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BAKALÁŘSKÁ PRÁCE

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Minimální *KC* prostory

Katedra teoretické informatiky a matematické logiky

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Prohlašuji, že jsem svou bakalářskou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

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Abstrakt: Prostory, ve kterých je každý kompaktní podprostor uzavřený, se nazývají KC prostory (nepředpokládáme žádné oddělovací axiomy). Zřejmě každý Hausdorffův prostor je KC a každý KC prostor je T_1. Práce odpovídá na otázku, zda-li je každý KC prostor, který nemá ostře slabší KC topologii, už nutně kompaktní. V roce 2002 T. Vidalis dokázal, že každý takový KC prostor je spočetně kompaktní, avšak jeho důkaz obsahuje chybu. Stejný problém úspěšně vyřešili v roce 2007 A. Bella a C. Constantini.

Klíčová slova: kompaktní prostor, spočetně kompaktní prostor, KC prostor, minimalní topologie, oddělující axiomy

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Abstract: Spaces, in which each compact subset is closed are called, KC spaces (we do not require any separation axioms). Obviously every Hausdorff space is KC and every KC space is T_1. This thesis answers the question, whetever every KC space, which has no strictly weaker KC topology, is necessary compact. In the year 2002 T. Vidalis proved that every such space is countably compact, however his proof contains an error. The same problem was affirmatively solved in 2007 by A. Bella and C. Constantini.

Keywords: compact space, countably compact space, KC space, minimal topology, separation axioms

1. Introduction

It is well-know that every Hausdorff compact space is both maximal compact and minimal Hausdorff, but in [5] the author has proven that there is minimal Hausdorff space, which is not compact, as well as maximal compact space, which is not Hausdorff. In the same paper was in fact proven, that maximal compact spaces are KC, but the definition of KC space is not cited.

The space is called KC (sometimes they are called T_B spaces) if every compact subspace is closed. It is easy to see that every Hausdorff space is KC and every KC space is T_1. So it can be viewed as a separation axiom between T_1 and T_2.

In [3] authors have proven that each countable minimal KC space is compact and they have posed a question, whether every minimal KC space is countably compact. T. Vidalis in [1] claims to answer affirmatively this question, but his proof contains an error.

The question posed in [3] was finally answered by A. Bella and C. Constantini in [2], who have proven that each minimal KC space is even compact. Together with results of [5] this gives that the class of minimal KC spaces and the class of maximal compact spaces are identical. In this thesis we give a shorter and simpler proof of this statement.

Related question to this is, whether a KC space have some KC compactification. In [3] authors have proven that a KC space has one-point compactification KC if and only if it is sequential. We'll prove, that any space, which has some KC compactification, has also one-point compactification KC. Contrary similar questions in Hausdorff spaces. If a Hausdorff space has some Hausdorff compactification then it is Tychonoff, and hence it has maximal Hausdorff compactification—Čech-Stone compactification. On the other hand, the one-point compactification does not need to be Hausdorff in that case.

2. Preliminaries

Definition 2.1. Let X is a topological space. It is said to be *compact* if every open cover of X has a finite subcover.

Note that contrary to the usual definition we don't require a compact space to be Hausdorff.

Definition 2.2. A collection $\mathcal{F}$ of subsets of X is called an *ultrafilter* in X if it satisfies:

(i) $\emptyset \notin \mathcal{F}$
(ii) if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$
(iii) if $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$ then $B \in \mathcal{F}$
(iv) if $A \cup B = X$ and $A \cap B = \emptyset$ then $\mathcal{F}$ contains exactly one of the sets A, B.

If $\mathcal{F}$ satisfies only conditions (i)–(iii), it is called a *filter* and if it satisfies only conditions (i) and (ii), it is called a *filter-base*.

An ultrafilter $\mathcal{F}$ is said to be *free* if $\bigcap \mathcal{F} = \emptyset$, i.e. there is no finite set $F \in \mathcal{F}$. An ultrafilter is an *uniform ultrafilter* in X if $|F| = |X|$ for every $F \in \mathcal{F}$.

It is easy to see that for every filter-base $\mathcal{G}$ the system $\mathcal{F}^{\mathcal{G}} = \{A \subseteq X : \exists G \in \mathcal{G}, G \subseteq A\}$ is a filter on X. Ultrafilters are maximal filters with respect to inclusion.

An example of a filter-base is the system $\mathcal{U}$ of all open neighbourhoods of a point x in a topological space X. It is so since for each $U \in \mathcal{U}$ we have $x \in U$ and if both U and V are open neighbourhoods of x, then $U \cap V$ is also one. Similarly, the filter generated by $\mathcal{U}$ is a filter of all (not necessary open) neighbourhoods of x.

Definition 2.3. Let X be a topological space and $\mathcal{F}$ a filter on X. A point $x \in X$ is called a *limit* of $\mathcal{F}$ if every neighbourhood U of x is contained in $\mathcal{F}$. In that case we say that the filter $\mathcal{F}$ *converges* to x.

If $\mathcal{F}$ is an ultrafilter, we can equivalently define that $x \in X$ is a limit of $\mathcal{F}$ if $x \in \bigcap\{\overline{F} : F \in \mathcal{F}\}$. Indeed if x is a limit from the previous definition, then $U \cap F \in \mathcal{F}$, and so it is non-empty for each neighbourhood U of x and each element F of the ultrafilter $\mathcal{F}$. On the other hand, let U be a neighbourhood of x and $x \in \overline{F}$ for each $F \in \mathcal{F}$, which means that $U \cap F$ is non-empty, and hence $U \in \mathcal{F}$ because $\mathcal{F}$ is a maximal filter.

All limits of ultrafilters define the topology on X, i.e. if $x \in \overline{A}$ then there is an ultrafilter $\mathcal{F}$ on X such that $A \in \mathcal{F}$ and it converges to x. Let $\mathcal{U}$ be the filter of all neighbourhoods of x. We have $A \cap U \neq \emptyset$ for every $U \in \mathcal{U}$ because $x \in \overline{A}$, hence $\{U \cap A : U \in \mathcal{U}\}$ is a filter-base and every ultrafilter which is above this filter-base converges to x.

Theorem 2.4. *A topological space X is Hausdorff if and only if every ultrafilter on X has at most one limit.*

Proof. If X is Hausdorff and $x, y \in X$, $x \neq y$, there are $U, V \subseteq X$ neighbourhoods of x and y which are disjoint. If for every ultrafilter $\mathcal{F}$ both x and y are its limits, then $U, V \in \mathcal{F}$, hence $\emptyset = U \cap V \in \mathcal{F}$ and $\mathcal{F}$ is not filter.

On the other hand, let X be space with unique limits of ultrafilters. Suppose for contradiction that there are points $x, y \in X, x \neq y$, with no disjoint neighbourhoods. Then

$$\mathcal{B} = \{U \cap V : U \text{ neighbourhood of } x, Y \text{ neighbourhood of } y\}$$

is a filter-base. Let $\mathcal{F}$ be an ultrafilter such that $\mathcal{B} \subseteq \mathcal{F}$ (it exists by the axiom of choice). Then both x and y are limits of $\mathcal{F}$, which is a contradiction. □

Theorem 2.5. *The following conditions are equivalent for a topological space X:*

(i) *X is compact.*
(ii) *Each ultrafilter in X has a limit.*
(iii) *If $\mathcal{F}$ is a filter-base of closed subsets of X, then $\bigcap \mathcal{F} \neq \emptyset$.*

Proof. (i) → (iii): Let $\mathcal{F}$ be a filter-base of closed sets. Suppose for contradiction $\bigcap \mathcal{F} = \emptyset$. Hence $\{X \setminus F : F \in \mathcal{F}\}$ is an open cover of X by De Morgan's laws. However, it has no finite subcover, since $(X \setminus F_1) \cup \cdots \cup (X \setminus F_n) = X \setminus (F_1 \cap \cdots \cap F_n)$ and $F_1 \cap \cdots \cap F_n$ is non-empty as an element of $\mathcal{F}$.

(iii) → (ii): For an ultrafilter $\mathcal{F}$, we have $\bigcap \{\overline{F} : F \in \mathcal{F}\} \neq \emptyset$ because $\{\overline{F} : F \in \mathcal{F}\}$ is a filter-base of closed subsets. Any point in this intersection is a limit point of the ultrafilter $\mathcal{F}$.

(ii) → (i): Suppose for contradiction that $\mathcal{U}$ is an open cover of X with no finite subcover. Hence $\{X \setminus U : U \in \mathcal{U}\}$ is a filter-base. Let $\mathcal{F}$ be an ultrafilter such that $\{X \setminus U : U \in \mathcal{U}\}$. By (ii) we get that $\mathcal{F}$ has a limit point x. But $x \notin U$ for any $U \in \mathcal{U}$, otherwise U would be a neighbourhood of x and $U \in \mathcal{F}$. However, this is a contradiction with $X \setminus U \in \mathcal{F}$. □

Corollary 2.6. *A topological space X is Hausdorff compact if and only if every ultrafilter on X has exactly one limit.*

3. KC SPACES

Definition 3.1. A topological space (X, τ) is said to be *KC space* if every compact subset $K \subseteq X$ is closed.

It is obvious that each Hausdorff space is KC and each KC space is T_1. There are also T_1 spaces which are not KC. We can take an infinite minimal T_1 space, i.e. cofinite topology on an infinite set X. It is obvious that this topology is T_1 and also that each subset of such space is compact, because every single open set covers the whole space except for finitely many points. But not all subsets of X are closed, since they are not all finite (note that X is infinite), and X isn't KC.

It's a bit harder to construct a KC space which is not Hausdorff.

Example 3.2. Consider the set $X = ([0,1]\times\omega)\cup\{a,b\}$ with topology on $[0,1]\times\omega$ being standard product topology and define the neighbourhoods of a and b by the following neighbourhood bases $\mathcal{B}_a$ and $\mathcal{B}_b$:

$$\mathcal{B}_a = \{U \subseteq X : a \in U, U \cap [0,1] \times \omega \text{ is open }, \exists N \in \omega \, \forall n > N, \langle 0, n\rangle \in U\}$$
$$\mathcal{B}_b = \{U \subseteq X : b \in U, \exists N \in \omega \, \forall n > N, (0,1] \times \{n\} \subseteq U\}$$

It is easy to see that a and b have no disjoint neighbourhoods, and hence X is not Hausdorff. On the other hand, let $K \subseteq X$ be compact. Consider two cases: If $K \subseteq [0,1] \times \{0,1,\ldots,n\}$ for some $n \in \omega$, then K is closed as a subset of a Hausdorff space.

On the other hand, let K contain a sequence $\langle\langle x_{n_k}, n_k\rangle : k \in \omega\rangle$ such that $n_1 < n_2 < \cdots < n_k < \cdots$. First, $K \cap ([0,1] \times \{n\})$ is closed for every $n \in \omega$ because it is a compact subspace of a Hausdorff space. Then for every point $x \in X$ which is neither a nor b holds $x \in K \iff x \in \overline{K}$. Now, let's consider points a and b.

If $\langle 0, n\rangle \in K$ for infinitely many n, then $a \in K$, otherwise K would not be compact. Suppose that $\langle 0, n\rangle \in K$ for only finitely many $n \in \omega$; let $N \in \omega$ be greater than all of these n. We have $\langle 0, n\rangle \notin K$ for $m > N$, hence there are $\varepsilon_n > 0$ such that $([0, \varepsilon_n) \times \{n\}) \cap K = \emptyset$. And $U = \{a\} \cup \bigcup_{n>N}([0, \varepsilon_n) \times \{n\})$ is a neighbourhood of a that is disjoint with K, and so $a \notin \overline{K}$.

Similarly if $\langle a_n, n\rangle \in K$ and $a_n \neq 0$ for infinitely many $n \in \omega$, then $b \in K$, otherwise this sequence would have no accumulation point, because a can't be accumulation point of it as in the previous case. On the other hand, there is $N \in \omega$ such that $((0,1] \times \{n\}) \cap K = \emptyset$ for every $n > N$. And finally the union of these sets is an open neighbourhood of b which is disjoint with K, and so $b \notin \overline{K}$.

Example 3.3. Consider set X such that $X = \omega_1$ with the following topology: A non-empty set $U \subseteq X$ is open if and only if $X \setminus U$ is countable or finite. It is easy to see that it is a T_1-topology on ω_1.

Furthermore only compact subsets of such space are finite. Indeed if $H \subseteq \omega_1$ is infinite, then there is a countable infinite subset H_0. Now consider a free ultrafilter $\mathcal{F}$ in H_0. It is an ultrafilter of closed sets, hence

$$\bigcap\{\overline{F} : F \in \mathcal{F}\} = \bigcap \mathcal{F} = \emptyset$$

Finally $\mathcal{F}$ can be viewed as ultrafilter in H and it has no limit point, hence H cannot be compact. Any finite set is closed in X, which means that X is a KC space.

Since $|X| = \omega_1$, it is easy to see that every two open subsets U, V of X intersects and X is not Hausdorff.

Every subset Y of a KC space X is also KC, because if we have $K \subseteq Y$ compact, then it is compact as subset of X, and so closed in X, hence in Y. On the other hand, if Y is KC for every compact subset Y of a space X, then X is KC. Simply let $K \subseteq X$ be compact, then $K \cup \{x\}$ is compact for each $x \in \overline{K}$. K is closed in $K \cup \{x\}$ and $x \in K$. This proves the following lemma.

Lemma 3.4. *A space X is KC if and only if every compact subset of X is KC.*

Definition 3.5. If $\mathcal{P}$ is a property of topological spaces (e.g. compactness, KC or T_2). A space (X, τ) is said to be maximal (minimal resp.) if for every strictly stronger (weaker resp.) topology τ' the space (X, τ') doesn't satisfy condition $\mathcal{P}$.

Theorem 3.6. ([5], Theorem 1) *A topological space X is maximal compact if and only if it is KC compact.*

Proof. Necessity: Suppose that (X, τ) is compact space and K is such compact subset X, which is not τ-closed. Define a new topology τ' on X by sub-basis $\tau \cup \{X \setminus K\}$. Since K is τ'-closed and not τ-closed, τ' is strictly stronger than τ. The τ'-open sets of X are of form $(U \setminus K) \cup V$, where U, V are τ-open sets.

Next, we'll prove that (X, τ') is still compact. Let $\mathcal{U}$ be a τ'-open cover. For each $U \in \mathcal{U}$ we have

$$U = \big(A(U) \setminus K\big) \cup B(U),$$

where $A(U), B(U)$ are τ-open. The system $\{B(U) : U \in \mathcal{U}\}$ covers K. Hence it has a finite subcover $\{B(U_1), \dots, B(U_n)\}$. Let $B = \bigcup\{B(U_1), \dots, B(U_n)\}$. Consider the system $\{A(U) : U \in \mathcal{U}\} \cup \{B\}$, it is a τ-open cover of a compact space X, hence it has a finite subcover $\{A(V_1), \dots, A(V_m)\} \cup \{B\}$. Finally $\{U_1, \dots, U_n, V_1, \dots, V_m\}$ is a subcover of $\mathcal{U}$, which covers X.

Sufficiency: Let (X, τ) be a KC compact space and let $\tau' \supseteq \tau$ be a compact topology on X. Every τ'-closed set is τ'-compact, which means it is especially τ-compact. Since τ is KC, it is τ-closed. We get $\tau' = \tau$, and so (X, τ) is maximal compact. □

Our goal is to prove that a topological space is maximal compact if and only if it is minimal KC. As corollary to the previous theorem we get that every maximal compact space (X, τ) is minimal KC. Indeed it is KC and if τ' is strictly weaker topology, then τ' is compact and cannot be KC, because τ is strictly stronger compact topology, and hence τ' is not maximal compact. Finally, if we prove that every minimal KC topology is compact, then it would be KC compact and by the previous theorem maximal compact. In the following chapters we'll prove that minimal KC spaces are compact.

Definition 3.7. Let X, Y be topological spaces. The map $f: X \to Y$ is said to be *closed* if for every closed $F \subseteq X$, its image $f[F]$ is also closed in Y.

Although in [4] the author defines compact spaces as Hausdorff, the following theorem holds even for non-Hausdorff spaces, because the proof doesn't use any separation axioms.

Theorem 3.8. ([4], Theorem 3.1.10) *Any continuous image of a compact space is compact.*

Proof. Let X be a compact space, Y be a topological space and $f: X \to Y$ be a continuous surjective map. And let $\mathcal{U}$ be an open cover of Y. Then

$$f^{-1}[\mathcal{U}] = \{f^{-1}[U] : U \in \mathcal{U}\}$$

is an open cover of X. It has a finite subcover $\{f^{-1}[U_1], \dots, f^{-1}[U_n]\}$. But since f is surjective, $\{U_1, \dots, U_n\}$ is an open cover of Y. Indeed, let $y \in Y$ be arbitrary. Then there is some $x \in X$ such that $f(x) = y$. Let $x \in f^{-1}[U_k]$, then $y = f(x) \in U_k$. □

Corollary 3.9. *If X is a compact space and Y is a KC space, then every continuous map $f: X \to Y$ is closed.*

The following two lemma's are quite well-known characterisation of non-compact spaces. We'll use both of them in the fifth chapter.

Lemma 3.10. *Let (X, τ) be a KC non-compact space. Then there is a discrete subset $D \subseteq X$, such that $\overline{D}$ is not compact. Furthermore there is an ultrafilter $\mathcal{F}$ in X, such that $D \in \mathcal{F}$ and $\mathcal{F}$ does not converge.*

Proof. Let $\mathcal{U} = \{U_i : i < \kappa\}$ be a strictly increasing open cover of X, where κ is an infinite regular cardinal. We'll construct sets $D_\lambda = \{x_i : i < \lambda\}$ by transfinite induction. First, let $D_0 = \{x_0\}$ for some $x_0 \in U_0$.

Let λ is ordinal successor. If $\overline{D_{\lambda-1}}$ is compact, then there is α_λ such that $\overline{D}_{\lambda-1} \subseteq U_{\alpha_\lambda}$. Let $x_\lambda \in U_{\alpha_\lambda+1} \setminus U_{\alpha_\lambda}$ and $D_\lambda = D_{\lambda-1} \cup \{x_\lambda\}$. For limit ordinals λ, let $D_\lambda = \bigcup_{i<\lambda} D_i$.

This process stops when $\overline{D_\lambda}$ is not compact, which holds at least for $\lambda = \kappa$, because then the open cover $\mathcal{U}$ witnesses that $\overline{D_\kappa}$ is not compact. It is easy to see that D_λ is discrete. The open set, which contains exactly one point x_{i+1} is $U_{\alpha_{i+1}} \setminus \overline{D_i}$.

Finally we'll prove that if D is discrete subset of a KC space X and every ultrafilter $\mathcal{F}$ in X, such that $D \in \mathcal{F}$, converges, then $\overline{D}$ is compact. Let βD be a set of all ultrafilters in D with Stone topology. Define a map $f: \beta D \to X$, such that for any ultrafilter $\mathcal{F} \in \beta D$, $f(\mathcal{F})$ is a limit of the ultrafilter $\mathcal{F}$. It is easy to see, that f is continuous. Because X is KC and $f[\beta D]$ is compact, it is closed, hence $\overline{D} \subseteq f[\beta D]$. But from definition of f we easily get $f[\beta D] \subseteq \overline{D}$, hence $f[\beta D] = \overline{D}$ and both are compact. □

Lemma 3.11. *Let (X, τ) be a topological space, which is not compact. Then there is $C \subseteq X$, such that C has no complete accumulation point.*

Proof. A space X is not compact, hence there is a strictly increasing infinite open cover $\mathcal{U} = \{U_i : i < \kappa\}$. Without loss of generality, we can assume, that $\mathcal{U}$ has the smallest cardinality, i.e. κ is an infinite regular cardinal. For any $i < \kappa$ let $x_i \in U_{i+1} \setminus U_i$. The set $C = \{x_i : i < \kappa\}$ has the requested property. Because if x is a complete accumulation point of C, then every open neighbourhood of x intersects $\{x_i : \alpha \leq i < \kappa\}$ for each $\alpha < \kappa$, because the complement in C has cardinality strictly smaller than κ. We get, that x can't be in any U_α, from

$$x \in \overline{\{x_i : \alpha \leq i < \kappa\}} \subseteq X \setminus U_\alpha$$

And finally $\mathcal{U}$ doesn't cover the point x. $\square$

4. Compactifications of KC spaces

Definition 4.1. Let X be a topological space. A space cX is a *compactification* of X if it is compact and X is dense subset of cX.

Definition 4.2. Let X be a non-compact KC space. We define the *one-point compactification* αX of X as the topological space such that $\alpha X = X \cup \{\infty\}$, and the neighbourhoods of each point inside X coincide with neighbourhoods in X and a set U is an open neighbourhood of ∞ if and only if $\infty \in U$ and $\alpha X \setminus U$ is a compact subset of X.

Note that αX is always compact. Since X is not compact, ∞ is not isolated point in αX, which means that αX is a compactification of X. For each KC space αX is always T_1 but it is not necessary KC.

The one-point compactification is such compactification of a space X, that for any other compactification cX there is at most one continuous map $\varphi\colon cX \to \alpha X$ such that $\varphi|_X = \mathbf{1}_X$. It is defined as $\varphi(y) = \infty$ for each $y \notin X$ (if it is continuous).

Theorem 4.3. *Let X be a KC space. If X has a KC compactification cX, then also one-point compactification αX is KC.*

Proof. Let $\varphi\colon cX \to \alpha X$ be defined as $\varphi|_X = \mathbf{1}_X$ and $\varphi(x) = \infty$ for any $x \in cX \setminus X$. Then φ is closed, i.e. for any $F \subseteq cX$ closed, also $\varphi[F]$ is a closed subset of αX.

Let $F \subseteq X$. It is a closed subset of X and $\overline{F}^{\alpha X} \subseteq F \cup \{\infty\}$. But F is also compact, because it is a closed subset of cX, hence $\alpha X \setminus F$ is a neighbourhood of ∞ which is disjoint with F. This gives that F is closed in αX.

On the other hand, let $F \setminus X \neq \emptyset$. Then $F \cap X$ is still closed subset of X and $\varphi[F] = (F \cap X) \cup \{\infty\}$. Since there is no other point, that could be an accumulation point of $\varphi[F]$, we get $\varphi[F]$ is closed.

Now, let $K \subseteq \alpha X$ be compact. If $K \subseteq X$ then it is closed from X is KC. Suppose $\infty \in K$ and consider set

$$K' = \left(\overline{K \cap X}^{cX} \cap (cX \setminus X)\right) \cup (K \cap X).$$

It must be compact, because if we have a filter-base $\mathcal{F}$ of closed sets then $\varphi[\mathcal{F}] = \{\varphi[F] : F \in \mathcal{F}\}$ is also a filter-base of closed sets in K, hence it has a limit in K. If this limit is in X, than $\mathcal{F}$ has the same limit. In the second case, its limit is ∞. Then $\mathcal{F}$ has limit in $\overline{K \cap X}^{cX}$ and because it has no limit in $K \cap X$ this limit lies in $\overline{K \cap X}^{cX} \cap (cX \setminus X)$.

Hence K' is closed, because cX is KC. Then also $\varphi[K']$ is closed and $\varphi[K'] = K$. $\square$

Corollary 4.4. *If X is KC compact space, then for each subspace $Y \subseteq X$, the one-point compactification αY is KC.*

The previous theorem gives us many examples of KC compact spaces, which are not Hausdorff. It is well-known that every Tychonoff space has a Hausdorff compactification. Hence for each Tychonoff space X we know that αX is KC. Furthermore αX is Hausdorff if and only if X is a locally compact Hausdorff space. Hence if X is Tychonoff and not locally compact, then αX is KC compact, which is not Hausdorff. An example of such space is $\alpha\mathbb{Q}$.

A related question is to characterise spaces, which have some KC compactifications. By theorem 4.3, we know that it is equivalent to characterise spaces, which has the one-point compactification KC. In [3] was proven that a countable KC space has this property if and only if it is sequential.

5. Minimal KC spaces are compact

Definition 5.1. Let (X, τ) be a topological space, which is not compact, $x_0 \in X$, and $\mathcal{F}$ an ultrafilter in X, which doesn't converge in τ. We define a new topology $\tau(\mathcal{F})$ on X, such that U is a $\tau(\mathcal{F})$-open set if it is τ-open and satisfies one of the following conditions:

(i) $x_0 \in U$ and $U \in \mathcal{F}$
(ii) $x_0 \notin U$

It's easy to see that $\tau(\mathcal{F})$ is a T_1-topology, which is strictly weaker than τ. Neighbourhoods of any point except for x_0 have not changed. The only new accumulation point of any set can be x_0. An ultrafilter $\mathcal{F}$ converges to x_0 in the new topology, as well as any ultrafilter containing each of the open set U, such that $x_0 \in U$ & $U \in \mathcal{F}$. The $\tau(\mathcal{F})$ topology may be also described by the system of its closed sets. A τ-closed set F is $\tau(\mathcal{F})$-closed if and only if whenever $F \in \mathcal{F}$, it contains also the point x_0.

Lemma 5.2. *Let (X, τ) be a non-compact KC-space, $\mathcal{F}$ a non-converging ultrafilter and $\sigma = \tau(\mathcal{F})$. If $K \subseteq X$ is τ-compact than it is σ-closed and topologies τ and σ agree on K.*

Proof. Since (X, τ) is KC, we know that K is τ-closed. It suffices to prove that $K \notin \mathcal{F}$. Indeed if $K \in \mathcal{F}$, then $\mathcal{F} \cap 2^K$ is an ultrafilter in K, which has no τ-limit and K is not τ-compact. □

Lemma 5.3. ([2], Corollary 2.2) *If (X, τ) is a minimal KC space, then for each $x, y \in X$ and each open neighbourhood V of x, there is an open neighbourhood W of y such that $\overline{W \setminus V}$ is compact.*

Proof. We'll describe just the idea of the proof, the complete proof can be found in [2].

First step is to prove, that for every KC space X and every two points $a, b \in X$ we have (not necessary strictly) weaker topology $\tau_{a,b}$ defined by: U is $\tau_{a,b}$-open if it is τ-open and $a \notin U$, or $a \in U$ and there are τ-open neighbourhoods V, W of points a, b and K compact, such that $U = V \cup (W \setminus K)$. The $\tau_{a,b}$-compact subsets are exactly those, which are also τ-compact.

Second step is straightforward. If τ is minimal KC, then for points x, y holds $\tau_{x,y} = \tau$. Hence, each neighbourhood V of x can be written as $U \cup (W \setminus K)$ for some U, W, and K as above. Especially, we get $W \setminus K \subseteq V$, which is equivalent to $W \setminus V \subseteq K$. By KC we finally get that $\overline{W \setminus V}$ is compact subset of K. □

Lemma 5.4. *Let (X, τ) be a minimal KC space, D a discrete subset with non-compact τ-closure, $\mathcal{F}$ an ultrafilter, such that $D \in \mathcal{F}$ and $\mathcal{F}$ does not converge in the topology τ. Let $\sigma = \tau(\mathcal{F})$. Then every σ-compact subset is also τ-compact.*

Proof. Suppose for contradiction, that M is a σ-compact set, which is not τ-compact. Then there is τ-open neighbourhood of x_0 such that $M \setminus U_0$ is not τ-compact either. Let $N = (M \setminus U_0) \cup \{x_0\}$.

Now, we'll prove that N is τ-closed. Let $x \in \overline{N}$. From lemma 5.3 let $V \ni x$ such that $K = \overline{V \setminus U_0}$ is τ-compact. Then topologies τ and σ agree on K. Since V is neighbourhood of x, we have $x \in \overline{V \cap N} \subseteq \overline{K \cap N} \cup \{x_0\}$. Note that $N \cap K$ is σ-compact because it is a closed subset of a compact space N. But σ and τ still agree on K, hence $N \cap K$ is τ-compact, and so τ-closed. This gives $x \in N$.

Finally we have two possibilities:

(a) If $X \setminus N \in \mathcal{F}$ then topologies σ and τ agree on N, and hence it is τ-compact.

(b) On the other hand, if $N \in \mathcal{F}$ then we have $D' \subseteq N$ for some $D' \subseteq D$. From D' is discrete, we know that $\overline{D'} \setminus D'$ is closed. Let W be such an open set, that $\overline{D'} \cap W = D'$. Then $W \cup U_0$ is a σ-open neighbourhood of x_0.

Now, suppose that $\overline{D'}$ is not τ-compact (otherwise N would be τ-compact). From lemma 3.11 we know that there is a set C without any complete τ-accumulation points. But C has a complete accumulation point in the topology σ, hence this point is x_0. Then $|(W \cup U_0) \cap C| = |C|$, beacuse $W \cup U_0$ is a σ-open neighbourhood of x_0. Since $(W \cup U_0) \cap C \subseteq D'$, we can suppose without loss of generality that $C \subseteq D'$.

Let $C = D_0 \cup D_1$, where D_0, D_1 are disjoint and have the same cardinality as C. At most one of these sets can be in $\mathcal{F}$. Without loss of generality assume that $D_1 \notin \mathcal{F}$. From D is discrete, we get $\overline{D_1}^{\tau} \notin \mathcal{F}$, and so D_1 has no σ-accumulation points, that are not τ-accumulation points. Hence D_1 has no complete σ-accumulation point. This contradicts N is σ-compact. □

Theorem 5.5. *Every minimal KC space is compact.*

Proof. Suppose for contradiction that (X, τ) is minimal KC space, which is not compact. From lemma 3.10 let D be discrete subset of X with non-compact closure and $\mathcal{F}$ a non-converting ultrafilter, such that $D \in \mathcal{F}$. Let $\sigma = \tau(\mathcal{F})$.

From lemma 5.2 we have that every τ-compact subset is also σ-closed. Finally lemma 5.4 says that there is no σ-compact subsets, which is not τ-compact. And together with the first fact this proves that σ is a KC topology. It contradicts τ is minimal KC. □

Corollary 5.6. *The following conditions are equivalent for a topological space X:*

(i) *X is maximal compact.*
(ii) *X is minimal KC.*
(iii) *X is KC compact.*

6. Vidalis's proof

In [1] the author claims to prove that every minimal KC space is countably compact, but his proof contains an error. In this chapter we'll describe the error.

The idea of the proof is, for every non-countable compact KC space (X, τ), to take F a countable infinite set with no accumulation point in (X, τ) and a free ultrafilter $\mathcal{F}$, such that $F \in \mathcal{F}$, then easily $\mathcal{F}$ has no limit since F has no accumulation points. And then prove that $\tau(\mathcal{F})$ topology is KC.

In lemma 3.5 in [1] author tries to prove that for some $F_1 \subseteq F$, $F_1 \in \mathcal{F}$, such that $F_1 \subseteq \overline{K}^{\tau}$, $K \cup F_1$ is τ-compact. But $F_1 \subseteq F$ and F has no τ-accumulation points, hence also F_1 has no τ-accumulation points and is infinite since $F_1 \in \mathcal{F}$ and $\mathcal{F}$ is free ultrafilter. After all $K \cup F_1$ can't be τ-compact.

The mistake in the proof is in the part, where he for some open cover $\mathcal{U}$ of $K \cup F_1$ chooses a finite subcover of $\mathcal{U}$. He first chooses $U'(x_0) = U(x_0) \cup \bigcup\{U_{i_n} : n \in \omega\}$ such that $x_0 \in U(x_0)$ and $\{U_{i_n} : n \in \omega\}$ covers F_1. And then he takes only such the rest of open sets in $\mathcal{U}$ getting a new subcover $\{V_j : j \in J\}$ of $K \setminus U'(x_0)$. Then there is a finite subcover $\{V_{j_1}, V_{j_2}, \ldots, V_{j_n}\}$ of $K \setminus U'(x_0)$. Finally he takes

$$\{U(x_0)\} \cup \{U_{i_n} : n \in \omega\} \cup \{V_{j_1}, V_{j_2}, \ldots, V_{j_n}\}$$

which is a countable τ-open cover of K as well as $K \cup F_1$. Now, he claims that since K is τ-countable compact then this cover has finite subcover. But this finite subcover is not always cover of $K \cup F_1$, the only thing we can say is that this subcover covers K.

Indeed, with assumptions of the lemma 3.5 of [1] let's construct a τ-open cover of $K \cup F_1$ with no finite subcover. Let $U = K \setminus F_1$, then U is τ-open set in K, because F_1 has no accumulation points. For each $x_i \in F_1$ let $V(x_i)$ be such open set, that contains exactly the point x_i of F_1, i.e. such an open set, that $V(x_i) \cap F_1 = \{x_i\}$ (it exists because F_1 is discrete). Finally $\mathcal{U} = \{U\} \cup \{V(x_i) : x_i \in F_1\}$ is an open cover of $K \cup F_1$, but has no finite subcover, because every finite subcover covers only finite number of points of the infinite set F_1.

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