

Qualifying Quiz

Following Qualifying Quiz is a set of 7 demanding problems. Try to solve all of them and send your solutions in PDF via e-mail to mathsbeyondlimits@gmail.com. The deadline for submitting solutions is **March 31, 2016**. Don't get upset if you find them difficult as they are meant to be demanding, thought-provoking and getting the best out of you. Also don't hesitate to submit just partial solutions as sometimes they may be very near the completion. At the same time we discourage you from googling solutions to these problems.

- Problem 1

A convex quadrilateral $ABCD$ is given. Let M, N be the midpoints of AB and CD respectively. Denote the intersections of DM and AN by O and CM and BN by P . Prove that the area of $NOMP$ is equal to the sum of areas of BPC and AOD .

- Problem 2

Prove that if p is an odd prime, then for any integer $k \in \{1, 2, \dots, p-2\}$:

$$p \mid \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq p-1} i_1 i_2 \dots i_k$$

- Problem 3

There are four congruent right triangles on the table. You are allowed to cut a triangle by its altitude producing two new right triangles. Show that after finite amount of steps, there will always be at least two congruent triangles on the table.

- Problem 4

Let ABC be a triangle with incircle and circumcircle respectively ω and o . Circle ω touches the side BC at point D . Denote by M the midpoint of arc BC of the circle o not containing A . Line MD intersects o at E . Suppose that I is the center of ω . Prove that $\angle AEI = 90^\circ$.

- Problem 5

Show that an equation $\frac{a^2+b^2+c^2}{3} = abc - 1$ has infinitely many solutions in integer numbers. Can you find them all?

- Problem 6

Burglars while breaking into a house encountered an alarm with n buttons - it is known that the code is of length k . The alarm is off once the code is entered no matter what was entered before. Prove that the burglars can check all the possibilities by pushing buttons $n^k + k - 1$ times.

- Problem 7

Excircles of triangle ABC touch sides CA, AB at points E, F respectively. Denote by J the center of A -excircle of triangle ABC and by A' point antipodal to A on circumcircle of triangle ABC . Prove that lines JA' and EF are perpendicular.