# Number Theory Seminar

NMAG470

## 10. a 11. dubna

### Minicourse: Sums of squares in fields

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In this course I want to address several aspects of sums of squares and their role in the development of modern algebra. The focus will be on sums of squares in commutative rings and in particular in fields. Some basics of the theory of quadratic forms over fields will be introduced. The course will only assume some general algebraic knowledge on bachelor level.

I Sums of squares in number theory
II Sums of squares in real algebra
III The Pythagoras number and the level of a field

The study of sums of squares is an old topic in number theory and algebra. In Brahmagupta's book an identity is given for writing a product of two sums of two squares again as a sum of two squares. In modern terms it can be obtained by using the norm form for the ring of Gaussian integers, or more generally for the quadratic extension of a given commutative ring by adjoining a square root of -1. Euler gave a similar identity for products of sums of four squares, which can likewise be viewed as the norm form of an algebraic structure, the quaternions. He probably did this with the goal to prove that every positive integer is a sum of four squares, which was one of Fermat's (correct) statements. This way the problem had been reduced to showing the claim for prime numbers, and on this basis the proof was later completed by Lagrange.

Later Degen and Cayley found independently a similar identity for sums of eight squares and a related algebraic structure, the octonions. On the other hand, attempts to find such an identity for sums of 16 squares failed and in the end Hurwitz showed that such an identity cannot exist. More generally, the problem is to express a product of two quadratic forms (homogeneous polynomials of degree two) as a quadratic form applied to bilinear expressions in the variables. If this is possible then a solution to this problem is called a composition formula. Using Clifford algebras one can explain why composition formulae do not exist for certain triples of forms, just by their dimensions.

Many problems of an analytic flavour can be described by asking whether a certain real valued function takes only non-negative values. An obviously sufficient condition is that the function can be written as a sum of squares of other functions. For rational functions (fractions of polynomials), Hilbert asked in his 17th problem whether the converse is true, namely whether any rational function taking only non-negative values can be written as a sum of squares of rational functions. The positive answer was found by Artin in 1927 and it opened the way to a new research area, called real algebra. Here different versions of the problem can be considered, for example, for a polynomial taking only non-negative values, one can ask whether it can be written as a sum of squares of polynomials. This is generally not true. But also the number of squares in such an expression can be studied. This gives rise to the definition of the Pythagoras number of a commutative ring, the smallest number p such that every sum of squares is equal to a sum of p squares. For a field we may ask whether -1 can be written as a sum of squares in it and if so, how many squares are needed. The smallest such number s, if it exists, is called the level of the field. Pfister's showed that the level of a field where -1 is a sum of squares is always a 2-power and that on the other hand every 2-power is the level of some field. The proofs of these results give a nice insight into the topics of quadratic form theory over fields.

## Dates and times:

**10. 4. 10:40 in K4**
**11. 4. 10:40 in seminar room KA**
**11. 4. 14:00 in seminar room MUUK**

**Web semináře:**
`sites.google.com/site/vitakala/teaching/number-theory-seminar`