that w_1 is adjacent to z_2. Since w_1 has degree at least three, (79) implies that w_5 is not adjacent to v_2, z_1 or z_2; therefore, $|L(w_5)| = 2$, and by (83) we have $w_5 = v_7$ and G satisfies (B3). If (Q) fails, then note that v_5 has degree one in $G - X$, hence $v_5 \notin V(G'')$ and consequently, $v_5 \neq w_5$. It follows that v_5 is adjacent to w_2, and by (T), we have $|L(w_5)| = 3$. However, by symmetry of the path $w_1w_2w_3w_4w_5$, we conclude that v_5 is also adjacent to w_4, which is a contradiction since $v_5 \neq w_3$.

Suppose now that $|L(w_3)| = 3$ and w_3 has a neighbor in $X \cup \{z_1, z_2\}$. If $|L(w_i)| = 3$ or $w_i \in \{v_1, v_5\}$ holds for each $i \in \{1, 5\}$, then since both z_1 and z_2 have degree at least three, (79) implies that (B4) holds. Therefore, by symmetry we may assume that $|L(w_1)| = 2$ and $w_1 \notin \{v_1, v_5\}$. Again, we consider the (≤ 5)-chord Q contained in $X \cup \{z_1, z_2, w_1, w_2, w_3, w_4\}$ and the subgraph F of G that is split off by Q containing neither P nor the triangle $w_2w_3w_4$. As in the previous paragraph, we conclude that F is a face and violates (S3) or (Q). If $|L(w_5)| = 2$, then by symmetry we can assume that $w_5 \in V(F)$, and thus $w_5 = v_5$. However, in that case $w_5 \notin V(G'')$, which is a contradiction. Therefore, $|L(w_5)| = 3$ and $w_5 \notin V(F)$. Since z_1 has degree at least three, w_5 is adjacent to z_1 by (79). However, v_5 is adjacent to w_2, and the path $v_5w_2w_3w_4w_5$ satisfies (B4).

It follows that G'' satisfies (T). Let us now show that G'' is L_1-colorable or L_2-colorable, thus obtaining an L-coloring of G and a contradiction. Suppose first that neither z_1 nor z_2 have a neighbor in P. Then both L_1 and L_2 satisfy

(S3). We conclude that G'' violates (OBSTa) or (OBSTb). Thus, G contains a (unique) near-obstruction H. The case that $|L_i(v)| = |L'(v)|$ for every $v \in V(G)$ is excluded similarly to (93), thus H has at least one vertex u_1 such that say $|L'(u_1)| = 3$ and $|L_i(u_1)| = 2$. Let K be the outer face of H, and let $q_0q_1 \ldots q_t = K - V(P)$, where q_0 is the neighbor of p_0 (or of p_1, if H is OBSTb1, OBSTb2 or OBSTx1 and $p_0 \notin V(H)$).

The vertex u_1 cannot be adjacent to both z_1 and z_2, thus $L_1(u_1) \neq L_2(u_1)$. Since H is neither L_1-colorable nor L_2-colorable and H is 1-determined by (94), it follows that H contains another vertex u_2 such that $|L'(u_2)| = 3$ and $|L_i(u_2)| = 2$. Suppose that u_1 and u_2 are both adjacent to z_1 or both adjacent to z_2. Since (B1) and (B2) are false, the distance between u_1 and u_2 must be at least three. Furthermore, we may assume that no other vertex between u_1 and u_2 in $K - V(P)$ has list of size two. This is only possible if H is OBSTa1, OBSTa5, OBSTx2a, or OBSTx3. Note that H is not OBSTa1, OBSTa5 or OBSTx3, since OBSTa1 is 2-determined and OBSTa5 and OBSTx3 are 4-determined. Therefore, either H is OBSTx2a or we may assume that u_1 is adjacent to z_1, u_2 is adjacent to z_2, and that $L_i(x) = L'(x)$ for $i \in \{1, 2\}$ and $x \in V(H) \setminus \{u_1, u_2\}$. In the latter case, we conclude that H is not 2-determined. By (94), H is one of OBSTa2, OBSTx1c, OBSTx2b, OBSTb1, OBSTb1a, OBSTb3, OBSTb5 or OBSTb6.

Let us make one more useful observation: suppose that $\ell(P) = 2$, q_0 is adjacent to p_0 and $|L_1(q_0)| = 2$. If $|L'(q_0)| = 3$, then consider the subgraph G_2 of G that is split off by the path $Q = p_0q_0zv$, where $z \in \{z_1, z_2\}$ and $v \in \{v_2, v_4\}$. By the minimality of G, there exists an L-coloring of this path that does not extend to G_2. Since H contains a triangle whose distance to Q is at most 3, we conclude that G_2 violates (Q), and thus v_5 is adjacent to p_0. However, then $\ell(C) \leq 8$, contradicting (82). Therefore, $|L'(q_0)| = 2$, and by (92), if (B1) and (B2) are false, then $|L(q_0)| = 2$. Since u_1 and u_2 exist, in this situation H has at least three vertices with list of size two. This implies that H is neither OBSTa2 not OBSTx1c. It also implies that H is not OBSTx2a, since OBSTx2a is 2-determined.

Let us consider other obstructions separately:

- *H is OBSTx2b:* If p_0 has degree two in H, then by the observation we have $|L(q_0)| = 2$, and thus H is a tame near-obstruction, contradicting (86). Thus, p_0 has degree three in H. Furthermore, (86) implies that $q_5 \notin V(C)$, and thus q_5 is adjacent to z_1 and q_3 is adjacent to z_2. If $|L(q_1)| = 2$, then by (85) applied to (a subpath of) $v_4z_2q_3q_2q_1$, v_5 is adjacent to q_2 (possibly $v_5 = z_1$). However, by (79) and (83) G does not contain any other vertices, and such a graph is L-colorable. Thus, $|L(q_1)| = 3$ and q_1 is adjacent to v_4. By (85) for $p_0q_0q_1v_4$, we conclude that v_5 is adjacent to p_0, contradicting (82).

- *H is OBSTb1 or OBSTb1a:* If $p_0 \in V(H)$, then by (85) for the path $v_4z_2u_2p_0$, we have that v_5 is adjacent to p_0. However, then G contains no other vertices and is L-colorable. Thus, $p_0 \notin V(H)$ and H is OBSTb1. In this case, we similarly conclude that the path $p_0p_1u_2z_2v_4$ splits off a face, OBSTb1 or OBSTb2. In all these cases, G is L-colorable.

- *H is OBSTb3:* This is excluded by (84).

- *H is OBSTb5:* Suppose that $u_2 = q_0$. Then $u_1 = q_2$ and $q_4 = v_1$, and by (85) applied to $v_4z_2q_0p_0$, we conclude that v_5 is adjacent to p_0. However, such a graph is L-colorable. So, $u_2 = q_2$ and $u_1 = q_4$. If $|L(q_0)| = 3$, then q_0 would be adjacent to v_4, contradicting (92). Thus, $|L(q_0)| = 2$. Consider the path $q_0q_1q_2z_2v_4$. By (85), v_5 is adjacent to q_1 (possibly $v_5 = q_0$). However, then G is L-colorable.

- *H is OBSTb6:* Let us note that only one two-element subset of vertices of H with list of size two does not have lists determined by the rest of H—the one consisting of the two rightmost square vertices in the depiction of OBSTb6 in Figure 10.2). So, we may assume that p_3 has degree 4 in H, $u_2 = q_4$ and $u_1 = q_6$, and $|L(q_0)| = 2$. If v_4 is adjacent to q_2, then considering the subgraph split off by the path $q_0q_1q_2v_4$, we conclude that $v_5 = q_2$ and $|L(q_2)| = 2$. If v_4 is not adjacent to q_2, then $|L(q_2)| = 2$ as well. By (85) applied to $q_2q_3q_4z_2v_4$, we have that v_5 is adjacent to q_3. And again, we conclude that G is L-colorable.

Let us now consider the case that z_1 or z_2 is adjacent to a vertex of P. By (92), this vertex must be an internal vertex of P. If exactly one of z_1 and z_2 has a neighbor in P, then by (84) at least one of L_1 and L_2, say L_1, satisfies (S3). It follows that G'' with the list assignment L_1 must violate (OBSTa) or (OBSTb), and contains a near-obstruction H. However, since one of z_1 and z_2 has an internal vertex $p \in P$ as a neighbor, p is a cut-vertex in G'', thus this is only possible if $p \in \{p_1, p_{m-1}\}$ and either $\ell(P) = 2$ and H is OBSTx1, or $\ell(P) = 5$ and H is OBSTb1 or OBSTb2. Suppose that there exists a vertex $v \in V(H)$ adjacent to p such that $|L_1(v)| = 2$. By (81), v is adjacent to v_2, v_4, z_1 or z_2. Since z_1 or z_2 is adjacent to p and neither z_1 nor z_2 is incident with a (≤ 4)-cycle, (79) implies that z_1 or z_2 has degree two. This is a contradiction. It follows that no vertex with list of size two is adjacent to p, hence $\ell(P) = 2$. By (86), the two vertices of H with list of size two are adjacent to z_2 and v_4, respectively. However, then p_0 and v_4 are joined by a 2-chord contradicting (92).

Finally, suppose that both z_1 and z_2 have a neighbor in P. Since neither (B1) nor (B2) holds, the neighbors of z_1 and z_2 are internal vertices of P by (92), and $\ell(P) \geq 4$. Let p_i be the neighbor of z_1 and p_j the neighbor of z_2. Suppose that $i < m-1$ or $j < m-3$. By (79), P contains two adjacent vertices of degree two that are not contained in any (≤ 5)-cycle. In that case, contract these two vertices into one (and change its color so that it is consistent with the colors of its neighbors). The resulting graph is a smaller counterexample to

Theorem 102, which is a contradiction. Therefore, $i = m-1$ and $j = m-3$. Let $Q = p_0p_1 \ldots p_{m-3}z_2v_4$, and let φ be an L-coloring of the subgraph of G induced by $V(P) \cup X \cup \{v_1, z_1, z_2\}$ that exists by the minimality of G. Let $G_3 = G - (V(P) \setminus V(Q)) - \{v_1, v_2, v_3, z_1\}$. Let L_3 be the list coloring such that $L_3(x) = \varphi(x)$ for $x \in V(Q)$ and $L_3(x) = L(x)$ otherwise. The graph G_3 is not L_3-colorable, thus it violates (Q) or contains OBSTb1 or OBSTb2. If G_3 violates (Q), then (92) implies that v_5 is adjacent to p_0 and G contains OBSTb2 or OBSTb2a. If G_3 contains OBSTb1, then G contains OBSTb6. Otherwise, G is L-colorable. $\square$

Let T be the 4-cycle in distance at most one or a triangle in distance at most two from X, which exists by (93) and (95). Since $d(P,T) \leq 4$, we have $\ell(P) = 2$.

Suppose that (A3) happens, i.e., T is a 4-cycle sharing an edge with the path $p_2v_1 \ldots v_k$. Let v_iv_{i+1} be such an edge with i minimal and let φ be an L-coloring of the path $p_2v_1 \ldots v_i$. Let G' be the graph obtained from $G - v_iv_{i+1}$ by adding a vertex v adjacent to v_i and v_{i+1}. Let c be a color that does not appear in the lists of v_i and v_{i+1}. Let L' be a list assignment such that $L'(x) = \{\varphi(x)\}$ for $x \in \{v_1, \ldots, v_i\}$, $L'(v) = \{c\}$ if $|L(v_{i+1})| = 2$ and $L'(v) = \{\varphi(v_i), c\}$ if $|L(v_{i+1})| = 3$, $L'(v_{i+1}) = (L(v_{i+1}) \setminus \{\varphi(v_i)\}) \cup \{c\}$ and $L'(x) = L(x)$ for other vertices $x \in V(G')$. Note that G' is not L'-colorable. Furthermore, by the choice of X, if $k = 4$ then $|L(v_k)| = 4$, hence a path R of length at most 5 is precolored in P. Furthermore, since T contains the edge v_iv_{i+1}, we have $d(R) \geq B - 5 \geq r(R)$. By (81) and (92), R is an induced path and no vertex with list of size two other than v_s, v_{i+1} and v is adjacent to it, and since $\ell(C) \geq 9$, it follows that (S3) and (Q) are satisfied. Since T is a 4-cycle, v cannot be in distance at most one from a triangle in G', thus (T) holds as well. By the minimality of G, we conclude that G' violates (OBSTb); let H be the minimal non-L'-colorable subgraph of G'. We have $\ell(R) \geq 4$, and consequently, $i \geq 1$. If $i = 1$, then we also have $|L'(v)| = 1$, $|L(v_2)| = 2$ and $|L(v_1)| = 3$; let $w = v_1$. If $i \geq 2$, then choose $w \in \{v_1, v_2\}$ such that $|L(w)| = 3$. Such a vertex w has degree at least three in G, and thus it has degree at least three in H (even if w is an endvertex of the precolored path of H, since then w has a neighbor x with list of size two in H, and the edge wx belongs to C by (92)). There exist L-colorings φ_1 and φ_2 of the path $p_2v_1 \ldots v_i$ such that $\varphi_1(w) \neq \varphi_2(w)$; let L'_1 and L'_2 be the corresponding list assignments to G'. Since G' is neither L'_1-colorable nor L'_2-colorable, the inspection of the graphs in Figure 10.2 shows that H is OBSTb1, OBSTb1a, OBSTb1b, OBSTb3 or OBSTb5. Since the edge $v_{i-1}v_i$ is not incident with T, the vertex v_i has degree at least three in G, and hence also in H; therefore, H is OBSTb3 and $|L'(v)| = 1$. However, (79) and (83) imply that $V(G) = V(H) \setminus \{v\}$, contradicting (82). We conclude that (A3) is false.

Now, suppose that (B2) happens. If $v_4 \in V(T)$, then let $Y = \{v_3, v_4\}$. If $v_4 \notin V(T)$ and $z_2 \in V(T)$, then let $Y = \{v_3, v_4, z_2\}$; otherwise let $Y = \{v_3, v_4, z_2, z_1\}$. Note that if $z_1 \in Y$, then z_2 is not incident with a 4-cycle, and since (A3) is false, at most one of z_1 and z_2 has a neighbor in P. Thus, there exists an L-coloring ψ of the subgraph G_0 of G induced by $Y \cup V(P) \cup \{z_1, v_1, v_2\}$ such that $\psi(v_4) \not\in L(v_5)$. Let $G' = G - Y$ and let L' be the list assignment such that $L'(x) = \{\psi(x)\}$ for $x \in \{v_1, v_2\}$, $L'(x) = L(x) \setminus \{\psi(y)\}$ if $x \in V(G') \setminus \{v_1, v_2\}$ has a neighbor $y \in Y$, and $L'(x) = L(x)$ otherwise. The graph G' is not L'-colorable. Since z_2 has degree at least three, (79) and (92) together with the choice of Y imply that G' satisfies (I) and (S2). Obviously, (T) is satisfied as well. Suppose that a vertex v with $|L'(v)| = 2$ has two neighbors in $p_0p_1p_2v_1v_2$. By (81), we have $|L(v)| = 3$, hence v is adjacent to a vertex in Y. Suppose that $v \neq z_1$. Since (A3) is false, v is not adjacent to z_1; but then (79) implies that z_1 has degree two, which is a contradiction. Therefore, $v = z_1$, and since ψ assigns a color to z_1, G' satisfies (Q). Hence, G' violates (OBSTb); let H be the subgraph of G' isomorphic to OBSTb1 or OBSTb2. Note that v_2 is adjacent to a vertex x such that $|L'(x)| = 2$. Since z_1 has degree at least three, (79) implies that $x = z_1$, and thus $Y = \{v_3, v_4, z_2\}$. Furthermore, note that neither z_1 nor z_2 has a neighbor in P, thus there exists an L-coloring ψ' of the subgraph of G_0 such that $\psi'(y) = \psi(y)$ for $y \in \{v_1, v_2, v_3, v_4\}$ and $\psi'(z_2) \neq \psi(z_2)$. Since both OBSTb1 and OBSTb2 are 1-determined, z_2 has a neighbor in H different from z_1. Furthermore, H is not OBSTb2, since OBSTb2 is 2-determined and z_2 cannot have more than two neighbors in H whose list according to L' has size two. However, if H is OBSTb1, then p_0 and v_4 are joined by a 3-chord, and by (85), v_5 is a common neighbor of p_0 and v_4. This contradicts (82).

Therefore, neither (A3) nor (B2) holds and T is a triangle. Let us consider the case that (B4) is true.

- *Suppose first that* $v_2w_1, z_1w_3, z_2w_5 \in E(G)$. Note that v_1 may be equal to w_1. Let $S = L(v_2) \setminus (L(v_1) \setminus L(p_2))$. If $S \not\subseteq L(z_1)$, then let L' be the list assignment such that $L'(v_1) = L(v_1) \setminus L(p_2)$, $L'(v_2) = S \setminus L(z_1)$ and $L'(x) = L(x)$ otherwise. Observe that the graph $G - \{z_1, w_3\}$ is not L'-colorable and that it satisfies the assumptions of Theorem 102 (it satisfies (OBSTb), since v_3 is the only neighbor of v_2 with list of size two and $v_1v_2v_3$ cannot be a subpath of a 5-cycle), contradicting the minimality of G. Thus, $S \subseteq L(z_1)$. If $S \neq L(v_3)$, then choose a color $c \in S \setminus L(v_3)$; let L' be the list assignment obtained from L by removing c from the lists of neighbors of v_2 other than v_1. Note that $G - v_2$ is not L'-colorable, and as in (93), we conclude that $G - v_2$ is a smaller counterexample to Theorem 102, which is a contradiction. Similarly, we exclude the case that a color $c' \in L(v_4) \setminus L(v_5)$ does not belong either to S or to $L(z_2)$. Therefore, there exists a color $c' \in S \cap L(z_2)$. By (79) and (92), z_2 is not adjacent to a vertex of P.

  Suppose that w_1 and w_5 do not have a common neighbor. Let G' be the graph obtained from $G - \{w_3, z_1, v_3\}$ by identifying v_2 with z_2 to a new vertex v. Let L' be the list assignment such that $L'(v_1) = L(v_1) \setminus L(p_2)$, $L'(v) = \{c'\}$, $L'(v_4) = \{c''\}$ for a color $c'' \in L(v_4) \setminus \{c'\}$ such that $L(v_3) \neq$

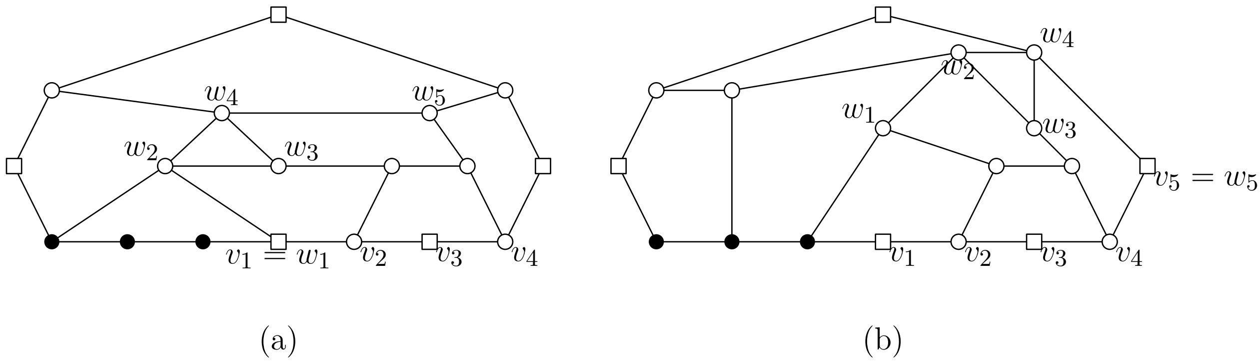


Figure 10.7: Configuration in case that (B4) holds.

$\{c', c''\}$ and $L'(x) = L(x)$ otherwise. Note that $t(G') \geq B$, since both v_2 and z_2 are in distance at least $B - 2$ in G from any (≤ 4)-cycle different from T. Since w_1 and w_5 do not have a common neighbor, (79) implies that v is not contained in any (≤ 4)-cycle in G'. Since G' is not L'-colorable, we conclude that it violates (OBSTb). Let H be the subgraph of G' isomorphic to one of the graphs in Figure 10.2. By (81), p_1 is not adjacent to a vertex with list of size two, hence v_4 belongs to H. Note that v has degree at least three in H, as otherwise G contains a cycle K of length at most 7 such that $v_1v_2v_3v_4 \subset K$ and the open disk bounded by K contains z_1, z_2 and w_3, contradicting (79). The inspection of the graphs in Figure 10.2 shows that v has degree exactly three and that both internal faces incident with v in H have length five. Similarly, (79) implies that $vw_5 \in E(H)$ and $w_1 = v_1$. But then $v_1vw_5w_4w_2$ is the only 5-cycle in G' containing the edge v_1v, thus $v_1w_2 \in E(H)$ and v_1 has degree at least three in H. This is only possible if H is OBSTb4. However, then H is the graph in Figure 10.7(a), which is L-colorable.

So, suppose that w_1 and w_5 have a common neighbor w, and thus by (79), w_2 and w_4 have degree three. By (92), $|L(w)| = 3$. Let ψ be an L-coloring of $p_2v_1v_2v_3v_4z_2$ such that $\psi(v_4) = c'$. Let d be a color in $L(z_1) \setminus \{\psi(v_2), \psi(z_2)\}$. Note that z_2 has no neighbor in P by (79). If $w_1 \neq v_1$, then let d' be a color in $L(w_1) \setminus \{\psi(v_2)\}$ such that $L(w_2) \setminus \{d'\} = L(w_3) \setminus \{d\}$, if such a color exists, and an arbitrary color in $L(w_1)$ otherwise. Among the possible choices of ψ, d and d', we choose them so that the following additional conditions hold:

- If w_1 is adjacent to p_1, then $L(w_1) \neq L(p_2) \cup \{\psi(v_2), d'\}$.
- If $w_1 = v_1$, then either $\psi(v_1) \notin L(w_2)$ or $L(w_2) \setminus \{\psi(v_1)\} \neq L(w_3) \setminus \{d\}$.
- If $w_1 \neq v_1$, w_1 is not adjacent to p_1 and p_1 has a neighbor $z \notin V(C)$, then $L(z) \setminus L(p_1) \neq L(w_5) \setminus \{\psi(z_2)\}$.

Let $G' = G - \{w_2, w_3, w_4, z_1, z_2, v_3, v_4\}$, with the list assignment L' such that $L'(x) = \{\psi(x)\}$ for $x \in \{v_1, v_2\}$, $L'(w_1) = L(w_1) \setminus \{d'\}$ if $w_1 \neq v_1$, $L'(x) = L(x) \setminus \{\psi(y)\}$ for every vertex x with a neighbor $y \in \{v_4, z_2\}$ and $L'(x) = L(x)$ otherwise. The graph G' is not L'-colorable. If w_1 had a common neighbor with v_4 or z_2, then (79) would imply that w has degree two; hence (92) implies that G' satisfies (I). If G' violated (Q), then (79) and (92) would imply that w_1 is adjacent to p_1. But, in that case the choice of ψ, d and d' ensures that (Q) holds. Hence, G' violates (OBSTb) and contains a subgraph H isomorphic to OBSTb1 or OBSTb2. Then v_2 is adjacent to a vertex with list of size two, and by (79), this vertex is w_1; hence, we have $w_1 \neq v_1$. Note that there exists a path w_1xy in H such that y has list of size two. By (92), we have $|L(y)| = 3$, hence y is adjacent to z_2 or v_4. Since w has degree at least three, (79) implies $x = w$ and $y = w_5$. If H were OBSTb1, then w_5 would be adjacent to p_0, and by (85) applied to $v_4z_2w_5p_0$, we would have that v_5 is adjacent to p_0, contradicting (82). It follows that H is OBSTb2. Note that w_1 is not adjacent to p_1, thus the unique neighbor z of p_1 in $V(H) \setminus V(C)$ satisfies $L'(z) \setminus L(p_1) \neq L'(w_5)$. However, then H is L'-colorable, contradicting the assumption that (OBSTb) does not hold.

- *Next, consider the case that* $z_1w_1, z_2w_3, v_4w_5 \in E(G)$. Note that w_5 may be equal to v_5. Similarly to the previous case, we conclude that $L(v_2) \setminus (L(v_1) \setminus L(p_2)) = L(v_3) \subseteq L(v_4)$, that each color $c' \in L(v_4) \setminus L(v_5)$ belongs to both $L(v_3)$ and $L(z_2)$ and that $L(z_1) = L(z_2)$—otherwise, we can color a subset Y of $X \cup \{z_2\}$, remove the colors of the vertices of Y from the lists of their neighbors, and obtain a smaller counterexample to Theorem 102.

  If $L(z_2) \neq L(v_4)$, then let ψ be an L-coloring of $p_2v_1v_2v_3v_4$ such that $\psi(v_4) \notin L(z_2)$. Let G' be the graph obtained from $G - \{v_3, z_2, w_3\}$ by adding the edge v_2v_4. Let c be a color that does not appear in any of the lists and L' the list assignment such that $L'(x) = \{\psi(x)\}$ for $x \in \{v_1, v_2\}$, $L'(v_4) = \{c\}$, $L'(x) = (L(x) \setminus \{\psi(v_4)\}) \cup \{c\}$ for all other vertices x adjacent to v_4, and $L'(x) = L(x)$ otherwise. Note that G' is not L'-colorable. Also, by (79), the edge v_2v_4 is not incident with a (≤ 4)-cycle, and thus $t(G') \geq B$. Furthermore, the distance from v_2 to T in G is three, thus $r(p_0p_1p_2v_1v_2v_4) \geq B - 7 \geq r(5)$. Since v_2 is not incident with a vertex with list of size two and every cycle containing the edge v_2v_4 has length at least seven, G' satisfies (OBSTb) and contradicts the minimality of G.

  Therefore, $L(z_2) = L(v_4)$. If p_1 is adjacent to z_1, then let $G' = G - \{p_2, v_1, v_2, v_3, v_4, z_2, w_3\}$. Let ψ be an L-coloring of the subgraph of G induced by $\{p_1, p_2, v_1, v_2, v_3, v_4, z_1, z_2, w_1, w_2\}$ such that $\psi(v_4) \notin L(v_5)$ and $\psi(w_2) \notin L(w_3) \setminus \{\psi(z_2)\}$. Let L' be the list assignment such that $L'(x) = \{\psi(x)\}$ for $x \in \{z_1, w_1, w_2\}$, $L'(x) = L(x) \setminus \{\psi(v_4)\}$ if x is a neighbor of v_4 and $L'(x) = L(x)$ otherwise. By (79), neither w_1 nor w_2 has a common neighbor with v_4 (since if $w_5 \neq v_5$, then w_5 has degree at least three). By (92), w_1 has no neighbor with list of size two in G', and since w_1 has degree at least three, (79) implies that G' satisfies (Q). Since G' is not L-colorable, by the minimality of G we conclude that G' violates (OBSTb). Because w_1 has degree at least three, (79) implies that G' contains OBSTb2. Let y be the neighbor of w_2 with list of size two and consider the path $Q = v_4w_5w_4w_2y$. If Q is not a subpath of C, then v_4 and w_2 have a common neighbor by (85), implying that w_2 has degree two, which is a contradiction. Therefore, $w_5 = v_5$ and $Q \subset C$. However, then there exists an L-coloring ψ' of the subgraph of G split off by the 3-chord $p_1z_1w_1w_2$ that differs from ψ exactly in the colors of w_1 and w_2, and at least one of ψ and ψ' extends to an L-coloring of G. This is a contradiction.

It follows that $p_1z_1 \notin E(G)$. Suppose now that w_1 and w_5 do not have a common neighbor. Then, let G' be the graph obtained from $G-\{v_3, z_2, w_3\}$ by identifying z_1 with v_4 to a new vertex v, with the list assignment L' such that $L'(v) = L(v_4)\setminus L(v_3)$, $L'(v_1) = L(v_1)\setminus L(p_2)$, $L'(v_2) \subseteq L(v_2)\setminus L'(v_1)$ has size one and $L'(x) = L(x)$ otherwise. Observe that G' satisfies $t(G') \geq B$ and that it is not L'-colorable. Also, since p_1 is not adjacent to z_1, (92) implies that G' satisfies (S3). No vertex with list of size two is adjacent to p_1 or v_2 and the only vertex with list of size two adjacent to v is v_5, thus G' satisfies (Q). We conclude that G' violates (OBSTb); let H be the subgraph of G' isomorphic to one of the graphs drawn in Figure 10.2. By (92), v_2 has degree two in H. If v had degree two, then $v_1v_2v_3v_4$ would be a subpath of a cycle K of length at most seven in G, such that the open disk bounded by K contains z_1, z_2 and w_3. This contradicts (79), hence v has degree three in H and H is OBSTb4. Let x be the common neighbor of p_2 and v in H. By (92), x is adjacent to z_1 in G. In H, there exists a path $xyzv_5$, and by (79) we have $x = w_1$, $y = w_2$, $z = w_4$ and $v_5 = w_5$. Then G is the graph depicted in Figure 10.7(b), which is L-colorable.

Therefore, w_1 and w_5 have a common neighbor w. By (92), $|L(w)| = 3$, and by (79), w_2 and w_4 have degree three. Suppose now that w_1 has no neighbor in P. Then there exists an L-coloring ψ of the subgraph G_0 of G induced by $V(P) \cup \{v_1, v_2, v_3, v_4, z_1, z_2, w_1\}$ such that $\psi(v_4) \notin L(v_5)$ and either $\psi(w_1) \notin L(w_2)$ or $L(w_2) \setminus \{\psi(w_1)\} \neq L(w_3) \setminus \{\psi(z_2)\}$. Let $G' = G-\{v_3, v_4, z_2, w_2, w_3, w_4\}$ with the list assignment L' such that $L'(x) = \{\psi(x)\}$ for $x \in \{v_1, v_2, z_1\}$, $L'(w_1) = \{\psi(z_1), \psi(w_1)\}$, $L'(x) = L(x)\setminus\{\psi(v_4)\}$ if x is a neighbor of v_4 and $L'(x) = L(x)$ otherwise. Note that G' is not L'-colorable. By (79) and (92), G' satisfies (I), and since p_1 is not adjacent to z_1, G' satisfies (S3). Since w_1 has no neighbor in P and v_2 has no neighbor with list of size two, G' also satisfies (Q). We conclude that (OBSTb) is violated and that G' contains one of the graphs depicted in Figure 10.2; let H be such a subgraph. The inspection of such graphs shows that if v_2 has degree three in H, then it is incident with a path v_2xyz with $|L'(z)| = 2$, where $z \neq w_1$. By (79), z is not a neighbor of v_4, hence $|L(z)| = 2$. However, that contradicts (92). Therefore, v_2 has degree two in H. Similarly, we conclude that v_1 has degree two in H, thus H is OBSTb1a, OBSTb1b or OBSTb4. Note that there exists an L-coloring ψ' of G_0 such that ψ' matches ψ on v_1, v_2, v_3 and v_4, either $\psi'(w_1) \notin L(w_2)$ or $L(w_2) \setminus \{\psi'(w_1)\} \neq L(w_3) \setminus \{\psi'(z_2)\}$, and $\psi'(z_1) \neq \psi(z_2)$ (ψ' may or may not differ from ψ on w_1). Note that ψ' does not extend to a coloring of H; that is only possible if H is OBSTb1a and $\psi(w_1) = \psi'(w_1)$. But then there exists a path $v_2z_1xyp_0$ with $|L'(y)| = 2$. By (92), we have $|L(y)| = 3$, thus y is adjacent to v_4. However, then v_4yp_0 is a 2-chord contradicting (92).

Finally, consider the case that w_1 has a neighbor $p_i \in V(P)$. By (79), z_1 has degree three. Observe that there exist colors $c_1 \in L(w_1) \setminus L(p_i)$ and $c_2 \in L(v_2) \setminus (L(v_1) \setminus L(p_2))$ such that $c_1 = c_2$ or $c_1 \notin L(z_1)$ or $c_2 \notin L(z_1)$. Let G' be the graph obtained from $G - \{p_{i+1}, \ldots, p_2, v_1, z_1, z_2, w_2, w_3, w_4\}$ by identifying w_1 with v_2 to a new vertex v. By (79), v is not incident with a (≤ 4)-cycle, thus $t(G') \geq B$ and $d(p_0 \ldots p_iv) \geq B-4 > r(3)$. Let c be a new color that does not appear in any of the lists and L' the list assignment such that $L'(v) = \{c\}$, $L'(v_3) = (L(v_3) \setminus \{c_2\}) \cup \{c\}$, $L'(x) = (L(x) \setminus \{c_1\}) \cup \{c\}$ if x is a neighbor of w_1 and $L'(x) = L(x)$ otherwise. Observe that G' is a counterexample to Theorem 102 smaller than G, which is a contradiction.

Therefore, (B4) is false.

Suppose that (A4) holds. Note that $w_1 \neq v_1$ and $w_5 \neq v_5$, since v_2 and v_4 have list of size three. Suppose first that there exists an L-coloring ψ of the subgraph induced by $V(P) \cup \{v_1, v_2, v_3, v_4, w_1, w_2\}$ such that $\psi(v_4) \notin L(v_5)$ and $|L(w_3) \setminus \{\psi(v_3), \psi(w_2)\}| \geq 2$. Then, let $G' = G - \{v_3, v_4, w_3\}$ with the list assignment L' such that $L'(x) = \{\psi(x)\}$ for $x \in \{v_1, v_2, w_1\}$, $L'(w_2) = \{\psi(w_1), \psi(w_2)\}$, $L'(x) = L(x) \setminus \{\psi(v_4)\}$ if x is a neighbor of v_4 and $L'(x) = L(x)$ otherwise. Note that G' is not L-colorable, and the choice of ψ ensures that (S3) holds. By (79), no neighbor of w_2 is adjacent to v_4, as otherwise w_5 would have degree two; thus, (92) implies that (I) holds. As w_1 has degree at least three, (79) implies that w_2 is not adjacent to a vertex of P and (Q) holds. Therefore, G' violates (OBSTb) and contains a subgraph H isomorphic to one of the graphs drawn in Figure 10.2. No neighbor of v_2 has list of size two, thus w_1 belongs to H. If v_1 or v_2 had degree greater than two in H, then G would contain a (≤ 3)-chord contradicting (83) or (92); hence, H is OBSTb1a, OBSTb1b or OBSTb4. Since w_1 has degree at least three, H is not OBSTb1a. If H were OBSTb1b, then G would contain a (≤ 3)-chord starting in v_2 contradicting (92). Finally, if H is OBSTb4, then let w_2yz be the path in the boundary of the outer face of H with $|L'(z)| = 2$. If z is a neighbor of v_4, then by (79) we have $y = w_4$ and $z = w_5$; however, then there exists a path $v_4w_5y'z'$ in the boundary of the outer face of H with $|L(z')| = 2$, contradicting (92). Otherwise, we have $|L(z)| = 2$. Consider the subgraph split off by $v_3w_3w_4w_2yz$. Since both v_3 and z have list of size two and w_3 and y have no common neighbor, this subgraph satisfies the assumptions of Theorem 102, contradicting the minimality of G.

Suppose now that such a coloring ψ does not exist. (79) and (92) show that can only happen if w_1 is adjacent to p_1. Since w_5 has degree at least three, (85) implies that w_4 has no neighbor in P. Let ψ' be an L-coloring of the subgraph induced by $V(P) \cup \{v_1, v_2, v_3, v_4, w_1, w_2, w_3, w_4\}$ such that $\psi'(v_4) \notin L(v_5)$, $G' = G - \{p_2, v_1, v_2, v_3, w_3\}$, $L'(x) = \{\psi'(x)\}$ for $x \in \{w_1, w_2, w_4\}$, $L'(x) = L(x) \setminus \{\psi'(v_4)\}$ if x is a neighbor of v_4 and $L'(x) = L(x)$ otherwise. By (79) and (92), w_2 is not adjacent to any vertex with list of size two and w_5 is the only neighbor of w_4 with list of size two. Furthermore, w_5 is not adjacent to p_0 by (92), and it is not adjacent to p_1, since (similarly to (85)) we would have that the path $p_0p_1w_5v_4$ splits off a 5-face, implying that v_5 is adjacent to p_0 and contradicting (82). It follows that G' satisfies (Q). Furthermore, G' satisfies (OBSTb), since by (92) it does not contain a path v_4w_5xy with $|L(y)| = 2$. Therefore, G' a counterexample to Theorem 102 smaller than G, which is a contradiction. Therefore, (A4) is false.

Suppose now that (B3) holds. Let ψ be an L-coloring of the subgraph G_0 of G induced by $V(P) \cup \{v_1, v_2, \ldots, v_6, w_2\}$ such that $\psi(v_6) \notin L(v_7)$ (w_2 has no neighbor in P by (79), thus such a coloring exists). Let L' be the list assignment such that $L'(x) = \{\psi(x)\}$ for $x \in \{v_1, v_2, v_3\}$, $L'(v_4) = \{\psi(v_3), \psi(v_4)\}$, $L'(x) = L(x) \setminus \{\psi(y)\}$ if x has a neighbor $y \in \{w_2, v_6\}$ and $L'(x) = L(x)$ otherwise. The graph $G' = G - \{v_5, v_6, w_2\}$ is not L'-colorable, and by (79) and (83), it satisfies (I) and (Q). Furthermore, note that there exists another L-coloring ψ' of G_0 such that $\psi'(v_6) = \psi(v_6)$, $\psi'(w_2) = \psi(w_2)$, $\psi'(v_4) \neq \psi(v_4)$ and $\psi'(v_2) \neq \psi(v_2)$, thus we can choose ψ so that (OBSTb) holds, unless G' contains OBSTb3. By (79) and (92), we then have that z_1 is adjacent to p_1 and w_1 is adjacent to p_0, and by (85) applied to $v_6w_2w_1p_0$, v_7 is adjacent to p_0. Nevertheless, such a graph is L-colorable. Therefore, G' contradicts the minimality of G. It follows that (B3) is false as well, hence

(96) *G satisfies (A2), (A5) or (B1).*

Suppose that there exists a vertex $t \in V(T) \cap (V(P) \cup \{v_1\})$. Let G' be the graph obtained from G by splitting t to two vertices t' and t'' and adding a new vertex v adjacent to t' and t'', so that T becomes a 5-face. Let ψ be an L-coloring of the subgraph of G induced by $V(P) \cup \{t\}$, c a color that does not appear in any of the lists, and let L' be the list assignment such that $L'(t') = L'(t'') = \{\psi(t)\}$, $L(v) = \{c\}$ and $L'(x) = L(x)$ otherwise. Note that G' is not L'-colorable, thus it must violate (OBSTb); let H be the subgraph of G' isomorphic to one of the graphs in Figure 10.2. In H, v has degree two and is incident with a 5-face. If $t \in V(P)$, then H is OBSTb1 or OBSTb2; but then G contains OBSTx1c or OBSTa6. Therefore, $t = v_1$. If H is OBSTb1, then G contains OBSTx1; if H is OBSTb1a, then G contains OBSTx1a; if H is OBSTb1b, then G contains

OBSTx1b; if H is OBSTb2b, then G contains OBSTx4; and if H is OBSTb5, then G contains OBSTx2b. It follows that H is OBSTb4 or OBSTb6. By (79) and (83), we conclude that G is equal to the graph obtained from H by removing v and identifying t' with t''. However, then G is L-colorable. Therefore,

(97) $V(T) \cap (V(P) \cup \{v_1\}) = \emptyset$.

Let X' be the subset of $\{v_s, v_{s-1}, v_{s-2}, v_{s-3}\}$ defined symmetrically to X on the other side of P. By symmetry and the assumption that $t(G) \geq B$, we conclude that T is also incident with a vertex of X' (the case (A2)) or one of the vertices z'_1 or z'_2 incident with the 5-face $v_{s-1}v_{s-2}v_{s-3}z'_2z'_1$ (the cases (B1) and (A5)). Let b be the first vertex in the sequence v_2, v_3, z_1, z_2 and v_4 that is incident with T, and let b' be the first such vertex among v_{s-1}, v_{s-2}, z'_1, z'_2 and v_{s-3}. Note that either $b = b'$ or b and b' are adjacent.

Suppose now that $V(T) \subseteq V(C)$. In this case (A5) does not hold. By (89), we have $b \in \{v_3, v_4\}$ and $b' \in \{v_{s-2}, v_{s-3}\}$. If $b' = v_{s-3}$, then $v_{s-3} \in X'$ and by the choice of X', we have $|L(v_{s-2})| = 2$. This contradicts (89). Thus $b' = v_{s-2}$ and symmetrically, $b = v_3$. By (89), we have $|L(v_2)| = |L(v_{s-1})| = 3$, and by (90), $|L(v_1)| = 2$. However, then $X = \{v_1\}$ and $b \notin X$, which is a contradiction. It follows that

(98) *T shares at most two vertices with C.*

Suppose that $v_{s-3} \in X' \cap V(T)$ and $v_{s-2} \notin V(T)$. The choice of X' implies that $|L(v_{s-3})| = 3$ and $|L(v_{s-2})| = |L(v_{s-4})| = 2$. If $\{v_2, v_3, v_4\} \cap V(T) = \emptyset$, then $b \in \{z_1, z_2\}$; let $v \in \{v_2, v_4\}$ be the neighbor of b. By (92) applied to vbv_{s-3}, we conclude that $T = vbv_{s-3}$, contrary to the assumption that $v \notin V(T)$. It follows that a vertex of $\{v_2, v_3, v_4\} \cap V(T)$ is equal to either v_{s-3} or v_{s-4}. By (82), we have $6 \leq s \leq 8$. If $s = 8$, then $v_4 = v_{s-4}$, which is only possible if both X and X' satisfy (A5). Let $z_1z_2z_3$ be the path such that $T = z_2v_4v_5$, z_1 is adjacent to v_2 and z_3 is adjacent to v_7. Let ψ be an L-coloring of the subgraph of G induced by $V(P) \cup \{v_1, v_2, v_3, v_6, v_7, v_8\}$ such that $\psi(v_3) \notin L(v_4)$ or $\psi(v_6) \notin L(v_5)$ or $L(v_4) \setminus \{\psi(v_3)\} \neq L(v_5) \setminus \{\psi(v_6)\}$. Let $G' = G - \{v_3, v_4, v_5, v_6\}$ with the list assignment L' such that $L'(x) = \{\psi(x)\}$ for $x \in \{v_1, v_8\}$, $L'(v_2) = \{\psi(v_1), \psi(v_2)\}$, $L'(v_7) = \{\psi(v_7), \psi(v_8)\}$ and $L'(x) = L(x)$ otherwise. Note that G' is not L'-colorable, and since v_2 and v_7 are the only vertices with list of size two, it is easy to see that it satisfies the assumptions of Theorem 102. This contradicts the minimality of G.

Therefore, $s \leq 7$. By (81) and (98), C has no chords. If $t \in V(T) \setminus V(C)$ has a neighbor $v \in V(C)$, then vt is an edge of T, as otherwise (79) would imply that v_{s-1} or v_{s-5} (which have lists of size three) has degree two. Note that there exists at most one vertex with two neighbors in the path $p_0p_1p_2v_1v_2$ and another neighbor in T. If such a vertex v exists, then v_{s-4} has degree two by (79), hence $V(T) \cap V(C) = \{v_{s-3}\}$. Therefore, there exists an L-coloring ψ of the subgraph of G induced by $V(P) \cup V(T) \cup \{v_1, v_2, \ldots, v_{s-4}, v\}$ such that $\psi(v_{s-3}) \notin L(v_{s-2})$.

Let $G' = G - V(T)$ and let L' be the list assignment given by $L'(x) = \{\psi(x)\}$ for $x \in \{v_1, v_2, \ldots, v_{s-5}\}$, $L'(v_{s-4}) = \{\psi(v_{s-5}), \psi(v_{s-4})\}$, $L'(x) = L(x) \setminus \{\psi(y)\}$ if x has a neighbor $y \in V(T)$, and $L'(x) = L(x)$ otherwise. Note that G' is not L'-colorable, and by (79) and (92), it satisfies (I). The choice of ψ ensures that (Q) holds as well. Thus, G' must violate (OBSTb), and in particular $s = 7$ and $v_3 \notin V(T)$. Let H be the subgraph of G' isomorphic to OBSTb1 or OBSTb2. By (79), v_s is the only vertex with list of size two adjacent to p_0, thus $v_s \in V(H)$. Let $v_s xy$ be the path in the outer face of H such that $|L'(y)| = 2$. By (79), we have $x = v_{s-1}$ and $y = v_{s-2}$. hence H is OBSTb2. But then there exists a path of length three joining v_2 with v_{s-2} and contradicting (92). Therefore, if $v_{s-3} \in X' \cap V(T)$, then $v_{s-2} \in V(T)$, and in particular, $b' \neq v_{s-3}$. Symmetrically,

(99) *if* $v_4 \in X \cap V(T)$*, then* $v_3 \in V(T)$*,*

and $b \neq v_4$.

If $b \notin \{z_1, z_2\}$ and $b' \notin \{z'_1, z'_2\}$, then since $\ell(C) > 8$, we have $b = v_3$ and $b' = v_{s-2} = v_4$. By symmetry, we may assume that $|L(v_4)| = 3$, and since $v_4 \in X'$, the choice of X' implies that $|L(v_5)| = 2$, $|L(v_6)| = 3$ and $|L(v_3)| = 2$. Consequently, $|L(v_2)| = 3$ and $|L(v_1)| = 2$. Let ψ be a coloring of the subgraph of G induced by $V(P) \cup V(T) \cup \{v_1, v_2\}$ such that $\psi(v_4) \notin L(v_5)$; note that (79) implies that the vertex of $V(T) \setminus V(C)$ is not adjacent to a vertex of P, ensuring that such a coloring exists. Let $G' = G - V(T)$ and let L' be the list assignment such that $L'(v_1) = \{\psi(v_1)\}$, $L'(v_2) = \{\psi(v_1), \psi(v_2)\}$, $L'(x) = L(x) \setminus \{\psi(y)\}$ if x has a neighbor $y \in V(T)$, and $L'(x) = L(x)$ otherwise. The graph G' is not L'-colorable, and by (79) and (92), it satisfies (I) and (Q). This contradicts the minimality of G. Thus, we may assume that say $b \in \{z_1, z_2\}$.

If $b = z_1$, then (92) implies that $b \neq b'$ and $b' \in \{z'_1, z'_2\}$. Let $V(T) = \{b, b', t\}$, let $v' \in \{v_{s-1}, v_{s-3}\}$ be the neighbor of b' and let G_2 be the subgraph split off by $v_2 z_1 b' v'$. If $T \not\subseteq G_2$, then (85) implies that v_2 and v' have a common neighbor with list of size two, hence $v' = v_4 = v_{s-3}$ and $b' = z'_2$. By (79), we have $z'_2 = z_2$. Note that $t \neq z'_1$, since $b' \neq z'_1$. If t has a neighbor in P, then since z'_1 has degree at least three, (79) implies that $tp_0, z'_1 p_1 \in E(G)$. However, such a graph is L-colorable. It follows that t has no neighbor in P. Similarly, z_1 and z_2 have no neighbors in C other than v_2 and v_4 and no neighbor of v_7 is adjacent to a vertex of T. There exists an L-coloring of the subgraph of G induced by $V(P) \cup V(T) \cup \{v_1, v_2, v_3\}$ such that $|L(v_4) \setminus \{\psi(v_3), \psi(z_2)\}| \geq 2$. Let $G' = G - (V(T) \cup \{v_3, v_4, v_5\})$ with the list assignment L' such that $L'(v_1) = \{\psi(v_1)\}$, $L'(v_2) = \{\psi(v_1), \psi(v_2)\}$, $L'(x) = L(x) \setminus \{\psi(y)\}$ if x has a neighbor $y \in V(T)$, and $L'(x) = L(x)$ otherwise. Observe that G' satisfies the assumptions of Theorem 102 and is not L'-colorable, contradicting the minimality of G.

Let us now consider the case that $T \subseteq G_2$. Since t has degree at least three, we conclude that the subgraph of G split off by the path $v_2 z_1 t b' v'$ is OBSTb1, $t = z_2$ and either $z'_2 = z_2$, $b' = z'_1$ and $s = 7$, or $b' = z'_2$ and $s = 9$. Suppose that b or b' has a neighbor in P. If $s = 7$, then the resulting graph would be L-colorable.

If $s = 9$, then (79) implies that z'_1 has degree two. This is a contradiction, hence neither b nor b' has a neighbor in P. Let ψ be an L-coloring of the subgraph of G induced by $V(P) \cup V(T) \cup \{v_1, v_2, v_3\}$ such that $|L(v_4) \setminus \{\psi(v_3), \psi(t)\}| \geq 2$. Let $G' = G - \{v_3, v_4, v_5, t\}$ if $s = 7$ and $G' = G - \{v_3, v_4, v_5, v_6, v_7, t\}$ if $s = 7$, with the list assignment L' such that $L'(x) = \{\psi(x)\}$ if $x \in \{v_1, v_2, z_1\}$, $L'(b') = \{\psi(b'), \psi(z_1)\}$ and $L'(x) = L(x)$ otherwise. Note that G' is not L'-colorable, thus it violates (OBSTb). Since b' and v_s are the only vertices with list of size two, G' contains OBSTb1a, OBSTb1b or OBSTb3 as a subgraph; and if $s = 9$, (79) implies that z'_1 belongs to this subgraph. However, in all the cases the resulting graph is L-colorable, which is a contradiction.

Therefore, we have $b = z_2$. Suppose that $b' \in V(C)$. If $b' = v_4$, then (99) implies that $v_4 \notin X$, thus (A5) holds and $v_5 \in V(T)$. This is a contradiction, as we would choose $b = v_5$. Therefore, $b' \neq v_4$, and (92) implies that the 2-chord v_4bb' splits off T, thus $b' = v_5$. Since $v_3 \notin V(T)$, we have $v_4 \notin X$ and (A5) holds by (99). However, since $|L(v_4)| = |L(v_5)| = 3$, we have $v_5 \notin X'$, and since $b' \in X'$, this is a contradiction.

Finally, consider the case that $b' \notin V(C)$. Note that $b' \neq z'_1$, since we already excluded the symmetric case $b = z_1$, hence $b' = z'_2$. Suppose first that $b = b'$. By (92), we have $v_{s-3} \in \{v_4, v_5\}$. If $v_{s-3} = v_4$, then let $V(T) = \{b, t, t'\}$, and note that $\{t, t'\} \cap \{z_1, z'_1\} = \emptyset$, by the choice of b and b'. Since z_1 and z'_1 have degree at least three, (79) implies that the vertices of T have no neighbors in P, and that the distance between T and $\{v_1, v_7\}$ is at least three. There exists an L-coloring ψ of the subgraph of G induced by $V(P) \cup V(T) \cup \{v_1, v_2, v_3\}$ such that $|L(v_4) \setminus \{\psi(v_3), \psi(b)\}| \geq 2$. Let $G' = G - \{v_3, v_4, v_5, b, t, t'\}$ and L' the list assignment such that $L'(v_1) = \{\psi(v_1)\}$, $L'(v_2) = \{\psi(v_2)\}$, $L'(x) = L(x) \setminus \{\psi(y)\}$ if x has a neighbor $y \in V(T)$ and $L'(x) = L(x)$ otherwise. Observe that G' is not L'-colorable and satisfies (I). Since z_1 has degree at least three, (79) implies that G' satisfies (Q). It follows that G' contains a subgraph H isomorphic to OBSTb1 or OBSTb2. By (79), we have $z_1, v_7 \in V(H)$. If H is OBSTb1, then C has a 3-chord $v_2z_1xv_7$ contradicting (92). If H is OBSTb2, then G contains a path $v_2z_1xyzv_7$, where y has a neighbor in T. However, then t or t' has degree two by (79), which is a contradiction.

If $v_{s-3} = v_5$, then both X and X' satisfy (A5). By (92), we have $z_1 \neq z'_1$. Since both z_1 and z'_1 have degree at least three, (79) implies that b has no neighbor in P and is in distance at least three from $\{v_1, v_7\}$. Let ψ be an L-coloring of the subgraph of G induced by $V(P) \cup V(T) \cup \{v_1, v_2, v_3\}$ such that $\psi(v_5) \notin L(v_6)$. Let $G' = G - \{v_3, v_4, v_5, v_6, b\}$ and L' the list assignment such that $L'(v_1) = \{\psi(v_1)\}$, $L'(v_2) = \{\psi(v_2)\}$, $L'(x) = L(x) \setminus \{\psi(y)\}$ if x has a neighbor $y \in V(T)$ and $L'(x) = L(x)$ otherwise. Observe that G' is not L'-colorable and satisfies (I) and (Q). By the minimality of G, G' contains a subgraph H isomorphic to OBSTb1 or OBSTb2. The distance between the neighbors of b is at least three, thus at most one of them belongs to H and has list of size two. It follows that H is OBSTb1 and $v_7 \in V(H)$. However, then z_1 or z'_1 has degree two by (79), which is a contradiction.

We conclude that $b \neq b'$. Since T has two vertices that do not belong to C, neither X nor X' satisfies (A5). Since $v_3 \not\in V(T)$, by (99) we have $v_4 \not\in V(T)$, and symmetrically, $v_{s-3} \not\in V(T)$; thus, $v_{s-3} \neq v_4$. Let $\{t\} = V(T) \setminus \{b, b'\}$. Consider the 3-chord $Q = v_4bb'v_{s-3}$ and the subgraph G_2 split off by it. If $T \not\subset G_2$, then (85) implies that v_4 and v_{s-3} have a common neighbor, and thus $s = 9$. If $T \subset G_2$, then we similarly conclude that $v_4btb'v_{s-3}$ splits off OBSTb1, i.e., $s = 11$ and t is adjacent to v_6.

Let $S_1 = L(v_2) \setminus (L(v_1) \setminus L(p_2))$ and $S_2 = L(v_{s-1}) \setminus (L(v_s) \setminus L(p_0))$. By the minimality of G, we have $|S_1| = |S_2| = 2$, as otherwise we can remove the edge v_1v_2 or $v_{s-1}v_s$. Suppose now that there exists an L-coloring ψ of T such that for every $c_1 \in S_1$ and $c_2 \in S_2$, there exists an L-coloring φ of the subgraph of G induced by $V(T) \cup \{v_2, v_3, \ldots, v_{s-1}\}$ such that $\varphi(v_2) = c_1$, $\varphi(v_{s-1}) = c_2$ and $\varphi(x) = \psi(x)$ for $x \in V(T)$. Let $G' = G - (V(T) \cup \{v_3, v_4, \ldots, v_{s-2}\})$ and let L' be the list assignment such that $L'(x) = L(x) \setminus \{\psi(y)\}$ if x has a neighbor y in $V(T)$ and $L'(x) = L(x)$ otherwise. The choice of ψ implies that every L'-coloring of G' corresponds to an L-coloring of G, thus G' is not L'-colorable. Note that no vertex of T is adjacent to a vertex of P and that the distance between T and $\{v_1, v_s\}$ is at least three, since otherwise (79) would imply that z_1 or z_1' has degree two. Thus, G' satisfies (S3) and (I). Furthermore, it satisfies (OBSTa), since otherwise a triangle of G' would be in distance at most 7 from T, contradicting $t(G) \geq B$. Therefore, G' would be a smaller counterexample to Theorem 102, which is a contradiction.

We conclude that no such L-coloring ψ exists. In particular, for any color $c \in L(b)$, the list $L(v_4) \setminus \{c\}$ has size two and intersects $L(v_3)$. It follows that $L(v_3) \subseteq L(v_4) = L(b)$, and symmetrically, $L(v_{s-2}) \subseteq L(v_{s-3}) = L(b')$. Similarly, we conclude that $L(v_3) = S_1$, $L(v_{s-2}) = S_2$, $L(v_5) \subseteq L(v_4)$, $L(v_{s-4}) \subseteq L(v_{s-3})$, and if $s = 11$, then $L(v_5), L(v_7) \subseteq L(v_6) = L(t)$. If $L(v_3) = L(v_5) = S_1$, then choose $\psi(b) \in S_1$ arbitrarily. Now, regardless of the values of c_1, c_2 and the rest of ψ, we can choose the color of v_4 to be the unique color in $L(v_4) \setminus S_1$, and the L-coloring φ will exist. Therefore, $L(v_5) \neq S_1$ and $L(v_{s-4}) \neq S_2$. Similarly, if $s = 11$, then $L(v_5) \neq L(v_7)$. Let $\{c_3\} = L(v_5) \cap S_1$. Let $\psi(b)$ be the unique color in $S_1 \setminus L(v_5)$. Furthermore, if $s = 11$ then let $\psi(t) = c_3$, and if $s = 9$ then let $\psi(b') = c_3$. Observe that ψ (extended to the third vertex of T arbitrarily) has the required property—if $c_1 \neq \psi(b)$, then we can color v_3 by $\psi(b)$, so that two neighbors of v_4 have the same color. And if $c_1 = \psi(b)$, then we can color v_3 by c_3, v_4 by the color in $L(v_4) \setminus S_1$ and v_5 with c_3, so that v_6 has two neighbors with the same color. This contradiction finishes the proof of Theorem 102. $\square$

10.2 Concluding remarks

The proof of Theorem 102 follows the lines of the original Thomassen's proof [69]. However, a basically unavoidable part of the proof—the need to handle 2-chords, so that we can color and remove a 5-face in (95)—forces us to deal with a large number of counterexamples to the claim "every precoloring of a path of length two can be extended." Especially painful is the obstruction OBSTx1, which even applies to a path of length one. One could ask whether we could not avoid this by forbidding vertices with list of size two in triangles completely. This cuts down the number of obstructions significantly, and indeed, this was our original aim. However, at the final stage of the proof, we would only end up knowing that there is a triangle whose distance is at most two from a vertex on each side of the precolored path P. This is a quite small amount of structure to work with, making the arising case analysis extremely difficult. Additionally, one runs into trouble if these two vertices are in fact identical, which would essentially force extending Corollary 103 to precolored cycles of length at most 10. The number of obstacles for such cycles then becomes rather large, and it is not quite clear how such an extension of Corollary 103 could be proved.

Another point where one could hope to save on obstructions is by only considering the precoloring of a path of length at most 4 in case that (≤ 4)-cycles are far enough from it. However, there are many places throughout the proof where it is useful to extend the coloring of a path of length two to a coloring of a path of length five, and it is unclear how to handle these situations using only paths of length four.

Consequently, we end up with a nontrivial number of obstructions, and the proof becomes rather technical. Despite the length of this paper, still a large amount of work is hidden in the need to carefully verify all the claims; in particular, we in general do not give detailed proofs of colorability of the described graphs. We feel that doubling the length of the paper by spelling out all these technical details would not make the exposition any clearer or more believable. Similar remarks apply to other results proved using this technique (even the original paper of Thomassen [24], although written quite briefly, becomes rather long when all details are worked out). Given the rather repetitive nature of the arguments, one wonders whether it would not be possible to apply computer to obtain such proofs. Let us however note that many of the reductions appearing in our proof are quite tricky and it is not immediately obvious how they could be obtained mechanically.

On the positive side, Theorem 102 is somewhat interesting even for graphs of girth five, since it describes which precolorings of a path of length at most five can be extended. This might be useful as a technical tool in further study of 4-critical graphs of girth five. Similarly, Theorem 102 and Corollary 103 give interesting information regarding graphs with exactly one cycle of length at most four.

Compared with the solution to Havel's problem [24], our proof is rather elementary, not using any deeper results. Would it be possible to apply the techniques of [24] instead? Possibly, but it would require developing a new proof of 3-list-colorability of planar graphs of girth 5 based on reducible configurations and discharging. While our initial inquiry in that direction was somewhat encouraging, it seems inevitable that the set of reducible configurations needed would be rather large (possibly hundreds as opposed to about 10 needed in [24] for the case of 3-coloring), so the proof would become of somewhat dubious value.

Finally, let us remark that we could require a much weaker assumption on the distance between 4-cycles, since in most of the arguments only triangles cause problems. However, for obvious reasons we did not want to complicate the proof any more.

Bibliography

Bibliography

[1] Aksenov, V. A., Borodin, O. V., and Glebov, A. N. Extending 3-colorability from a 7-face to a planar triangle-free graph. *Sib Elektron Mat Izv 1* (2004), 117–128. in Russian.

[2] Aksionov, V. A. On continuation of 3-colouring of planar graphs. *Diskret. Anal. Novosibirsk 26* (1974), 3–19. in Russian.

[3] Aksionov, V. A., and Mel'nikov, L. S. Some counterexamples associated with the Three Color Problem. *J. Combin. Theory Ser. B 28* (1980), 1–9.

[4] Albertson, M., and Hutchinsonn, J. The three excluded cases of Dirac's map-color theorem. *Annals of the New York Academy of Sciences 319* (1979), 7–17.

[5] Albertson, M. O. You can't paint yourself into a corner. *J. Combin. Theory Ser. B 73*, 2 (1998), 189–194.

[6] Appel, K., and Haken, W. Every planar map is four colorable, Part I: discharging. *Illinois J. of Math. 21* (1977), 429–490.

[7] Appel, K., Haken, W., and Koch, J. Every planar map is four colorable, Part II: reducibility. *Illinois J. of Math. 21* (1977), 491–567.

[8] Axenovich, M., Hutchinson, J. P., and Lastrina, M. A. List precoloring extension in planar graphs. *Discrete Mathematics 311*, 12 (2011), 1046–1056.

[9] Ballantine, J. P. A postulational introduction to the four color problem. *Publ. in Math., Univ. of Washington, Seattle* (1930), 1–16.

[10] Barát, J., and Tóth, G. Towards the Albertson Conjecture. *Electronic Journal of Combinatorics 17* (2010), R73.

[11] Borodin, O. V. A new proof of grünbaum's 3 color theorem. *Discrete Mathematics 169* (1997), 177–183.

[12] Borodin, O. V., Glebov, A. N., Montassier, M., and Raspaud, A. Planar graphs without 5- and 7-cycles and without adjacent triangles are 3-colorable. *Journal of Combinatorial Theory, Series B 99* (2009), 668–673.

[13] Borodin, O. V., Glebov, A. N., Raspaud, A., and Salavatipour, M. R. Planar graphs without cycles of length from 4 to 7 are 3-colorable. *Journal of Combinatorial Theory, Series B 93* (2005), 303–311.

[14] Campos, V., and Havet, F. 5-choosability of graphs with 2 crossings. *Research Report RR-7618, INRIA 05* (2011).

[15] Chen, M., Raspaud, A., and Wang, W. Three-coloring planar graphs without short cycles. *Information Processing Letters 101* (2007), 134–138.

[16] Chen, M., and Wang, W. Planar graphs without 4,6,8-cycles are 3-colorable. *Science in China Series A: Mathematics 50* (2007), 1552–1562.

[17] Chen, M., and Wang, W. On 3-colorable planar graphs without short cycles. *Appl. Math. Letter 21* (2008), 961–965.

[18] Chenette, N., Postle, L., Streib, N., Thomas, R., and Yerger, C. Five-coloring graphs in the klein bottle. submitted.

[19] Diestel, R. *Graph Theory*, third ed., vol. 173 of *Graduate Texts in Mathematics.* Springer-Verlag, Heidelberg, 2005.

[20] Dirac, G. A. The structure of k-chromatic graphs. *Fund. Math. 40* (1953), 42–55.

[21] Dvořák, Z. 3-choosability of planar graphs with (≤ 4)-cycles far apart. submitted.

[22] Dvořák, Z., and Kawarabayashi, K. Choosability of planar graphs of girth 5. manuscript.

[23] Dvořák, Z., Kawarabayashi, K., and Thomas, R. Three-coloring triangle-free planar graphs in linear time. *ACM Transactions on Algorithms 7* (2011), article no. 41.

[24] Dvořák, Z., Kráľ, D., and Thomas, R. Coloring planar graphs with triangles far apart. manuscript.

[25] Dvořák, Z., Kráľ, D., and Thomas, R. Three-coloring triangle-free graphs on surfaces I. Extending a coloring to a disk with one triangle. submitted.

[26] Dvořák, Z., Kráľ, D., and Thomas, R. Three-coloring triangle-free graphs on surfaces II. 4-critical graphs in a disk. manuscript.

[27] Dvořák, Z., Kráľ, D., and Thomas, R. Three-coloring triangle-free graphs on surfaces III. graphs of girth five. manuscript.

[28] Dvořák, Z., Kráľ, D., and Thomas, R. Coloring triangle-free graphs on surfaces. In *Proceedings of the twentieth Annual ACM-SIAM Symposium on Discrete Algorithms* (Philadelphia, PA, USA, 2009), SODA '09, Society for Industrial and Applied Mathematics, pp. 120–129.

[29] Dvořák, Z., and Lidický, B. 4-critical graphs on surfaces without contractible (≤ 4)-cycles. manuscript.

[30] Dvořák, Z., Lidický, B., and Mohar, B. 5-choosability of graphs with crossings far apart. submitted.

[31] Dvořák, Z., Lidický, B., Mohar, B., and Postle, L. 5-list-coloring planar graphs with distant precolored vertices. *in preparation* (2012).

[32] Dvořák, Z., Lidický, B., and Škrekovski, R. 3-choosability of triangle-free planar graphs with constraints on 4-cycles. *SIAM Journal on Discrete Mathematics 24* (2010), 934–945.

[33] Dvořák, Z., Lidický, B., and Škrekovski, R. Graphs with two crossings are 5-choosable. *SIAM J. Discrete Math. 25* (2011), 1746–1753.

[34] Eppstein, D. Diameter and treewidth in minor-closed graph families. *Algorithmica 27* (2000), 275–291.

[35] Erdős, P., Rubin, A. L., and Taylor, H. Choosability in graphs. *Congr. Numer. 26* (1980), 125–157.

[36] Erman, R., Havet, F., Lidický, B., and Pangrác, O. 5-colouring graphs with 4 crossings. *SIAM Journal on Discrete Mathematics 25* (2011), 401–422.

[37] Fisk, S. The nonexistence of colorings. *J. Combinatorial Theory Ser. B 24* (1978), 247–248.

[38] Franklin, P. A Six Colour Problem. *J. Math. Phys. 13* (1934), 363–369.

[39] Gallai, T. Kritische Graphen I. *Publ. Math. Inst. Hungar. Acad. Sci. 8* (1963), 265–292.

[40] Garey, M., and Johnson, D. *Computers and Intractability: A Guide to the Theory of NP-completeness.* WH Freeman & Co. New York, NY, USA, 1979.

[41] Gimbel, J., and Thomassen, C. Coloring graphs with fixed genus and girth. *Trans. Amer. Math. Soc. 349* (1997), 4555–4564.

[42] Gonthier, G. Formal proof–the four-color theorem. *Notices of the AMS 55*, 11 (2008), 1382–1393.

[43] Grötzsch, H. Ein Dreifarbenzatz für Dreikreisfreie Netze auf der Kugel. *Math.-Natur. Reihe 8* (1959), 109–120.

[44] Grünbaum, B. Grötzsch's theorem on 3-colorings. *Michigan Math. J. 10* (1963), 303–310.

[45] Kawarabayashi, K., Kral, D., Kynčl, J., and Lidický, B. 6-critical graphs on the klein bottle. *SIAM J. Discrete Math. 23* (2008), 372–383.

[46] Kawarabayashi, K.-i., and Mohar, B. List-color-critical graphs on a fixed surface. In *Proceedings of the twentieth Annual ACM-SIAM Symposium on Discrete Algorithms* (Philadelphia, PA, USA, 2009), SODA '09, Society for Industrial and Applied Mathematics, pp. 1156–1165.

[47] Kostochka, A. Color-critical graphs and hypergraphs with few edges: a survey. In *More sets, graphs and numbers.* Springer, 2006, pp. 175–197.

[48] Kowalik, L. Fast 3-coloring triangle-free planar graphs. In *ESA* (2004), S. Albers and T. Radzik, Eds., vol. 3221 of *Lecture Notes in Computer Science*, Springer, pp. 436–447.

[49] Kowalik, L., and Kurowski, M. Oracles for bounded length shortest paths in planar graphs. *ACM Trans. Algorithms 2* (2006), 335–363.

[50] Král', D., and Stacho, L. Coloring plane graphs with independent crossings. *Journal of Graph Theory 64* (2010), 184–205.

[51] Lam, P. C. B., Shiu, W. C., and Song, Z. M. The 3-choosability of plane graphs of girth 4. *Discrete Math. 294* (2005), 297–301.

[52] Li, X. On 3-choosable planar graphs of girth at least 4. *Discrete Math. 309* (2009), 2424–2431.

[53] Lidický, B. On 3-choosability of plane graphs without 6-, 7- and 8-cycles. *Australasian J. Combin. 44* (2009), 77–86.

[54] Mohar, B. 7-critical graphs of bounded genus. *Discrete Mathematics 112* (1993), 279–281.

[55] Mohar, B., and Thomassen, C. *Graphs on Surfaces.* The Johns Hopkins University Press, Baltimore and London, 2001.

[56] Montassier, M. The 3-color problem. `https://www.labri.fr/perso/montassi/pmwiki.php?n=Site.ThreeColorProblem`.

[57] Postle, L., and Thomas, R. A Linear Upper Bound for 6-Critical Graphs on Surfaces. manuscript.

[58] Ringel, G., and Youngs, J. W. T. Solution of the Heawood map-coloring problem. *Proc. Nat. Acad. Sci. U.S.A. 60* (1968), 438–445.

[59] Robertson, N., Sanders, D. P., Seymour, P., and Thomas, R. The four colour theorem. *J. Combin. Theory Ser. B. 70* (1997), 2–44.

[60] Shih, W.-K., and Hsu, W.-L. A new planarity test. *Theoret. Comp. Sci. 223* (1999), 179–191.

[61] Steinberg, R. The state of the three color problem. Quo Vadis, Graph Theory? *Ann. Discrete Math. 55* (1993), 211–248.

[62] Steinberger, J. P. An unavoidable set of D-reducible configurations. *Trans. Amer. Math. Soc 362* (2010), 6633–6661.

[63] Thomas, R., and Walls, B. Three-coloring Klein bottle graphs of girth five. *J. Combin. Theory Ser. B 92* (2004), 115–135.

[64] Thomassen, C. Five-Coloring Maps on Surfaces. *Journal of Combinatorial Theory, Series B 59* (1993), 89 – 105.

[65] Thomassen, C. Every planar graph is 5-choosable. *J. Combin. Theory Ser. B 62* (1994), 180–181.

[66] Thomassen, C. Five-coloring graphs on the torus. *J. Combin. Theory Ser. B 62* (1994), 11–33.

[67] Thomassen, C. Grötzsch's 3-color theorem and its counterparts for the torus and the projective plane. *J. Combin. Theory Ser. B 62* (1994), 268–279.

[68] Thomassen, C. Grötzsch's 3-color theorem and its counterparts for the torus and the projective plane. *J. Combin. Theory Ser. B 62* (1994), 268–279.

[69] Thomassen, C. 3-list-coloring planar graphs of girth 5. *J. Combin. Theory Ser. B 64* (1995), 101–107.

[70] Thomassen, C. Color-critical graphs on a fixed surface. *J. Combin. Theory Ser. B 70* (1997), 67–100.

[71] Thomassen, C. The chromatic number of a graph of girth 5 on a fixed surface. *J. Combin. Theory Ser. B 87* (2003), 38–71.

[72] Thomassen, C. A short list color proof of Grotzsch's theorem. *J. Combin. Theory Ser. B 88* (2003), 189–192.

[73] Thomassen, C. Exponentially many 5-list-colorings of planar graphs. *J. Comb. Theory, Ser. B 97* (2007), 571–583.

[74] Vizing, V. G. Vertex colorings with given colors (in russian). *Metody Diskret. Analiz, Novosibirsk 29* (1976), 3–10.

[75] Voigt, M. List colourings of planar graphs. *Discrete Math. 120* (1993), 215–219.

[76] Voigt, M. A not 3-choosable planar graph without 3-cycles. *Discrete Math. 146* (1995), 325–328.

[77] Walls, B. *Coloring girth restricted graphs on surfaces.* PhD thesis, Georgia Institute of Technology, 1999.

[78] Williamson, S. G. Depth-first search and kuratowski subgraphs. *J. Assoc. Comput. Mach. 31* (1984), 681–693.

[79] Zhang, H. On 3-choosability of plane graphs without 5-, 8- and 9-cycles. *J. Lanzhou Univ., Nat. Sci. 41* (2005), 93–97.

[80] Zhang, H., and Xu, B. On 3-choosability of plane graphs without 6-, 7- and 9-cycles. *Appl. Math. J. Chinese Univ., Ser. B (Engl. Ed.) 19* (2004), 109–115.

[81] Zhang, H., Xu, B., and Sun, Z. Every plane graph with girth at least 4 without 8- and 9-circuits is 3-choosable. *Ars Comb. 80* (2006), 247–257.

[82] Zhu, X., Lianying, M., and Wang, C. On 3-choosability of plane graphs without 3-, 8- and 9-cycles. *Australas. J. Comb. 38* (2007), 249–254.