ERRATA

Jakub Opršal: Minimal KC spaces

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Theorem 3.6. *([5], Theorem 1) A topological space X is maximal compact if and only if it is KC compact.*

Proof. Necessity: Suppose that (X, τ) is compact space and K is such compact subset X, which is not τ-closed. Define a new topology τ' on X by sub-basis $\tau \cup \{X \setminus K\}$. Since K is τ'-closed and not τ-closed, τ' is strictly stronger than τ. The τ'-open sets of X are of form $(U \setminus K) \cup V$, where U, V are τ-open sets.

Next, we'll prove that (X, τ') is still compact. Let $\mathcal{U}$ be a τ'-open cover. For each $U \in \mathcal{U}$ we have

$$U = \big(A(U) \setminus K\big) \cup B(U),$$

where $A(U), B(U)$ are τ-open. The system $\{B(U) : U \in \mathcal{U}\}$ covers K. Hence it has a finite subcover $\{B(U_1), \dots, B(U_n)\}$. Let $B = \bigcup\{B(U_1), \dots, B(U_n)\}$. Consider the system $\{A(U) : U \in \mathcal{U}\} \cup \{B(U) : U \in \mathcal{U}\} \cup \{B\}$, it is a τ-open cover of a compact space X, hence it has a finite subcover

$$\{A(V_1), \dots, A(V_m), B(U_{n+1}), \dots, B(U_k)\} \cup \{B\}.$$

Finally $\{U_1, \dots, U_k, V_1, \dots, V_m\}$ is a finite subcover of $\mathcal{U}$, which covers X.

Sufficiency: Let (X, τ) be a KC compact space and let $\tau' \supseteq \tau$ be a compact topology on X. Every τ'-closed set is τ'-compact, which means it is especially τ-compact. Since τ is KC, it is τ-closed. We get $\tau' = \tau$, and so (X, τ) is maximal compact. $\square$

Lemma 3.10. *Let (X, τ) be a KC non-compact space. Then there is a discrete subset $D \subseteq X$, such that $\overline{D}$ is not compact. Furthermore there is an ultrafilter $\mathcal{F}$ in X, such that $D \in \mathcal{F}$ and for every $C \in \mathcal{F}$, $\overline{C}^{\tau}$ is not compact.*

Proof. Let $\mathcal{U} = \{U_i : i < \kappa\}$ be a strictly increasing open cover of X, where κ is an infinite regular cardinal. We'll construct sets $D_\lambda = \{x_i : i < \lambda\}$ by transfinite induction. First, let $D_0 = \{x_0\}$ for some $x_0 \in U_0$.

Let λ be an ordinal successor. If $\overline{D_{\lambda-1}}$ is compact, then there is α_λ such that $\overline{D}_{\lambda-1} \subseteq U_{\alpha_\lambda}$. Let $x_\lambda \in U_{\alpha_\lambda+1} \setminus U_{\alpha_\lambda}$ and $D_\lambda = D_{\lambda-1} \cup \{x_\lambda\}$. For limit ordinals λ, let $D_\lambda = \bigcup_{i<\lambda} D_i$.

This process stops when $\overline{D_\lambda}$ is not compact, which holds at least for $\lambda = \kappa$, because then the open cover $\mathcal{U}$ witnesses that $\overline{D_\kappa}$ is not compact. It is easy to see that D_λ is discrete. The open set, which contains exactly one point x_{i+1} is $U_{\alpha_{i+1}} \setminus \overline{D_i}$.

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Let's construct the ultrafilter. The family $\mathcal{F}' = \{C \subseteq D : \overline{D \setminus C} \text{ is compact}\}$ is closed under finite intersection, because if $E_0, E_1 \subseteq D$ are such that $\overline{E_0}$ and $\overline{E_1}$ are compact, then $\overline{E_0 \cup E_1} = \overline{E_0} \cup \overline{E_1}$ is compact. Let $\mathcal{F}$ be an ultrafilter above this family. Then for any $C \in \mathcal{F}$, there is $C' \cap D \in \mathcal{F}$ and $\overline{C'}$ is not compact, otherwise $\overline{D}$ would be compact. Finally this leads to that $\overline{C}$ is not compact either. □

5. Minimal KC spaces are compact

Definition 5.1. Let (X, τ) be a T_1 space, which is not compact, $x_0 \in X$, and $\mathcal{F}$ an ultrafilter in X, such that x_0 is not a τ-limit of $\mathcal{F}$. We define a new topology $\tau(\mathcal{F})$ on X, such that U is a $\tau(\mathcal{F})$-open set if it is τ-open and satisfies one of the following conditions:

(i) $x_0 \in U$ and $U \in \mathcal{F}$
(ii) $x_0 \notin U$

It's easy to see that $\tau(\mathcal{F})$ is a T_1-topology, which is strictly weaker than τ. Neighbourhoods of any point except for x_0 have not changed. The only new accumulation point of any set can be x_0. An ultrafilter $\mathcal{F}$ converges to x_0 in the new topology, as well as any ultrafilter containing each of the open set U, such that $x_0 \in U$ & $U \in \mathcal{F}$. The $\tau(\mathcal{F})$ topology may be also described by the system of its closed sets. A τ-closed set F is $\tau(\mathcal{F})$-closed if and only if $F \in \mathcal{F}$ and it contains also the point x_0 or $F \notin \mathcal{F}$.

Lemma 5.2. *Let (X, τ) be a KC-space and $\mathcal{F}$ be an ultrafilter, such that $\overline{F}^{\tau}$ is not compact for any $F \in \mathcal{F}$. Let $\sigma = \tau(\mathcal{F})$. If $K \subseteq X$ is a τ-compact then it is σ-closed and topologies τ and σ agree on K.*

Proof. Since (X, τ) is KC, we know that K is τ-closed. It suffices to prove that $K \notin \mathcal{F}$. Indeed if $K \in \mathcal{F}$, then $\overline{K}^{\tau}$ is not compact, but $\overline{K}^{\tau} = K$. This gives a contradiction with K is τ-compact.

Lemma 5.3. ([2], Corollary 2.2) *If (X, τ) is a minimal KC space, then for each $x, y \in X$ and each open neighbourhood V of x, there is an open neighbourhood W of y such that $\overline{W \setminus V}$ is compact.*

Lemma 5.4. *Let (X, τ) be a minimal KC space, D a discrete subset with non-compact τ-closure, $\mathcal{F}$ an ultrafilter, such that $D \in \mathcal{F}$ and $\mathcal{F}$ contains only sets with non-compact τ-closures. Let $\sigma = \tau(\mathcal{F})$. Then every σ-compact subset is also τ-compact.*

Proof. Suppose for contradiction, that M is a σ-compact set, which is not τ-compact. Then there is τ-open neighbourhood of x_0 such that $M \setminus U_0$ is not τ-compact either. Let $N = (M \setminus U_0) \cup \{x_0\}$.

Now, we'll prove that N is τ-closed. Let $x \in \overline{N}$. From lemma 5.3 let $V \ni x$ such that $K = \overline{V \setminus U_0}$ is τ-compact. Then topologies τ and σ agree on K. Since V is neighbourhood of x, we have $x \in \overline{V \cap N} \subseteq \overline{K \cap N} \cup \{x_0\}$. Note that $N \cap K$ is σ-compact because it is a closed subset of a compact space N. But σ and τ still agree on K, hence $N \cap K$ is τ-compact, and so τ-closed. This gives $x \in N$.

Finally we have two possibilities:

(a) If $X \setminus N \in \mathcal{F}$ then topologies σ and τ agree on N, and hence it is τ-compact.

(b) On the other hand, if $N \in \mathcal{F}$ then let $D' = N \cap D$. From D' is discrete, we know that $\overline{D'} \setminus D'$ is closed. Let W be such an open set, that $\overline{D'} \cap W = D'$. Then $W \cup U_0$ is a σ-open neighbourhood of x_0.

Now, suppose that $\overline{D'}$ is not τ-compact (otherwise N would be τ-compact). From lemma 3.11 we know that there is a set C without any complete τ-accumulation points. But C has a complete accumulation point in the topology σ, hence this point is x_0. Then $|(W \cup U_0) \cap C| = |C|$, beacuse $W \cup U_0$ is a σ-open neighbourhood of x_0. Since $(W \cup U_0) \cap C \subseteq D'$, we can suppose without loss of generality that $C \subseteq D'$.

Let $C = D_0 \cup D_1$, where D_0, D_1 are disjoint and have the same cardinality as C. At most one of these sets can be in $\mathcal{F}$. Without loss of generality assume that $D_1 \notin \mathcal{F}$. From D is discrete, we get $\overline{D_1}^{\tau} \notin \mathcal{F}$, and so D_1 has no σ-accumulation points, that are not τ-accumulation points. Hence D_1 has no complete σ-accumulation point. This contradicts N is σ-compact. □

Theorem 5.5. *Every minimal KC space is compact.*

Proof. Suppose for contradiction that (X, τ) is a minimal KC space, which is not compact. From lemma 3.10 let D be discrete subset of X with non-compact closure and $\mathcal{F}$ an ultrafilter, such that $D \in \mathcal{F}$ and for every $C \in \mathcal{F}$, $\overline{C}$ is not compact. It is easy to see that $\mathcal{F}$ does not converge to any point in D, so let $x_0 \in D$ and $\sigma = \tau(\mathcal{F})$.

From lemma 5.2 we have that every τ-compact subset is also σ-closed. Finally lemma 5.4 says that there is no σ-compact subsets, which is not τ-compact. And together with the first fact this proves that σ is a KC topology. It contradicts τ is minimal KC. □