Charles University in Prague
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MASTER THESIS

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Categorial methods in structure theory

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Study programme: Mathematics
Specialization: Mathematical structures

Prague 2011

I would like to thank attendees of Prague Seminar on general mathematical structures and especially my supervisor prof. Trnková who were kind enough to offer valuable suggestions and ideas for the thesis.

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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In Prague, 5 August 2011 Jakub Opršal

Název práce: Kategoriální metody v teorii struktur

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Abstrakt: V první části práce se věnujeme funktorovým algebrám. Výjmečnou roli hrají iniciální funktorové algebry, které lze získat tzv. konstrukcí iniciální algebry. V tomto roce Adámek a Trnková dokázali, že v kategorii množin se konstrukce může zastavit pouze po nejvýše třech krocích, nebo až na libovolném regulárním kardinálu. My na tento výsledek navazujeme a zkoumáme souvislost délky konstrukce a velikosti iniciální algebry. Ukazujeme, že délka konstrukce nikdy nepřesáhne kardinalitu iniciální algebry.

Jinou transfinitiní konstrukci studoval Kelly v roce 1980. Popsal konstrukci volných algeber pro pointované funktory a definoval třídu dobře pointovaných funktorů, pro které je konstrukce obzvláště jednoduchá (a ve skutečnosti je zvláštním případem konstrukce relativně terminální algebry, kterou nedávno zkoumali Adámek a Trnková). V poslední kapitole popisujeme všechny dobře pointované funktory v kategorii množin a v kategorii k ní duální. Dále se věnujeme dobře pointovaným funktorům v mnohasortových množinách a popíšeme všechny možné třídy algeber pro takové funktory.

Klíčová slova: funktorové algebry, konstrukce iniciální algebry, konstrukce volné algebry, kategorie množin, mnohasortové množiny

Title: Categorial methods in structure theory

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Abstract: In the first part of the thesis we investigate functor algebras. Initial algebras have distinguished role in the study of these structures, and it can be constructed by certain transfinite construction, which is called initial algebra construction. Sooner this year Adámek and Trnková have prooved, that the construction stops in either at most three, or in κ steps where κ is a regular cardinal. We continue with their work, and we study the relation between the size of the algebra and the length of the convergence. We prove that the length of the convergence never exceeds the cardinality of the initial algebra.

Another transfinite construction has been studied in 1980 by Kelly. He has described the construction of free algebras for a pointed functor and defined a class of well-pointed functors for which the construction is especially simple (and is in fact special case of the construction of relatively terminal coalgebra which has been recently defined by Adámek and Trnková). In the last chapter we describe all well-pointed functors in categories of sets and the dual category, and we provide list of well-pointed functors of many-sorted sets which is comprehensive enough to describe all possible classes of well-pointed algebras in many-sorted sets.

Keywords: functorial algebras, initial algebra construction, free algebra construction, category of sets, many-sorted sets

Contents

1. Introduction

In this thesis, we study various transfinite constructions for initial algebras and free algebras which are based on the proof of famous Knaster-Tarski theorem about the least fixed point of non-decreasing map in a complete lattice. The thesis is divided into two parts. First of them is focused on initial algebras and the second one on pointed functors and free algebras.

Functorial algebras, later functorial coalgebras, and constructions of initial algebra and terminal coalgebra have been traditionally studied with connection to data types, automata, and systems (see fundamental study [R] by Rutten).

Some of the questions about the convergence of the terminal coalgebra construction, for example characterization of all endofunctors F of the category of sets such that the terminal coalgebra construction stops, are still open. However, the same question in the dual category (for initial algebra construction) was solved a long time ago in [TAKR] by Trnková at al.—they proved that the initial algebra construction in category of sets stops for a functor F if and only if F has a fixed point (i.e. an object X such that $FX \simeq X$). Some sufficient condition for the terminal coalgebra construction have been given by Adámek and Koubek in [AKo].

Recently, also length of the length of the construction has been object of interest. Adámek and Trnková characterized all possible ordinals π such that the initial algebra construction may end after exactly π steps. In this thesis we follow this result by study of the relationship between the size of the initial algebra and the length of the convergence. Usually, the size of the sets in the construction grows, until it reaches the size of the initial algebra then the construction still doesn't stop because the connecting morphism is usually not isomorphism, and continues to stabilize. A result about the length of the first part of the construction (where the size grows) is also given.

In the second part, we study pointed functors (endofunctors F of some category together with a natural transformation φ from identity functor to F). The pointed functorial algebras are little more complicated because they satisfy additional condition—a pair (A, a) where A is an object and $a\colon FA \to A$ is a morphism is pointed algebra, if $a\varphi_A = 1_A$ (the operation a respects the point φ). These structures have been defined and extensively studied by Kelly in [K]. The free algebra construction can be simplified for so-called well-pointed functors which are pointed functors (F, φ) that satisfy the condition $F\varphi = \varphi_F$ (they have been also defined by Kelly in his study).

In this thesis, we connect Kelly's results to a recent development of relatively initial algebras which has been defined by Adámek and Trnková in [AT_3], and we further study the class of well-pointed functors which appears to be very rare in the categories of sets, the dual category, and the category of many-sorted sets.

2. A short comment on notation

We use a little unusual notation about natural numbers. We view each natural number as ordinal, i.e. the set of all lesser natural numbers, especially 0 is empty set, $1 = \{0\}$ is one-point set which is usually used as general one-point set, $2 = \{0, 1\}$, and n is an n-element set.

If $\mathcal{K}$ is a category then we use symbol $\mathcal{K}$ for the class of all objects of $\mathcal{K}$. The set of morphisms from A to B is denoted $\mathrm{Hom}_{\mathcal{K}}(A, B)$ or just $\mathrm{Hom}(A, B)$ and a single morphism $f \in \mathrm{Hom}(A, B)$ is denoted $f\colon A \to B$.

Possibly the most misleading notation is the one for identities. Identities are always denoted 1 or 1_X, if it is identity on X. The identity functor is then denoted 1, do not confuse it with constant functor to one-point set, which is denoted C_1. Hence $C_1 X = 1$ for each set X and $C_1 f = 1_1$ for each map f.

The list of all used symbols can be found at the end of the thesis.

3. Some facts about set functors

This chapter constains some well known results about set functors which can be also found in [AT_1]. We'll need them in the following chapters especially in the chapter about well-pointed functors in the category of sets.

All functors in this chapter are endofuntors of **Set**.

Let M and N be two sets and a map $m\colon M \to N$. We define *constant* functor $C_{M,N}$ on objects by

$$C_{M,N}X = \begin{cases} M & \text{if } X \text{ is empty,} \\ N & \text{otherwise.} \end{cases}$$

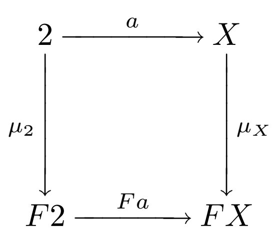
The image of empty map $f\colon 0 \to X$ (for X non-empty) is a map m. If both domain and codomain of f is non-empty then $C_{M,N} = 1_N$. The image of the identity on empty set is obviously 1_M. If $M = N$ and $m = 1_M$ then $C_{M,M}$ is denoted just C_M.

The most important of the constant functors is $C_{0,1}$ which maps empty set to empty set and any other set to one-point set, and C_1 which maps each set to ene-point set.

Lemma 3.1. *Let $\mu\colon 1 \to F$ be a natural transformation then either for each set X, μ_X is constant, or μ is monotransformation.*

Proof. For arbitrary set X and two points a_0, $a_1 \in X$ we can define a map $a\colon 2 \to X$ s.t. $a(0) = a_0$ and $a(1) = a_1$. Furthermore this map is a monomorphism with non-empty domain and every set funtor preserves such monomorphisms hence Fa is also monomorphism.

We have commutative square



From this square we get $Fa\mu_2(0) = \mu_X(a_0)$ and $Fa\mu_2(1) = \mu_X(a_1)$. Since Fa is injective we know that $\mu_X(a_0) = \mu_X(a_1)$ if and only if $\mu_2(0) = \mu_2(1)$. In the case that $\mu_2(0) = \mu_2(1)$ we get that all μ_X are constant for any set X. In the other case, when $\mu_2(0) \neq \mu_2(1)$ we get that all μ_X are injective. $\square$

Any transformation $\varepsilon: C_{0,1} \to F$ is called *a distinquished point (element)* of F. The unique element in $\operatorname{im} \varepsilon_X$ is usually denoted just ε_X. We call an element $x \in FX$ distinguished if for each set Y there is $x_Y \in FY$ such that for any map $f: X \to Y$ holds $Ff(x) = x_Y$. Note that for any non-empty set X, $x \in FX$ is distinguished if and only if it is in the image of some distinguished point $\rho: C_{0,1} \to 1$.

We call transformation $\mu: 1 \to F$ a *distinguished point* if it is constant, i.e. there is $\rho: C_{0,1} \to F$ such that $\mu = \varepsilon\vartheta$ where ϑ is the unique transformation from 1 to $C_{0,1}$. This is a little abuse of notation but (hopefully) it can't lead to misunderstanding beacause the meaning of the distinguished point is always the same.

The previous lemma can be then restated as: Every transformation $\mu: 1 \to F$ is either distinguished, or mono.

Lemma 3.2. *For any* **Set** *functor F holds that any point of $F0$ is distinguished.*

Proof. Let $x \in F0$, Y be arbitrary non-empty set. There is a unique map $\vartheta_Y: 0 \to Y$. So, we define $x_Y = \vartheta_Y(x)$. It's obvious that for any map $f: X \to Y$ we have $f(x_X) = f\vartheta_X(x) = \vartheta_Y(x) = x_Y$. □

We say that set-functor F is *connected* if $|F1| = 1$. The meaning of this definition will be revealed in the following lemma.

Lemma 3.3. *For any set-functor F there exists connected functors F_i, $i \in F1$ such that*

$$F = \coprod_{i \in F1} F_i.$$

Proof. For a set X denote f_X the (unique) map from X to 1. Let $i \in F1$. Define the functor F_i on abjects as

$$F_i X = Ff_X^{-1}[i]$$

and for $f: X \to Y$ define

$$F_i f = Ff|_{F_i X}.$$

First, we need to proove that F_i are well-defined, i.e. for any $f: X \to Y$ and $x \in F_i X$ holds $Ff(x) \in F_i Y$. The following holds

$$Ff_Y(Ff(x)) = F(f_Y f)(x) = Ff_X(x) = i$$

hence $Ff(x) \in F_i Y$.

It is easy to see that all F_i's are connected and that FX is a disjoint union of $F_i X$'s. □

Each of functors F_i from the last lemma is called *a component* of F. Precisely G is a component of F if it is a connected subfunctor of F and there is no other connected subfunctor H of F such that G is a proper subfunctor of H.

Lemma 3.4. *For any connected functor F there is a unique transformation $\mu: 1 \to F$.*

Proof. Denote $*$, the only element of $F1$. For a set X and a point $x \in X$ define $c_x \colon 1 \to X$ as $c_x(0) = x$. Finally set

$$\mu_X(x) = Fc_x(*).$$

Note that any transformation $\mu \colon 1 \to F$ must satisfy previous condition because it is equivalent to the commuting square

$$\begin{array}{ccc} 1 & \xrightarrow{c_x} & X \\ {\scriptstyle \mu_1}\downarrow & & \downarrow{\scriptstyle \mu_X} \\ F1 & \xrightarrow{Fc_x} & FX \end{array}$$

At last, we have to prove that μ is natural. Let $f \colon X \to Y$ be arbitrary map and $x \in X$ be an arbitrary point. The following equations holds

$$Ff\mu_X(x) = Ff(Fc_x(*)) = F(fc_x)(*) = F(c_{f(x)})(*) = \mu_Y f(x).$$

and that proves that the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle \mu_Y} \\ FX & \xrightarrow{Ff} & FY \end{array}$$

commutes and μ_X is natural. □

Corollary 3.5. *Any transformation $\mu \colon 1 \to 1$ is identity.* □

Proposition 3.6. *A functor F is faithful if and only if there is a component of F with no distinguished element.*

Proof. Note that from previous lemma we know that for every component G of F there is a unique transformation $\mu \colon 1 \to G$ which is either monotranformation, or distinguished element. If at least one of the tranformations μ_i is mono it is obvious that F is faithful.

On the other hand if F is faithful then at least one of the components of F is faithful. Let G be such a component. We'll prove that $\mu \colon 1 \to G$ is mono. The component μ_X can be defined as $\mu_X(x) = Gc_x(*)$ (where $*$ is the only element in $G1$ and $c_x(0) = x$) so $\mu_X(x) = \mu_X(y)$ if and only if $Gc_x(*) = Gc_y(*)$ that is if and only if $Gc_x = Gc_y$ but that's not true for $x \neq y$ because G is faithful. □

Corollary 3.7. *A functor F is faithful if and only if identity is a subfunctor of F.* □

Example 3.8. (Power-set functor) We define power-set functor $\mathcal{P}$ as $\mathcal{P}(X) = \{A \subseteq X\}$ and for a map $f: X \to Y$, $\mathcal{P}f(A) = f[A]$. The power-set functor is not connected, and since $P(1) = \{0, 1\}$, it has two components. One of the componenets is C_1 which maps each set to $\{0\}$. The second one is $\mathcal{P}^{\neq 0}$ which maps each set to set of all its non-empty subsets. Functor $\mathcal{P}^{\neq 0}$ is defined on maps same as $\mathcal{P}$ (note that for non-empty A, $f[A]$ is always non-empty).

It's obvious that C_1 is not faithful but $\mathcal{P}$ is faithful hence the natural transformation $\rho: 1 \to \mathcal{P}^{\neq 0}$ is monotransformation. It's easy to see that ρ can be defined as $\rho_X(x) = \{x\}$.

Example 3.9. Another important example is the functor which is defined as $FX = X^n$ (where n is a fixed cardinal), and $Ff(x_i : i < n) = (f(x_i) : i < n)$ for a map $f: X \to Y$. It's in fact the covariant Hom-functor $\mathrm{Hom}(n, -)$.

This functor is connected (surely $|1^n| = 1$ for any n) and it is faithful. The unique monotransformation $\rho: 1 \to F$ is defined as $\rho(x) = (x : i < n)$ (i.e. the diagonal map).

PART I. Initial Algebras

4. Initial algebra construction

The initial algebra construction as well as most of this chapter is described in $[AT_1]$.

Let F be an endofunctor of a category $\mathcal{K}$. Define F-*algebra* as a pair (A, a) where A is an object of $\mathcal{K}$ and a is a morphism from FA to A. Sometimes we denote F-algebra (A, a) only as A, and if there couldn't be any misunderstanding we call F-algebras just algebras. A *homomorphism* h of F-algebras (A, a) and (B, b) is a morphism $h \in \mathcal{K}(A, B)$ such that the square

$$\begin{array}{ccc} A & \xleftarrow{\;a\;} & FA \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Fh} \\ B & \xleftarrow{\;b\;} & FB \end{array}$$

commutes, i.e. the category of F-algebras is a full subcategory of comma-category $\mathcal{K} \downarrow F$. We denote the category of F-algebras as F-**Alg**.

Dually, we can define F-coalgebras. F-*coalgebra* is a pair (A, a) of an object $A \in \mathcal{K}$ and a morphism $a \colon A \to FA$. A *homomorphism* of F-coalgebras (A, a) and (B, b) is a morphism $h \in \mathrm{Hom}(A, B)$ such that the following square commutes.

$$\begin{array}{ccc} A & \xrightarrow{a} & FA \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Fh} \\ B & \xrightarrow{b} & FB \end{array}$$

The category of F-coalgebras is denoted F-**Coalg**.

The notion of homomorphisms between algebras and coalgebras has appered in [AT$_3$] Having an algebra (A, a) and a coalgebra (B, b) we can also define *algebra to coalgebra homomorphism* (or just algebra-coalgebra homomorphism) as such a morphism $h \in \mathrm{Hom}(A, B)$ that the corresponding square commutes:

$$\begin{array}{ccc} A & \xleftarrow{a} & FA \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Fh} \\ B & \xrightarrow{b} & FB \end{array}$$

The *coalgebra-algebra homomorphism* can be defined the same way (but dually).

Example 4.1. (Universal algebra) The F-algebras are generalization of universal algebras. If we are given a signature Σ (the set of operations Σ and a map ar from Σ to ω or more generaly to **Card**, mapping each operation $f \in F$ to its arity) we take $\mathcal{K} = \mathbf{Set}$ and

$$FX = \coprod_{f \in \Sigma} X^{\operatorname{ar} f}.$$

If we have an universal algebra $(A, f_A : f \in \Sigma)$ we define an F-algebra (A, a) as

$$\forall \overline{x} = (x_1, \ldots, x_{\operatorname{ar} f}) \in X^{\operatorname{ar} f} : a(\overline{x}) = f_A(x_1, \ldots, x_{\operatorname{ar} f}).$$

If we have an F-algebra we get a universal algebra in the same manner.

It is easy to see that F-homomorphisms are exactly those of universal algebras.

Example 4.2. (β-algebras and compact Hausdroff spaces) Recall that functor β can be defined on **Set** as $\beta X = \{\mathcal{F} : \mathcal{F} \text{ is an ultrafilter on } X\}$ on objects and if $f\colon X \to Y$ then $A \in \beta f(\mathcal{F})$ if and only if $f^{-1}[A] \in \mathcal{F}$. The functor β is essetially Čech-Stone compactification of the discreet space.

β-algebras are pairs (X, l) where X is a set and $l\colon \beta X \to X$. Some of these algebras desribes compact Hausdorff spaces—if (X, τ) is a compact Hausdorff space then each ultrafilter $\mathcal{F}$ on X has exactly one limit (a point $x \in X$ is a limit of an ultrafilter if and only if each neighbourhood U of x is in the filter $\mathcal{F}$), hence we can define β algebra $(X, \lim)$ where X is underlying set of the space (X, τ) and $\lim\colon \beta X \to X$ is the map which maps each ultrafilter to its limit. Recall that a map $f\colon X \to Y$ of Hausdroff spaces is continuous if and only if $f(\lim \mathcal{F}) = \lim \beta f(\mathcal{F})$. Hence continuous maps of compact Hausdorff spaces corresponds to homomorphisms of β-algebras.

There are also some β-algebras which are not compact Hausdorff spaces. For example in each compact Hausdorff space holds that if $\mathcal{F} = \{F : F \ni x\}$ where $x \in X$ then $\lim \mathcal{F} = x$. Nevertheless, this condition need not to be satisfied in general β-algebra.

An *initial algebra* is the initial object of the category of F-algebras, i.e. an algebra I s.t. for any other algebra A there is exactly one homomorphism $h\colon I \to A$.

Note that an object $A \in \mathcal{K}$ is a *fixed point* of a functor $F\colon \mathcal{K} \to \mathcal{K}$, if $FA \simeq A$.

Lemma 4.3. *Initial F-algebra is a fixed point of F.*

Proof. Let (I, i) be the initial algebra we claim that i is isomorphism. If we consider F-algebra (FI, Fi) then there is exactly one homomorphism $h\colon I \to FI$. We'll prove that h is an inverse of i.

Following two squares commute.

$$\begin{array}{ccccc} FI & \xrightarrow{Fh} & F^2 I & \xrightarrow{Fi} & FI \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle Fi} & & \downarrow{\scriptstyle i} \\ I & \xrightarrow{h} & FI & \xrightarrow{i} & I \end{array}$$

Hence ih is a homomorphism from I to I and it must be identity because I is initial algebra. We have $ih = 1_I$. On the other hand we have $hi = FiFh$ from the left square, and $FiFh = F(ih) = F1_I = 1_{FI}$. $\square$

The initial algebra construction is the generalization of the proof of Knaster-Tarski theorem of the fixed point of non-decreasing map. This theorem was proved by Knaster in [Kn] and by Tarski in [Ta].

Theorem 4.4. (Knaster-Tarski) *Let L be a complete lattice and $f\colon L \to L$ be a non-decreasing map. Then f has the least fixed point. Precisely, there exists $x \in L$ such that $f(x) = x$, and $x \leq y$ for any other point y such that $f(y) = y$.*

Proof. We'll construct a non-decreasing sequence

$$\bot \leq f(\bot) \leq f^2(\bot) \leq \cdots \leq f^\omega = \sup_{\beta < \omega} f^\beta \leq f^{\omega+1}(\bot) = f(f^\omega(\bot)) \leq \ldots$$

by transfinite induction. Start with $\bot$, the least element of L.

On non-limit steps let $f^{\lambda+1}(\bot) = f(f^\lambda(\bot))$. Note that $f^\lambda(\bot) \leq f^{\lambda+1}(\bot)$. For a limit ordinal α let $f^\alpha(\bot) = \sup\{f^\beta(\bot) : \beta < \alpha\}$.

Finally let ι be the least ordinal such that $f^\iota(\bot) = f^{\iota+1}(\bot)$. Such ordinal has to exists because if $f^\lambda(\bot) < f^{\lambda+1}(\bot)$ holds for each λ then $\{f^\lambda(\bot) : \lambda \in \mathbf{Ord}\}$ is a proper class and a subset of L in the same time.

The point $x = f^{\iota}(\bot)$ is the least fixed point. Surely, it is a fixed point. If we have any other fixed point y we get that $f^{\lambda}(\bot) \leq y$ for each ordinal λ. The last claim can be proven by tranfinite induction: $\bot \leq y$ and if $f^{\lambda}(\bot) \leq y$ then $f^{\lambda+1} \leq f(y) = y$. The limit step is obvious because $f^{\lambda}(x)$ is defined as supremum of $f^{\alpha}(x)$'s for $\alpha < \lambda$ which satisfies $f^{\alpha}(x) \leq y$. □

To get the initial algebra construction we replace complete lattice L by cocomplete category $\mathcal{K}$, map f by endofunctor F of $\mathcal{K}$, and the least fixed point by initial algebra. In general categories this construction doesn't work so nicely as in lattices because the argument about the smallness cannot be used in categories, and the construction doesn't have to stop.

(The initial algebra construction) We suppose that $\mathcal{K}$ is a cocomplete[1] category and F an endofunctor of $\mathcal{K}$.

Initial chain is certain chain of the form

$$W_0 \xrightarrow{w_{0,1}} W_1 \xrightarrow{w_{1,2}} W_2 \longrightarrow \cdots \to \quad W_\omega \xrightarrow{w_{\omega,\omega+1}} W_{\omega+1} \longrightarrow \cdots,$$

i.e. a functor $W\colon \mathbf{Ord} \to \mathcal{K}$. Formally, we define *Initial chain* for a functor F as a cocontinuous functor $W\colon \mathbf{Ord} \to \mathcal{K}$ which satisfies $W(\alpha+1) = FW(\alpha)$.

The initial chain can be constructed by transfinite induction. We denote $W_\alpha = W(\alpha)$ and $w_{\alpha,\beta}$, the image of inclusion $\alpha \subseteq \beta$. W_0 is chosen as the initial object of $\mathcal{K}$, $W_1 = FW_0$, and $w_{0,1}$ is the only morphism from W_0 to W_1.

For ordinal successor λ we choose $W_{\lambda+1} = FW_\lambda$ and $w_{\lambda,\lambda+1} = Fw_{\lambda-1,\lambda}$. For the limit ordinal λ we take

$$W_\lambda = \operatorname*{colim}_{\alpha<\lambda} W_\alpha$$

where the colimit is taken with the connecting morphisms $w_{\alpha,\beta}$ for $\alpha \leq \beta < \lambda$. We define $W_{\lambda+1} = FW_\lambda$, and the connecting map $w_{\lambda,\lambda+1}$ is the factoring morphism for the cocone $Fw_{\alpha,\lambda}w_{\alpha,\alpha+1}$, $\alpha < \lambda$.

Proposition 4.5. *We say that the initial algebra construction converges in precisely π steps if π is the smallest ordinal such that $w_{\pi,\pi+1}$ is isomorphism. In that case, the construction is stalled (for each $\beta > \alpha \geq \pi$, $w_{\alpha,\beta}$ is an isomorphism), and $(W_\pi, w_{\pi,\pi+1}^{-1})$ is the initial algebra.*

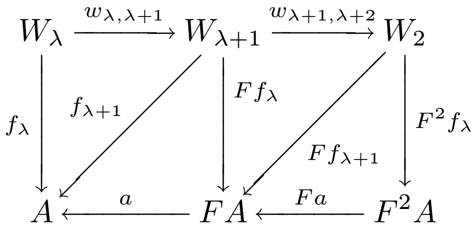
Proof. Let (A, a) be arbitrary F-algebra. We'll prove (by induction on λ) that for every ordinal λ there is a unique coalgebra-algebra morphism f_λ from coalgebra $(W_\lambda, w_{\lambda,\lambda+1}$ to algebra (A, a). It is obvious for $\lambda = 0$ because W_0 is initial object of $\mathcal{K}$, hence the square

$$\begin{array}{ccc} W_0 & \xrightarrow{w_{0,1}} & W_1 \\ {\scriptstyle f_0}\big\downarrow & & \big\downarrow{\scriptstyle Ff_0} \\ A & \xleftarrow{\;a\;} & FA \end{array}$$

[1] The presumption of cocompletness is not actually needed—we need only colimits of certain chains.

commutes ($aFf_0w_{0,1}$ is the unique morphism from W_0 to A and so is f_0).

For an ordinal succesor $\lambda + 1$, we define $f_{\lambda+1} = aFf_\lambda$. Both squares in the following diagram commutes, and so does both lower triangles (by the definition of $f_{\lambda+1}$).

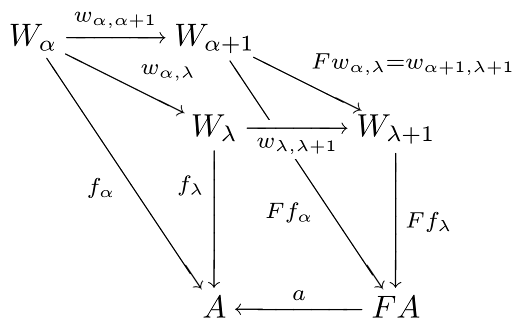


Hence whole diagram is commutative—especially the designed paleogram with both diagonals commutes, and $f_{\lambda+1}$ is coalgebra-algebra homomorphism.

It is also unique. If $f_{\lambda+1}$ was arbitrary coalgebra-algebra homomorphism we get that $f_{\lambda+1}w_{\lambda,\lambda+1}$ is coalgebra-algebra homomorphism from W_λ to A, hence from the induction prerequisite $f_{\lambda+1}w_{\lambda,\lambda+1} = f_\lambda$ holds which precisely says that the upper triangles commute. Using the presumption that $f_{\lambda+1}$ is homomorphism and the right upper triangle we get that

$$f_{\lambda+1} = aFf_{\lambda+1}w_{\lambda+1,\lambda+2} = aFf_\lambda.$$

If λ is limit ordinal we define f_λ as the factoring morphism for the cocone $\{f_\alpha : \alpha < \lambda\}$. For any $\alpha < \lambda$ the upper palleogram and both triangles in the following diagram commute,



hence the front square commutes as well. That proves that f_λ is a homomorphism.

The uniqueness of f_λ is given by uniqueness of factoring morphism. If f_λ is any coalgebra-algebra homomorphism. Note that $w_{\alpha,\lambda}$ is coalgebra homomorphism, hence the composition $f_\lambda w_{\alpha,\lambda}$ is coalgebra-algebra homomorphism and by uniqueness of f_α we get that $f_\alpha = f_\lambda w_{\alpha,\lambda}$. So f_λ is the unique factoring morphism.

Finally, remark that any homomorphism of F-algebras $(W_\pi, w^{-1}_{\pi,\pi+1})$ and (A, a) is in fact a coalgebra-algebra homomorphism of $(W_\pi, w_{\pi,\pi+1})$ and (A, a) and vice-versa. $\square$

From previous prosposition we know that if the construction stops for a functor F then there is the initial F-algebra and from lemma 4.3 we know that the initial algebra is a fixed point of the functor F. Hence we know that if the construction stops there is a fixed point of F. The converse is not generaly true (see the following examples) but it is true in the category of sets.

Both of these examples appears in book [AT_1].

Example 4.6. (Initial F-algebra exists but the initial chain doesn't stop) Let $\mathcal{K} = \mathbf{Ord} \cup \{\infty\}$ is the category of all ordinals with one more object ∞ which is greater then any ordinal λ. $\mathcal{K}$ is narrow category with morphism from λ to κ if and only if $\lambda \leq \kappa$ (λ and κ are ordinals or ∞).

We define endofuntor F of $\mathcal{K}$ as

$$F\lambda = \begin{cases} \lambda + 1 & \text{if } \lambda \text{ is a morphism,} \\ \infty & \text{if } \lambda = \infty. \end{cases}$$

It's obvious that the initial algebra goes as follows

$$W_0 = 0 \longrightarrow W_1 = 1 \longrightarrow W_2 = 2 \longrightarrow \cdots \to W_\omega = \omega \longrightarrow W_{\omega+1} = \omega{+}1 \longrightarrow \cdots,$$

i.e. $W_\lambda = \lambda$ for each ordinal λ. Hence the construction doesn't stop. But $\infty = F\infty$ is the only F-algebra and hence it is initial.

Example 4.7. (Functor F has a fixed point but there is no initial F-algebra) We'll construct this example in the category of partial grupoids. Object of this category are partial grupoids, i.e. pairs $(X, \cdot)$ where X is a set and $\cdot$ is a partial map from X^2 to X. Morphisms from $(X, \cdot)$ to $(Y, \circ)$ are those maps $f\colon X \to Y$ which satisfies $f(x) \circ f(y) = f(x \cdot y)$ whenever $x \cdot y$ is defined. Let T be a grupoid with one element x such that $x \cdot x = x$.

We define an endofunctor F of this category on objects as

$$F(X, \circ) = \begin{cases} T & \text{if } \circ \text{ is non-empty,} \\ (2^X, \emptyset) & \text{if } \circ = \emptyset. \end{cases}$$

The definition of F on objects determines Ff to be a constant map to x whenever f is morphism to $(Y, \circ)$ where $\circ$ is non-empty operation. And for a morphism $f\colon (X, \emptyset) \to (Y, \emptyset)$ we define $Ff(A) = f[A]$.

Grupoid T is obviously fixed point od F. But F has no initial algebra because if $(I, \circ)$ is an initial algebra, then $\circ = \emptyset$ (otherwise, there would be no morphism from I to any partial grupoid with empty operation) but no grupoid with empty operation is fixed point of F, since $|FX| > |X|$ for any grupoid $(X, \emptyset)$.

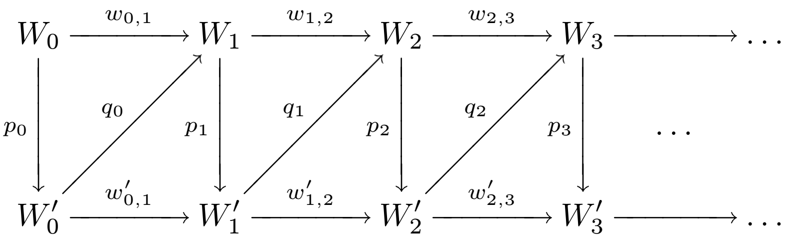
Lemma 4.8. *For each endofunctor F of* **Set** *such that $F0 \neq 0$ there is an endofunctor F' of* **Set** *such that F' preserves all monomorphisms and the initial chain for F' consist only of monomorphisms and both chains coincides from step ω onwards.*

Proof. Define F' as the standartization of F, i.e. $F'X = FX$ for all non-empty sets X and $F'0$ is the set of all distinguished points of F. An morphisms with non-empty domains F' is defined the same way as F. And if $f\colon 0 \to Y$ then

$$F'f\colon \rho \mapsto \rho_Y$$

for any distinguished point ρ of F. It's easy to see that $F'f$ is monomorphism, hence F' preserves all monomorphisms because those with non-empty domains are preserved by any set functor.

Let W denote initial chain for F and W' denote initial chain for F'. We'll define two natural transformation p and q between the initial segments $W|_\omega$ and $W'|_\omega$ of W and W'.



The maps p_0 and q_0 are empty maps. Define $p_1(x) = \rho$ where ρ is such distinguished element that for a map $f\colon 0 \to 1$ holds $\rho_1 = Ff(x)$. Furthermore define $p_{n+1} = Fp_n = F'p_n$ for any $n \leq 1$ and $q_{n+1} = Fq_n = F'q_{n+1}$ for any $n \leq 0$. Observe that $q_1 p_1 = w_{1,2}$ and $p_1 q_0 = w'_{0,1}$.

It holds

$$q_1 p_1(x) = \rho_{W_1}$$

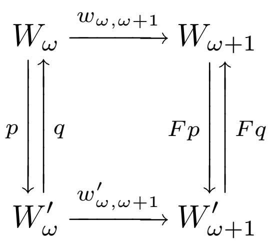
where ρ is such that $Ff(x) = \rho_1$ but it is easy to see that then for any map $g\colon 0 \to Y$ holds $Fg(x) = \rho_Y$ because ρ is distinguished. Especially $w_{1,2}(x) = Fw_{0,1}(x) = \rho_{W_1}$. The second condition $p_1 q_0 = w'_{0,1}$ is obvious because both sides are empty maps.

By induction we know that $q_n p_n = w_{n,n+1}$ and $p_{n+1} q_n = w'_{n,n+1}$ for any $n < \omega$. Define $p\colon W_\omega \to W'_\omega$ and $q\colon W'_\omega \to W_\omega$ as $p = \lim p_n$ and $q = \lim q_n$. Then p and q are mutually inverse:

$$pq = \lim_{n<\omega} p_n q_n = \lim_{n<\omega} w_{n,n+1} = 1_{W_\omega},$$

and the same holds for qp.

Finally the square



commutes, because $p_{n+1} = Fp_n$ and $q_{n+1} = Fq_n$. And p, q can be extended to natural isomorphisms of chains $W_\omega \to W_{\omega+1} \to \cdots$ and $W'_\omega \to W'_{\omega+1} \to \cdots$. $\square$

Theorem 4.9. *Let F be an endofunctor of* **Set**. *The following conditions are equivalent*

(1) *The initial chain for F converges.*

(2) *Initial F-algebra exists.*
(3) *There exists a fixed point of F.*

Proof. The implications (1) → (2) and (2) → (3) are corollaries of proposition 4.5 and lemma 4.3.

We only need to prove (3) → (1). If $F0 = 0$ then the statement is obvious. Without loss of generality we can assume that F is standard and the initial chain consists only of monomorphisms, otherwise we use lemma 4.8. Suppose that X is the fixed point of F and x an isomorphism $X \simeq FX$.

(X, x) is an F-algebra. Let $f_\alpha : W_\alpha \to X$ be the unique cocone. First prove that all f_α are monomorphisms, f_0 is. And if f_λ is a monomorphism then so is $f_{\lambda+1} = xFf_\lambda$ because x is an isomorphism and Ff_λ is monomorphism.

For a limit ordinal λ, W_λ is the union of W_α, $\alpha < \lambda$ and f_λ is the union of f_α, $\alpha < \lambda$ and hence mono because ny two points $a, b \in W_\lambda$ lies already in W_α for some $\alpha < \lambda$, and $f_\lambda(a) = f_\alpha(a) \neq f_\alpha(b) = f_\lambda(b)$.

Finally all W_λ are subobjects of X, hence for some α and $\beta \in \mathbf{Ord}$, $W_\alpha \simeq W_\beta$ and $w_{\alpha,\beta}$ is an isomorphism because **Set** is well-powered. In that case also $w_{\alpha,\alpha+1}$ is isomorphism and the construction stops in at most α steps. □

5. Length of the initial algebra construction

The length of the convergence has been recently studied by several authors. In [W] author proves that for finitary set functors the dual construction of terminal coalgebra in the category of sets stops after $\omega+\omega$ steps. The initial construction was then studied in the category of sets and many-sorted sets in [AT_2]. They have shown that the initial algebra construction can stop after arbitrary ordinal in many-sorted sets and that the same construction in sets stops after at most 3, or exactly κ steps where κ is a regular cardinal.

Here we show the proof of the second proposition and then we investigate the relation between the length of the convergence and the size of the initial algebra.

To prove the theorem about length of the construction in sets we will need some results about set functors which have been shown in [Ko]. However, the following lemma is from [AT_2].

Lemma 5.1. *If F is an endofunctor of* **Set** *and $1 < k < \omega$ such that*

$$Fk - \bigcup_{f\colon (k-1)\to k} \operatorname{im} Ff \neq 0$$

then there for any finite $n > k$ holds $|Fn| \geq |F1| + n - k + 1$.

Proof. Let

$$a \in Fk - \bigcup_{f:(k-1)\to k} \operatorname{im} Ff \neq 0.$$

Suppose that $n > k$. For each inclusion $m\colon k \to n$ we have a point $Fm(a) \in Fn$. All these points are pairwise different because if m and m' are two such inclusions then $Fm(a) \notin Fm \cap Fm'$. If $m \cap m'$ is empty then there are only distinguished elements in $Fm \cap Fm'$ and a is not one. On the other hand, if $m \cap m'$ is non-empty then F preserves this intersection and $m \cap m'$ has at most $k-1$ elements, hence $Fm(a) \notin Fm \cap Fm'$.

Moreover none of the $Fm(a)$'s lies in $\operatorname{im} f$ for any $f\colon 1 \to X$—especially for each component G of F there is at least one point different from all $Fm(a)$'s; hence

$$|Fn| \geq |F1| + \binom{n}{k}.$$

Finally, the inequality $\binom{n}{k} \geq n - k + 1$ can be easily proved by induction.

Proposition 5.2. (Koubek, 1971) *If F is a set functor and X a set such that $|FX| < |X|$ then F is constant on subcategory of sets with lesser cardinality then $|X|$.* $\square$

Theorem 5.3. (Adámek, Trnková, 2010) *Let F be an endofunctor of* **Set** *such that the initial chain converges in π steps then π is either at most 3, or an infinite regular cardinal.*

Proof. We disect the proof to several steps. In first step, we discuss the finite π, and in other steps, $\pi > \omega$. In the case $\pi = \omega$, there is nothing to prove, because ω is a regular cardinal.

Denote

$$\overline{W}_\lambda = W_{\lambda+1} - \operatorname{im} w_{\lambda,\lambda+1},$$

and note that $\overline{W}_\lambda = 0$ if and only if the construction stops in at most λ steps.

(1) If π is finite we want to prove that $\pi \leq 3$. Suppose that $F0 \neq 0$, otherwise the statement is obvious.

We claim that F is not faithful, if it would be then both 1 and C_1 (since $F0 \neq 0$) are subfunctors of F; hence $|FX| \geq |X|+1$ for any finite X. Let r denote number of components of F.

(a) If $|Fr| > r$ and r is finite then there is a component G of F such that $|Gr| > 1$; hence there is a point a outside of image of any monomoprhism from 1 to r. That implies that some $1 < k \leq r$ satisfies presumptions of previous lemma. Hence for any finite set X holds $|FX| \geq r + |X| - k + 1 \geq |X| + 1$.

If all W_n for finite n are finite then the construction doesn't stop in less then ω steps because none of finite sets is a fixed point of F. If at least one of W_n is infinite then $\overline{W}_n$ is infinite for some n and we can use part (4). The same holds, if r is infinite then $W_2 = FW_1 = Fr$ has greater cardinality then W_1, and hence $\overline{W}_1$ is infinite.

(b) If $|Fr| \leq r$ then obviously $|Fr| = r$ beacause $r \neq 0$ and each of r components of F maps r to a non-empty set. Furthermore, in this case the set of all r distinguished points of F is the initial algebra because $F0 \neq 0$ and for any $k < r$ holds $|Fk| = r$. The set Fr contains only distinguished points; hence we can define the operation o on r as $o(\rho_r) = \rho$.

We have several possibilities. First, if $|F0| \leq r$ then for any W_n holds $|W_n| \leq r$. But any distinguished point of F lies in the image of map $w_{2,3}$; hence $|W_3| = r$ and obviously $w_{3,4}$ is isomorphism. Secondly, if $|FF0| < |F0|$ then (as proved in [Ko]) F is constant on subcategory of sets of lesser cardinality then $F0$, especially (since $|W_2| < |W_1|$) $w_{3,4}$ is isomorphism. Finally, if $|FF0| \geq |F0| > r$ then $|W_n| > r$ for all finite n, hence the construction doesn't stop in finitely many steps.

(2) Suppose that $\pi > \omega$ and claim that W_ω is infinite.

We can assume (without loss of generality) that F is standart and the initial chain consists only of monomorphisms.

For contradiction suppose that W_ω is finite. Because F is standart, $W_\omega = \bigcup_{n<\omega} W_n$. Hence there exists $n < \omega$ such that $w_{n,\omega}$ is isomorphism and $\pi = n < \omega$.

(3) If all $\overline{W}_n$ are finite for $n < \omega$ then $\overline{W}_\omega$ is infinite.

Note that $\pi > \omega$, hence there is $x \in \overline{W}_\omega$. Further, let $f_i\colon W_\omega \to W_\omega$, $i < \omega$ be pairwise disjoint monomorphisms. We claim that for any f_i, $Ff_i(x)$ is in $\overline{W}_\omega$. If that's not true then $Ff_i(x) = Fw_{n,\omega}(y)$ for some $n < \omega$. Note that $V = w_{n,\omega} \cap f_i$ is finite and preserved by F (since it's non-empty) and $x \in Ff_i^{-1}[FV]$ but that's not possible because x doesn't lie in the image of any finite subobject of W_ω. If $x \in Fh[FV]$ for some finite $V \subseteq W_\omega$ and h monomorphism then $\operatorname{im} h \subseteq \operatorname{im} w_{n,\omega}$ for some finite n and $x \in \operatorname{im} Fh \subseteq \operatorname{im} Fw_{n,\omega} \subseteq \operatorname{im} w_{\omega,\omega+1}$.

Finally, $Ff_i(x)$ are not distinguished; hence they are pairwise distinct and $\{Ff_i(x) : i < \omega\}$ is an infinite subset of $\overline{W}_\omega$.

(4) If $\overline{W}_\lambda$ is infinite then $|\overline{W}_{\lambda+1}| \geq |\overline{W}_\lambda|$.

From $\pi > 1$ we know that there is $x \in FF0$ which is not distinguished. Let G be component of F such that $x \in GF0$. Consequently, $|GX| \geq |X|$ for any infinite X such that $|X| \geq |F0|$.

Let $m\colon \overline{W}_\lambda \to W_{\lambda+1}$ denote an inclusion then m and $w_{\lambda,\lambda+1}$ are disjoint, hence $Gm \cap Gw_{\lambda,\lambda+1}$ contains only distinguished elements but G has at most one.

Hence

$$|\overline{W}_{\lambda+1}| \leq |Gm| - 1 \leq |G\overline{W}_\lambda| - 1 \leq |\overline{W}_\lambda| - 1 \leq |\overline{W}_\lambda|$$

because $\overline{W}_\lambda$ is infinite and $|\overline{W}_\lambda| \geq |F0|$.

(5) If $\alpha < \pi$ is limit and there is k such that the cardinalities of $\overline{W}_\lambda$ are non-decreasing and infinite for $k < \lambda < \alpha$ then $|\overline{W}_\alpha| \geq |\overline{W}_\lambda|$ for each λ, $k < \lambda < \alpha$.

We'll construct monomorphisms $s_i\colon W_\lambda \to W_\lambda$, $i < |\overline{W}_k|$ with the following properties:

(i) $s_i^{-1}[\operatorname{im} w_{j,\lambda}] \subseteq \operatorname{im} w_{j,\lambda}$ for each i and $j < \lambda$,
(ii) $s_i \cap s_j \subseteq w_{k,\lambda}$ for each i, j,
(iii) s_i is identity on W_k for each i.

We define s_i on $\operatorname{im} w_{j,\lambda}$ inductively by j. $s_i|_{W_k} = 1_{W_k}$ and for j, $k < j < \lambda$ we know that $|\overline{W}_j| \geq |\overline{W}_k| \geq \omega$; so we can define $s_i|_{\overline{W}_j}$ as $|\overline{W}_k|$ disjoint monomorphisms.

If $x \in \overline{W}_\lambda$ we claim that $Fs_i(x)$, $i < |W_k|$ are pairwise distinct elements of $\overline{W}_\lambda$. Surely, each $Fs_i(x)$ lies in $\overline{W}_\lambda$, otherwise $Fs_i(x) \in \operatorname{im} Fw_{j,\lambda}$ and $x \in Fs_i^{-1}[\operatorname{im} Fw_{j,\lambda}]$ for some $j < \lambda$ but $s_i^{-1}[\operatorname{im} Fw_{j,\lambda}] \subseteq \operatorname{im} Fw_{j,\lambda}$ which is the contradiction with $x \in \overline{W}_\lambda$.

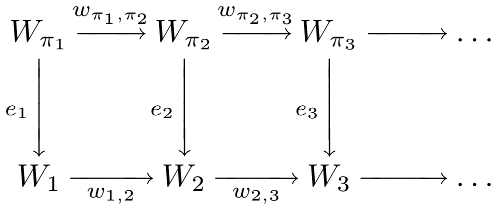
If $y = Fs_i(x) = Fs_j(x)$ for some $i \neq j$ then $y \in \operatorname{im} F(s_i \cap s_j) \subseteq Fw_{k,\lambda}$ because F preserves non-empty intersection $s_i \cap s_j$. And it's the contradiction with $y \in \overline{W}_\lambda$. Hence we found subset $\{Fs_i(x) : i < |\overline{W}_k|\}$ of $\overline{W}_\lambda$ with cardinality $|\overline{W}_k|$ which means that $|\overline{W}_k| \leq |\overline{W}_\lambda|$.

If $k \leq \omega$ such that $|\overline{W}_k|$ is infinite (such k exists from (3)) then from (4) and (5) we know that $|\overline{W}_\lambda|$ is non-decreasing for $k \leq \lambda < \pi$.

(6) π is regular cardinal.

From part (4) we already know that π is limit ordinal. Suppose that $\pi_i, i < \operatorname{cf} \pi$ is cofinal sequence in π such that for limit α, $\pi_\alpha = \sup_{k<\alpha} \pi_k$ and $\overline{W}_{\pi_1}$ is infinite.

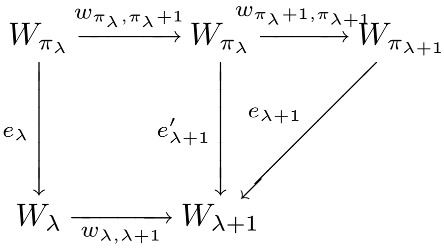
We'll construct epimorphisms $e_i\colon W_{\pi_i} \to W_i$, $1 \leq i < \operatorname{cf} \pi$ which form natural transformation of the corresponding chains, i.e. the following commutes.



Note that $|W_{\pi_1}| \leq |W_1| > 0$, hence we can choose e_1 to be an arbitrary epimorphism. On limit ordinals $\lambda + 1$ choose arbitrary epimorphism $\varepsilon\colon \overline{W}_{\pi_\lambda} \to \overline{W}_\lambda$ (such epimorphism exists because either $\overline{W}_\lambda$ is finite and $\overline{W}_{\pi_\lambda}$ is infinite, or $\overline{W}_\lambda$ is infinite and then $|\overline{W}_{\pi_\lambda}| \geq |\overline{W}_\lambda|$ from parts (4) and (5)), and define $e'_{\lambda+1}\colon W_{\pi_\lambda+1} \to W_{\lambda+1}$ such that

$$e'(x) = \begin{cases} w_{\lambda,\lambda+1} e_\lambda(y) & \text{if } x \in \operatorname{im} w_{\pi_\lambda,\pi_\lambda+1}, \\ \varepsilon(x) & \text{if } x \in \overline{W}_{\pi_\lambda}. \end{cases}$$

Such e' is epimorphism and the left square in the following diagram commutes.



Finally, define $e_{\lambda+1}$ such that the right triangle commutes, i.e. $e_{\lambda+1}$ is $e'_{\lambda+1}$ on $\operatorname{im} w_{\pi_\lambda+1,\pi_{\lambda+1}}$. For limit λ define e_λ as the colimit of e_α, $\alpha < \lambda$.

Denote $e = \lim_{\lambda < \operatorname{cf}\pi} e_\lambda$. We'll prove that $w_{\operatorname{cf}\pi,\pi}$ is epimorphism, in that case equality $\operatorname{cf}\pi = \pi$ holds. Let $x \in W_{\operatorname{cf}\pi+1}$ be arbitrary. Fe is epimorphism; hence there is $y \in W_{\pi+1}$ such that $Fe(y) = x$. But since $w_{\pi,\pi+1}$ is isomorphism then $y \in \operatorname{im} Fw_{k,\pi}$ for some $k < \pi$. Especially $y \in \operatorname{im} Fw_{\pi_i,\pi}$ for any $\pi_i > k$. There are i and z such that $y = Fw_{\pi_i,\pi}(z)$. Hence

$$x = Fe(y) = Fe\big(Fw_{\pi_i,\pi}(z)\big) = Fw_{i,\operatorname{cf}\pi} Fe_i(z)$$

and $x \in \operatorname{im} Fw_{i,\operatorname{cf}\pi} \subseteq \operatorname{im} w_{\operatorname{cf}\pi,\operatorname{cf}\pi+1}$. □

In the rest of this chapter we study the dependence of the length of the initial algebra construction on the size of the initial algebra. Althougth, the following theorem is new, it is direct corollary of the previous one.

Theorem 5.4. *Let F be an endofunctor of* **Set** *s.t. the initial F-algebra has cardinality exactly κ and the initial chain converges in π steps. If κ is infinite then $\pi \leq \kappa$, if κ is finite then π is either ω, or $\pi \leq 3$.*

Proof. For contradiction suppose that $\pi > \kappa$ and κ is infinite, that is (by previous theorem) $\pi \geq \kappa^+$. Without loss of generality we can assume that the initial chain is composed of monomorphisms.

Consider the part of the initial chain from κ to κ^+.

$$W_\kappa \xrightarrow{w_{\kappa,\kappa+1}} W_{\kappa+1} \xrightarrow{w_{\kappa+1,\kappa+2}} W_{\kappa+2} \longrightarrow \ldots \longrightarrow W_{\kappa^+}$$

For each λ in the interval $[\kappa, \kappa^+)$ we choose a point $x_\lambda \in W_{\lambda+1} - w_{\lambda,\lambda+1}[W_\lambda]$. We claim that all $w_{\lambda+1,\kappa^+}(x_\lambda)$ are pairwise distinct. Surely W_{κ^+} is a union of W_λ for $\kappa^+ > \lambda \geq \kappa$ because all $w_{\lambda,\lambda+1}$ are monomorphisms. Further, if $\alpha > \beta$ then $w_{\alpha+1,\beta+1}(x_\alpha) \neq x_\beta$ because $w_{\alpha+1,\beta+1}(x_\alpha) \in \operatorname{im} w_{\beta,\beta+1}$ and $x_\beta \notin \operatorname{im} w_{\beta,\beta+1}$.

Hence the set $\{w_{\lambda+1}(x_\lambda) : \kappa \leq \lambda < \kappa^+\}$ is a subset of W_{κ^+} of cardinality κ^+ which is the contradiction with the size of the initial F-algebra.

If κ is finite then W_ω is finite. We know that the initial chain for F coincides with initial chain of its standardization from the step ω onwards. We can suppose that F is standard, and the chain consists only of monomorphisms. Since W_ω is finite and $W_\omega = \bigcup_{n<\omega} W_n$, we get that $W_\omega = W_n$ for some finite n, so the construction (for the standardization) stops in at most n steps. Especially $w_{\omega,\omega+1}$ is an isomorphism. The initial construction for the original functor stops in at most ω steps. ☐

Example 5.5. It is easy to give an example of functor F such that the initial algebra is finite and the construction stop precisely after ω steps. Define $F0 = \omega$, for non-empty set X let

$$FX = \{A \subset X : |A| \geq \omega\} \cup \{*\},$$

and for any map $f\colon X \to Y$ with non-empty domain define $Ff(*) = *$ and

$$Ff(A) = \begin{cases} f[A] & \text{if } f[A] \text{ is infinite,} \\ * & \text{otherwise.} \end{cases}$$

For empty map $f\colon 0 \to X$ define $Ff(x) = *$ for all $x \in F0$.

The construction goes as follows. First $W_0 = 0$ and $W_1 = F0 = \omega$. Note that W_1 is infinite and for any infinite set X holds $|FX| > |X|$; hence the cardinality of W_n increases with $n < \omega$.

On morphisms the contruction looks a lot different. The morphism $w_{0,1}$ is the empty morphism and $w_{1,2} = Fw_{0,1}$ is constant on $*$. Moreover, $w_{2,3} =$

$Fw_{1,2}$ is also constantly $*$, etc. Consequently, it's easy to see that $W_\omega = \{*\}$, $W_{\omega+1} = \{*\}$, $w_{\omega,\omega+1}$ is isomorphism, and the construction stops in precisely ω steps.

These examples of functors for which the size of the algebra is finite and the construction stops in 1, 2, or 3 step are from [AT$_2$].

Example 5.6. Define F as $F0 = 3$ and $FX = \{A \subseteq X : |A| = 3\} \cup \{*\}$. For any map $f\colon X \to Y$ we define

$$Ff(x) = \begin{cases} f[x] & \text{if } f \text{ is monomorphism and } x \text{ is a three pointed subset of } X, \\ * & \text{otherwise.} \end{cases}$$

It's easy to see that $W_1 = 3$, $|W_2| = |\{W_1, *\}| = 2$, and $W_n = \{*\}$ for any $n > 2$; hence the construction stops after 3 steps.

For the construction of length 2 just take constant functor $C_{k,l}$ (i.e. $C_{k,l}0 = k$ and $C_{k,l}X = l$ for non-empty X) where $k \neq 0$. Then obviously $W_1 = k$ and $W_n = l$ for any $n > 1$. The similar example works for the length 1—if we take functor $C_{k,k}$ then $W_n = k$ for each n.

The following examples requires generalized continuum hypothesis. We'll use a theorem which describes cardinal exponentiation under GCH (it can be found for example in [BŠ]).

Theorem 5.7. (GCH) *Let κ and λ be two cardinals such that at least one of them is infinite then*

$$\kappa^\lambda = \begin{cases} \kappa^+ & \textit{if } \operatorname{cf}\kappa \leq \lambda < \kappa, \\ \kappa & \textit{if } \lambda < \operatorname{cf}\kappa, \\ \lambda^+ & \textit{if } \kappa \leq \lambda. \end{cases}$$

Example 5.8. (GCH) Let κ be a regular cardinal. We'll construct a functor F such that the initial chain converges in κ steps and the size of the initial algebra is κ. Simply let

$$FX = \coprod_{\lambda<\kappa} X^\lambda$$

Then the smallest fixed point of F is κ because for any $\lambda < \kappa$, we have $|F\lambda| \geq \lambda^\lambda > \lambda$ and it's easy to see that

$$|F\kappa| = \sum_{\lambda<\kappa} \kappa^\lambda \overset{\text{(GCH)}}{=} \sum_{\lambda<\kappa} \kappa = \kappa.$$

As noted in [AT$_1$] the construction stops in π steps, where π is the first regular cardinal bigger than all the arities; hence $\pi = \kappa$.

Example 5.9. (GCH) Let κ be any cardinal and $\lambda \leq \operatorname{cf}\kappa$ be a regular cardinal. Define functor F as follows

$$FX = \coprod_{\alpha<\lambda} X^\alpha \cup \coprod_{\alpha<\lambda} 1$$

then the size of the initial algebra is obviously κ because for any $\alpha < \kappa$ holds $|F\alpha| = \kappa$ and $|F\kappa| = \kappa$. Furthermore, as in the previous example, the initial algebra construction stops in λ steps because it is the least regular cardinal greater than all arities.

So (under GCH) we have proved for the size of initial algebra κ that the construction can stop in arbitrary regular cardinal which is lesser then $\operatorname{cf}\kappa$. If κ is not regular we know that the construction must stop after some regular cardinal λ which is smaller then κ. The natural choice of such cardinal would be $\operatorname{cf}\kappa$ as the above example concurs. It leads to a hypothesis: (Note that we have already proven the hypothesis for regular cardinal κ, and it's obvious for finite κ.)

Hypothesis 5.10. *Let F be a standart endofunctor of* **Set** *such that the inital algebra has cardinality exactly κ then the initial algebra construction stops in at most $\operatorname{cf}\kappa$ step. (We define $\operatorname{cf} n = 1$ for any $0 < n < \omega$, and $\operatorname{cf} 0 = 0$.)*

Further, we investigate how many steps does the construction take before the size of the initial algebra is obtained. Usually W_λ has the cardinality of the initial algebra much sooner then the construction stops (see functors above where the size is obtained in 1 or 2 steps but the construction then follows another κ steps). In the following example the size of the initial algebra is obtained in (much) more steps—the theorem below shows that the number of steps is in fact maximal possible.

Example 5.11. (GCH) For any regular cardinal $\kappa = \aleph_\alpha$ we can define subfunctor $P^{<\kappa}$ of power-set functor as

$$P^{<\kappa}X = \{A \subseteq X : |A| < \kappa\}.$$

Let W be initial chain for $P^{<\kappa}$ then W_n are finite for $n < \omega$, $W_\omega = \omega$. $|W_{\lambda+1}| = |PW_\lambda| = |W_\lambda|^+$ for infinite λ such that W_λ has cardinality lesser than κ. Then it's obvious that the cardinality of the initial algebra is obtained in exactly $\omega+\alpha$ steps.

Proposition 5.12. *Let F be such endofunctor of* **Set** *that the initial F-algebra has cardinality exactly $\aleph_\alpha$ then the initial algebra construction obtains this cardinality in at most $\omega + \alpha$ steps. Furthermore, if W_i is the initial chain then for each $\beta \leq \alpha$, $W_{\omega+\beta}$ has the cardinality at least $\aleph_\beta$.*

Proof. Note that W_ω is infinite, otherwise the construction would stop at the step ω with the finite initial algebra W_ω. And suppose that F is standart, i.e. that all conection morphisms of the initial chain are monomorphisms. We'll prove that W_λ's have increasing cardinalities for $\omega \leq \lambda$ until it reaches the cardinality $\aleph_\alpha$.

We only need to prove that if $|W_\lambda| = |FW_\lambda|$ then $|W_\lambda| = \aleph_\alpha$ but that's true because the initial algebra is the smallest fixed point of the functor F. The cardinalities of W_λ are strictly increasing because if $\lambda < \kappa$ then $\lambda + 1 \leq \kappa$ and $|W_\lambda| < |W_\lambda + 1| \leq |W_\kappa|$.

Finally, we prove by induction that $|W_{\omega+\lambda}| \geq \aleph_\lambda$ for each $\lambda \leq \alpha$. We know that for $\lambda = 0$. If $\lambda + 1$ is ordinal succesor then

$$\aleph_\lambda \leq |W_{\omega+\lambda}| < |W_{\omega+\lambda+1}|;$$

hence $|W_{\omega+\lambda+1}| \geq \aleph_\lambda^+ = \aleph_{\lambda+1}$. For limit ordinal λ we know that for each $\beta < \lambda$:

$$\aleph_\beta \leq |W_{\omega+\beta}| < |W_{\omega+\lambda}|,$$

and that implies $|W_{\omega+\lambda}| \geq \aleph_\lambda$. $\square$

6. Adjunction and algebras

In this chapter, we investigate the possibilities of pushing the initial algebra construction to another category via some functor G. The interesting case will be when G is a left adjoint. In that case we have a "reverse" functor U which pushes back algebras.

The author believes that the following propositions can be used to solve some convergence issues of the initial chain in the categories like $\mathbf{Set}^{op}$ where there is a little known about the convergence of construction by using suitable adjunction (in case of $\mathbf{Set}^{op}$ it may be some contravariant adjunction), and pushing the construction to a category where the convergence issues have been already solved (for example category $\mathbf{Set}$).

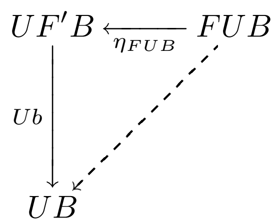
The following lemma is obvious corollary of the fact that the initial chain is defined as cocontinuous functor W which satisfies $FW = W+$ where $+$ is an endofunctor of the category of ordinals such that $+(\lambda) = \lambda + 1$ for each λ.

Proposition 6.1. *Let F and F' be an endofunctors of $\mathcal{K}$ and $\mathcal{H}$ respectively and let $W: \mathbf{Ord} \to \mathcal{K}$ be the initial chain for F. Suppose that we have a cocontinuous functor $G: \mathcal{K} \to \mathcal{H}$ which satisfies $GF = F'G$ then $GW: \mathbf{Ord} \to \mathcal{H}$ is the initial chain for F'.*

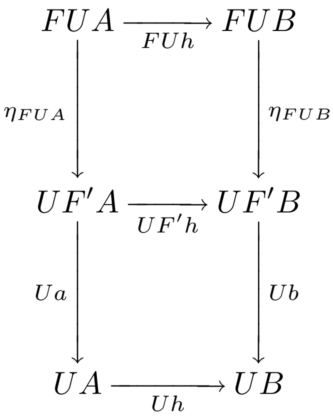
Especially, if the initial chain W converges so does the initial chain for F'. And if (I, i) is initial F-algebra then (GI, Gi) is initial F'-algebra. □

Proposition 6.2. *Suppose that we have adjoint functors $G \dashv U: \mathcal{K} \to \mathcal{H}$ and an endofunctor F of $\mathcal{K}$. Define new endofunctor F' of $\mathcal{H}$ as $F' = GFU$. Then there is a functor $\tilde{U}: F'\text{-}\mathbf{Alg} \to F\text{-}\mathbf{Alg}$ such that $\operatorname{im} \tilde{U}$ is the set of all F-algebras of the form (UB, a) where $B \in \mathcal{H}$ and a is arbitrary. Moreover, $\tilde{U}$ is full.*

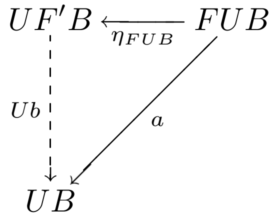
Proof. Let $\eta: 1 \to UG$ be the unit of adjunction $G \dashv U$, and define $\tilde{U}(B, b) = (UB, Ub \circ \eta_{FUB})$.



On morphism $\tilde{U}$ is defined as $\tilde{U}f = Uf$. For any $h: (A, a) \to (B, b)$, morphism of F'-algebras, $Uh: UA \to UB$ is morphism of F-algebras because the following diagram commutes.



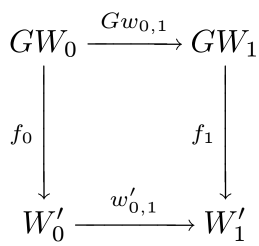
Now, suppose that we have an F-algebra (UB, a). Then by the universal property of the unit of adjunction η there exists $b \in \mathrm{Hom}_{\mathcal{H}}(GFUB, B) = \mathrm{Hom}(F'B, B)$ such that the following triangle commutes.



I.e. $\tilde{U}(B, b) = (UB, a)$. $\square$

Proposition 6.3. *Suppose that $G \dashv U$ is adjoint situation which is same as in the above theorem, F is an endofunctor of $\mathcal{K}$, and $F' \colon \mathcal{H} \to \mathcal{H}$ is defined as $F' = GFU$. Let W denote initial F-chain and W' denote initial F'-chain. Then there is natural transformation $f \colon GW \to W'$.*

Proof. We'll define $f_n \colon GW_n \to W'_n$ by transfinite induction. Note that functor G preserves colimits—especially G-image of initial object I of $\mathcal{K}$ is initial object of $\mathcal{H}$. Hence we can define $f_0 \colon GW_0 \to W'_0$ as 1_{GI}. For limit ordinal $n+1$ define $f_{n+1} = F' f_n \circ GF\eta_{W_n}$, and observe that the following square commutes



because GW_0 is initial object. Further, we'll prove by induction that f is natural. Suppose that the square

$$\begin{array}{ccc} GW_n & \xrightarrow{Gw_{n,n+1}} & GW_{n+1} \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ W'_n & \xrightarrow{w'_{n,n+1}} & W'_{n+1} \end{array} \qquad (*)$$

commutes and consider the diagram below.

$$\begin{array}{ccc} GW_{n+1} & \xrightarrow{Gw_{n+1,n+2}} & GW_{n+2} \\ {\scriptstyle GF\eta_{W_n}}\downarrow & & \downarrow{\scriptstyle GF\eta_{W_{n+1}}} \\ F'GW_n & \xrightarrow{F'Gw_{n,n+1}} & F'GW_{n+1} \\ {\scriptstyle F'f_n}\downarrow & & \downarrow{\scriptstyle F'f_{n+1}} \\ W'_{n+1} & \xrightarrow{w'_{n+1,n+2}} & W'_2 \end{array}$$

Upper square in this diagram commutes because $Gw_{n+1,n+2} = GFw_{n,n+1}$ and $GF\eta$ is natural, and the lower square is F' image of the square $(*)$. Finally, f_{n+1} is the composition on the left side of the diagram, and f_{n+2} is composition on the right side of the diagram by definition.

For limit λ, f_λ is defined as colimit of f_α, $\alpha < \lambda$ (functor G preserves colimits; hence $GW_\lambda = \operatorname{colim}_{\alpha<\lambda} GW_\alpha$). The square in the following diagram commutes

$$\begin{array}{ccccc} GW_\lambda & \xrightarrow{Gw_{\lambda,\lambda+1}} & GW_{\lambda+1} & & \\ {\scriptstyle f_\lambda}\downarrow & & \downarrow{\scriptstyle f_{\lambda+1}} & \searrow{\scriptstyle GF\eta_{W_\lambda}} & \\ & & & & F'GW_\lambda \\ & & & \swarrow{\scriptstyle F'f_\lambda} & \\ W'_\lambda & \xrightarrow{w'_{\lambda,\lambda+1}} & W'_{\lambda+1} & & \end{array}$$

because both ways are factoring morphisms for cocone

$$Fw'_{\alpha,\lambda}w'_{\alpha,\alpha+1}f_\alpha = Fw'_{\alpha,\lambda}f_{\alpha+1}Gw_{\alpha,\alpha+1} = Fw'_{\alpha,\lambda}F'f_\alpha GF\eta_{W_\alpha}Gw_{\alpha,\alpha+1}, \ \alpha < \lambda.$$

□

PART II. Free Algebras

7. Pointed and well-pointed functors

Pointed and well-pointed functors have been comprehensively studied in [K]. Most of the results in this chapters come from there.

If $\mathcal{K}$ is a category, then a *pointed functor* in $\mathcal{K}$ is a pair (F, φ) where F is an endofunctor of $\mathcal{K}$ and φ is a natural transformation from the identity functor to F. For a pointed functor (F, φ) we define (F, φ)-*algebras* (sometimes noted only F-algebras or algebras) as pairs (A, a) where $A \in \mathcal{K}$ and $a\colon FA \to A$ such that $a\varphi_A = 1_A$.

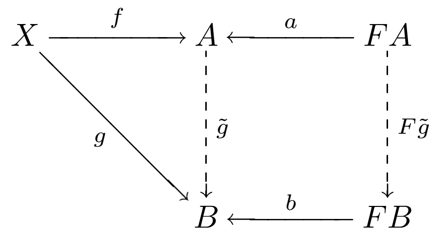
$$A \underset{a}{\overset{\varphi_A}{\rightleftarrows}} FA$$

A *homomorphism* of pointed algebras (A, a) and (B, b) is a morphism $f\colon A \to B$ such that the square

$$\begin{array}{ccc} A & \xleftarrow{\ a\ } & FA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Ff} \\ B & \xleftarrow{\ b\ } & FB \end{array}$$

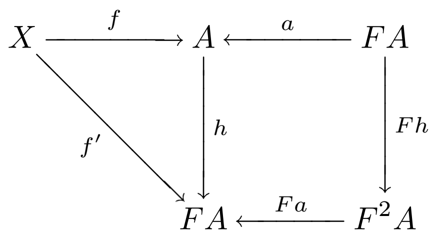
commutes. We'll denote the category of all (F, φ)-algebras with homomorphisms as (F, φ)-**Alg**.

A *free algebra* above $X \in \mathcal{K}$ is such (F, φ)-algebra (A, a) and morphism $f\colon X \to A$ that for any other algebra (B, b) and morphism $g\colon X \to B$ there is a unique homomorphism $\tilde{g}$ satisfying $g = \tilde{g}f$.



Lemma 7.1. *For a pointed functor (F, φ) and a free algebra (A, a) above X, the operation a is isomorphism.*

Proof. The proof is similar to the proof of lemma 4.3. Let $f: X \to A$ denote the free algebra morphism. First we consider algebra (FA, Fa) and define $f' = \varphi_A f$. And let $h: A \to FA$ be the unique algebra homomorphism for which the left triangle in the following diagram commutes.



We claim that $ah = 1_A$. Morphism $a: FA \to A$ is homomorphism and also $af' = a\varphi_A f = f$; hence $ha: A \to A$ is the unique factoring homomorphism and $ah = 1_A$. To prove $ha = 1_{FA}$ we use the square in the upper diagram: $ha = FaFh = F1_A = 1_{FA}$. $\square$

We'll construct free algebra by transfinite construction which is similar to initial algebra construction. First, we'll describe two simple, naïve approches to the problem.

The first construction is the free algebra construction for well-pointed functors as described in [K], the second construction is similar to the construction of relatively initial algebra which is described in [AT$_2$].

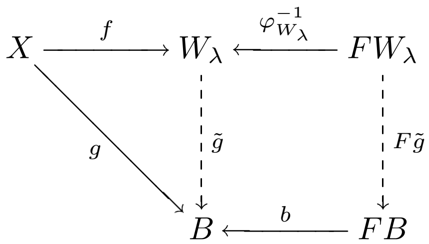
Both of these constructions have not been designed to solve this problem—in the first one additional condition for the functor (F, φ) is needed, and in the second one the pointed functors (and pointed algebras) are not considered.

Let X be an object of $\mathcal{K}$. We'll try to construct free algebra above X. Both construction starts with $W_0 = X$, $W_1 = FX$, and $w_{0,1} = \varphi_X$ but then we put $W_2 = FW_1$, and we have two chooses of $w_{1,2}$ either $\varphi_{W_1} = \varphi_{FX}$, or $Fw_{0,1} = F\varphi_X$.

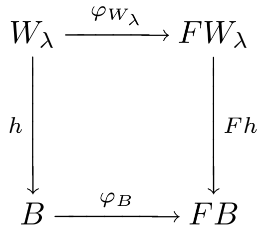
$$X \xrightarrow{\varphi_X} FX \underset{F\varphi_X}{\overset{\varphi_{FX}}{\rightrightarrows}} F^2X$$

The first construction continues as follows. For ordinal successors $\lambda + 1$ we choose $W_{\lambda+1} = FW_\lambda$ and $w_{\lambda,\lambda+1} = \varphi_{W_\lambda}$. For limit ordinals let $W_\lambda = \operatorname{colim}_{\alpha<\lambda} W_\alpha$ be the colimit of the so far constructed chain and $w_{\lambda,\lambda+1} = \varphi_{W_\lambda}$.

We say that the construction stops if φ_{W_λ} is isomorphism. In that case we get an (F, φ)-algebra $(W_\lambda, \varphi^{-1}_{W_\lambda})$ and a morphism $f = w_{0,\lambda}$ with the following property: For any other (F, φ)-algebra (B, b) and $g\colon X \to B$ there exists some $\tilde{g}$ (which do not need to be unique) such that $g = \tilde{g}f$ and the following square commutes.



Existence of $\tilde{g}$ is easy to prove. Construct $g_\lambda\colon W_\lambda \to B$, $\lambda \in \mathbf{Ord}$ inductively. Let $g_0 = g$, for ordinal successor $\lambda + 1$ we choose $g_{\lambda+1} = bFg_\lambda$, and for limit ordinals λ we choose g_λ to be the colimit of g_α, $\alpha < \lambda$. We know, there is some $\tilde{g} = g_\lambda$ such that $\tilde{g}f = g$. The designed square actually commutes for any map $h\colon W_\lambda \to B$ because



commutes and $b\varphi_B = 1_B$; hence

$$\varphi^{-1}_{W_\lambda} h = b(\varphi_B h)\varphi^{-1}_{W_\lambda} = bFh\varphi_{W_\lambda}\varphi^{-1}_{W_\lambda} = bFh. \qquad \square$$

The second construction starts with F-coalgebra $(X, \varphi_X) = (W_0, w_{0,1})$ and then for limit ordinals $\lambda + 1$ we define $W_{\lambda+1} = FW_\lambda$ and $w_{\lambda+1,\lambda+2} = Fw_{\lambda,\lambda+1}$. For limit λ, W_λ is the colimit of so far constructed chain and $w_{\lambda,\lambda+1}$ is the factoring morphism for the cocone $W_{\lambda+1} = FW_\lambda$ with $Fw_{\alpha,\lambda}w_{\alpha,\alpha+1}$ (i.e. it is defined the same way as in the initial algebra construction).

If this construction stops we get an F-algebra (which is not necessary pointed algebra) $(W_\lambda, w^{-1}_{\lambda,\lambda+1})$ and $f\colon X \to W_\lambda$ with the universal property that for any pointed algebra (B, b) and $g\colon X \to B$ there is unique $\tilde{g}\colon W_\lambda \to B$ which is homomorphism of (non-pointed) F-algebras and $g = \tilde{g}f$.

The universal property of the limit can be proved similar way as in the case of initial algebras—we'll prove that for every ordinal λ there is a unique coalgebra-algebra morphism from coalgebra $(W_\lambda, w_{\lambda,\lambda+1}$ to algebra (B, b). The only step which differs from the initial construction is the first one. We have to prove that $g\colon X \to B$ is a coalgebra-algebra morphism.

To prove that consider following diagram.

$$\begin{array}{ccc} X & \xrightarrow{\varphi_X} & FX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Ff} \\ B & \underset{b}{\overset{\varphi_B}{\rightleftarrows}} & FB \end{array}$$

The square with sides φ_B and φ_X commutes because φ is natural and $b\varphi_B = 1_B$, hence

$$bFf\varphi_X = b\varphi_B f = f. \qquad \square$$

Example 7.2. (The result of the second construction is not a pointed-algebra) Define endofunctor F of **Set** as $F = 1 + C_1$ and for transformation φ take the distinguished point of component C_1 of F. If we start with $X = 1$ then the above construction is in fact identical to initial construction of functor F, hence it stops after ω steps and the limit is (ω, x) where $x\colon \omega \cup \{*\} \to \omega$ is defined by $x(n) = n + 1$ and $x(*) = 0$. It's easy to see that this F-algebra is not pointed because $\varphi_\omega(n) = *$ for every n and we get $x\varphi_\omega(n) = x(*) = 0$.

The interesting case is when these two construction coincides. In that case, the equality $F\varphi = \varphi_F$ must hold. We'll show that the condition $F\varphi = \varphi_F$ is enough.

We'll use transfinite induction on λ. We know that both constructions start the same, and for ordinal successor $\lambda + 1$ they continue the same (because $\varphi_F = F\varphi$). For limit ordinal λ, it suffices to prove that φ_{W_λ} is factoring morphism for the corresponding cocone, i.e. that the following square commutes for any $\alpha < \lambda$.

$$\begin{array}{ccc} W_\alpha & \xrightarrow{w_{\alpha,\lambda}} & W_\lambda \\ {\scriptstyle w_{\alpha,\alpha+1}=\varphi_{W_\alpha}}\downarrow & & \downarrow{\scriptstyle \varphi_{W_\lambda}} \\ W_{\alpha+1} & \xrightarrow{Fw_{\alpha,\lambda}} & W_{\lambda+1} \end{array}$$

But that's obvious because $\varphi\colon 1 \to F$ is natural.

If both constructions coincides we say that functor (F, φ) is *well-pointed*. We can equivalently define well pointed functors as such pointed functors (F, φ) that $F\varphi = \varphi_F$.[2]

In this case if the construction stops then its limit $(W_\lambda, w^{-1}_{\lambda,\lambda+1})$ is the free algebra because it has the required universal property (from the second construction) and it is a pointed algebra (from the first one).

Lemma 7.3. *For any pointed functor (F, φ) following conditions are equivalent*

(1) *(F, φ) is well-pointed.*
(2) *For any $f\colon FX \to Y$ holds $F(f\varphi_X) = \varphi_Y f$.*

[2]The second definition is the original definition which apeared in [K].

Proof. (1) → (2) The following square commutes because φ is natural.

$$\begin{array}{ccc} F^2X & \xrightarrow{Ff} & FY \\ {\scriptstyle F\varphi_X=\varphi_{FX}}\uparrow & & \uparrow{\scriptstyle \varphi_Y} \\ FX & \xrightarrow{f} & Y \end{array}$$

But the upper way is $F(f\varphi_X)$ and the lower way $\varphi_Y f$.
(2) → (1) Let $Y = FX$ and $f = 1_X$, then the condition yields $F(\varphi_X) = \varphi_{FX}$. □

Lemma 7.4. *Let (F, φ) be a well pointed functor of $\mathcal{K}$ and $A \in \mathcal{K}$ then following conditions are equivalent.*

(1) *There exists an (F, φ)-algebra (A, a).*
(2) *There is a unique (F, φ)-algebra on A.*
(3) *φ_A is isomorphism.*

Proof. The implication (2) → (1) is trivial.
(3) → (2) For any algebra (A, a) we have $a\varphi_A = 1_A$; hence if φ_A is isomorphism then $a = \varphi_A^{-1}$.
(1) → (3) $a\varphi_A = 1_A$ holds for algebra (A, a). We'll prove that $a = \varphi_A^{-1}$. From the previous lemma we know that $F(a\varphi_A) = \varphi_A a$ but $F(a\varphi_A) = F(1_A) = 1_{FA}$. □

Corollary 7.5. *If (F, φ) is well pointed in $\mathcal{K}$ then the forgetful functor U which assigns each algebra (A, a) the underlying object A is full embedding.*

Proof. Every forgetful functor is faithful; hence we only need to prove that U is full and injective on objects.
(Full) Let (A, a) and (B, b) be two algebras and $f \colon A \to B$ be any morphism. We want to prove that f is homomorphism, i.e. that the square

$$\begin{array}{ccc} A & \xleftarrow{a} & FA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Ff} \\ B & \xleftarrow{b} & FB \end{array}$$

commutes. But from previous lemma we know that $a = \varphi_A^{-1}$ and $b = \varphi_B^{-1}$, hence the square commutes because φ is natural.
(Injective on objects) It is obvious corollary of the previous lemma because if (A, a) and (A, b) are two algebras then $a = b = \varphi_A^{-1}$. □

Lemma 7.6. *If (F, φ) is well-pointed functor and $\varepsilon \colon F \to G$ an epitransformation then $(G, \varepsilon\varphi)$ is also well-pointed. Moreover, $(G, \varepsilon\varphi)$-algebras are precisely those (F, φ)-algebras A such that ε_A is isomorphism.*

Proof. Since φ and ε are natural and $\varphi_F = F\varphi$, following diagram commutes

$$
\begin{array}{ccccc}
1 & \xrightarrow{\varphi} & F & \xrightarrow{\varepsilon} & G \\
\downarrow{\scriptstyle \varphi} & \swarrow{\scriptstyle 1_F} & \downarrow{\scriptstyle F\varphi} & & \downarrow{\scriptstyle G\varphi} \\
F & \xrightarrow{\varphi_F} & F^2 & \xrightarrow{\varepsilon_F} & GF \\
\downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle F\varepsilon} & & \downarrow{\scriptstyle G\varepsilon} \\
G & \xrightarrow{\varphi_G} & FG & \xrightarrow{\varepsilon_G} & G^2
\end{array}
$$

Note that $G(\varepsilon\varphi) = G\varepsilon G\varphi$ is the right side of the great square and $(\varepsilon\varphi)_G = \varepsilon_G \varphi_G$ is the bottom side of the great square. And from commutativity of the upper diagram we know that $G\varepsilon G\varphi\varepsilon = \varepsilon_G\varphi_G\varepsilon$ but ε is epitransformation; hence $G(\varepsilon\varphi) = (\varepsilon\varphi)_G$ and $(G, \varepsilon\varphi)$ is well-pointed.

For the second part, suppose that (A, a) is G-algebra, i.e. $a\varepsilon_A\varphi_A = 1_A$. Obviously, $(A, a\varepsilon_A)$ is F-algebra; hence $\varphi_A a\varepsilon_A = 1_{FA}$. By composing this equation with ε_A from the left we get $\varepsilon_A\varphi_A a\varepsilon_A = \varepsilon_A$ but since ε is epi we get $\varepsilon_A\varphi_A a = 1_{GA}$, hence $\varphi_A a$ is inverse to ε_A. That proves that any G-algebra A is F-algebra such that ε_A is isomorphism. The converse is obvious. □

Corollary 7.7. *Every factor (F, φ) of the identity functor is well-pointed.* □

Example 7.8. In the category of all groups we can take functor

$$FG = G/[G, G] = G/\{ghg^{-1}h^{-1} : g, h \in G\}$$

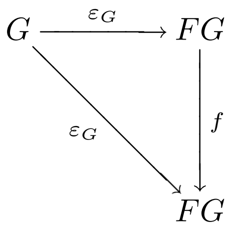
which assigns each group the largest commutative factor. Denote $\varepsilon_G : G \to G/[G, G]$ the factoring epimorphism. Functor F is well-defined on morphisms because if $f : G \to H$ is a group homomorphism and H is commutative group then $\ker f \geq [G, G]$. Hence f factors through ε_G.

For $f : G \to H$ we define Ff as the unique morphism such that the square

$$
\begin{array}{ccc}
G & \xrightarrow{\varepsilon_G} & G/[G, G] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle F} \\
H & \xrightarrow{\varepsilon_H} & H/[H, H]
\end{array}
$$

commutes. The fact that ε is natural transformation is obvious from this definition. Moreover, ε is also epitransformation because all components of ε are epimorphisms, hence (F, ε) is well-pointed.

The fact that (F, ε) is well-pointed can be easily proved directly. Observe that $\varepsilon_{FG} = 1_{FG}$ and as well $F\varepsilon_G = 1_{FG}$ since FG is commutative, and $\varepsilon_{FG} = 1_{FG}$ so the triangle



commutes for $f = 1_{FG}$ as well as $f = F\varphi_G$. From uniqness of f we get $1_{FG} = F\varphi_G$.

Example 7.9. (Varieties of universal algebras) Similar functors can be found in general case of any two varieties $\mathcal{A}$ and $\mathcal{B}$ of universal algebras such that $\mathcal{A} \subseteq \mathcal{B}$. The functor F is simply taken as the reflexion of $\mathcal{B}$ to $\mathcal{A}$. The reflexion of algebra $B \in \mathcal{B}$ is defined as factor of B by all equations which are satisfied in $\mathcal{B}$.

This shows that any variety of universal algebras is a class of pointed algebras for a suitable endofunctor of the free variety (i.e. variety where no non-trivial equations are satisfied).

8. Free algebra construction for pointed functors

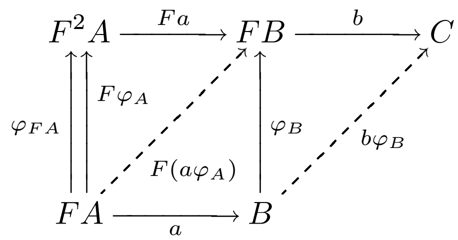
We have described the free algebra construction for well-pointed functors. The construction for pointed functors can be described by reduction of the pointed case to well-pointed case. In [K] author first describe the reduction via adjunctions and then explicitly. We'll show only the second, simpler reduction.

Suppose that (F, φ) is a pointed functor (which is not well-pointed) in category $\mathcal{K}$. We'll construct a well pointed functor (T, τ) of comma-category $F \downarrow \mathcal{K}$, such that T-algebras are such objects of $F \downarrow \mathcal{K}$ which are isomorphic to some F-algebra (note that F-**Alg** can be viewed as a full subcategory of $F \downarrow \mathcal{K}$).

Given an object (A, a, B) of $\mathcal{K}$, i.e. A, $B \in \mathcal{K}$ and $a\colon FA \to B$ we define $T(A, a, B) = (B, b, C)$, where (C, b) is the coequalizer of morphisms $FaF\varphi_A$ and $Fa\varphi_{FA}$.

$$FA \underset{\varphi_{FA}}{\overset{F\varphi_A}{\rightrightarrows}} F^2A \xrightarrow{\quad Fa \quad} FB \xrightarrow{\quad b \quad} C$$

The (A, a, B) component of natural transformation τ is defined as $\tau_{(A,a,B)} = (a\varphi_A, b\varphi_B)$. In the diagram below τ is marked by dashed arrows.

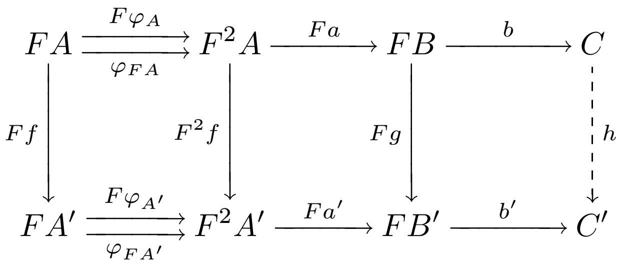


To show that $\tau_{(A,a,B)}$ is morphism in $F \downarrow \mathcal{K}$ note that the square with side φ_{FA} commutes; hence

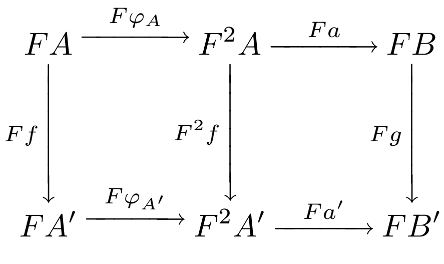
$$\varphi_B a = Fa\varphi_{FA}, \quad \text{and} \quad (b\varphi_B)a = bFa\varphi_{FA} \overset{(*)}{=} bFaF\varphi_A = bF(a\varphi_A).$$

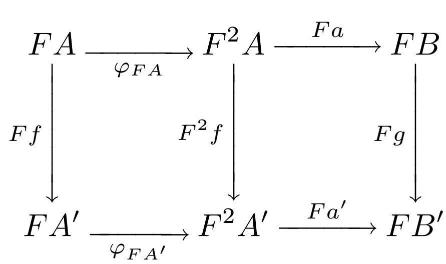
The equality $(*)$ holds because b is coequalizer of those two morphisms.

Suppose that (A, a, B) and (A', a', B') are two objects of $F \downarrow \mathcal{K}$ and (f, g) is a morphism between them. Furthermore, let $T(A, a, B) = (B, b, C)$ and $T(A', a', B') = (B', b', C')$, then $T(f, g)$ is defined as $T(f, g) = (g, h)$ where h is factoring morphism of $b'Fg$.



To show that this definition is correct we only need to ensure that $b'Fg$ equalizes $FaF\varphi_A$ and $Fa\varphi_{FA}$. But we know that b' equalizes $Fa'F\varphi_{A'}$ and $Fa'\varphi_{FA'}$. Further, the following two diagrams commute because all the squares are either F-image of commuting square, or commuting itselves.

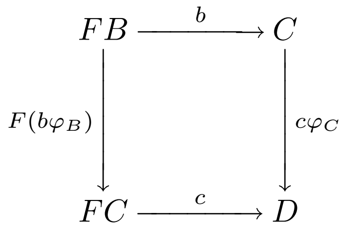




Hence

$$(b'Fg)(Fa\varphi_{FA}) = b'(Fa'\varphi_{FA'})Ff = b'(Fa'F\varphi_{A'})Ff = (b'Fg)(FaF\varphi_A).$$

Finally, we show that (T, τ) is well-pointed. Let $T(A, a, B) = (B, b, C)$ and $T(B, b, C) = (C, c, D)$ then $T\tau_{(A,a,B)} = T(a\varphi_A, b\varphi_B) = (b\varphi_B, h)$ for some h and $\tau_{(C,c,D)} = (b\varphi_B, c\varphi_C)$. To prove that $h = c\varphi_C$ it suffices to show that the square

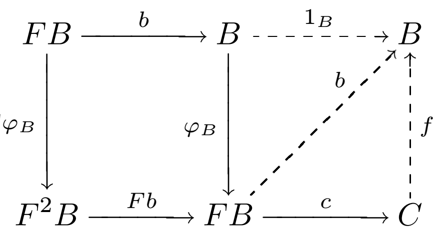


commutes but that's true because c equalizes $FbF\varphi_B$ and $Fb\varphi_{FB}$ which implies (together with the assumption that φ is natural)

$$cF(b\varphi_B) = c(Fb\varphi_{FB}) = c(\varphi_C b) = (c\varphi_C)b. \qquad \square$$

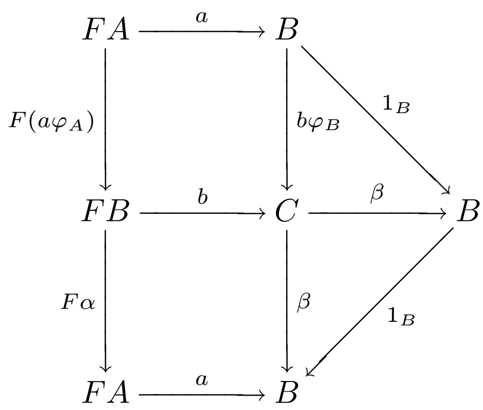
Proposition 8.1. *Let (F, φ) be a pointed functor and (T, τ) be a well-pointed functor in $F \downarrow \mathcal{K}$ defined as above. Then (T, τ)-algebras are precisely those objects of $F \downarrow \mathcal{K}$, which are isomorphic to some F-algebra. Formally, $(A, a, B) \in (T, \tau)$-**Alg** if and only if there is an F-algebra (C, c), such that $(A, a, B) \simeq (C, c, C)$.*

Proof. First, suppose that (B, b) is an F-algebra. We claim that (B, b, B) is T-algebra. We need to show that $\tau_{(B,b,B)} = (b\varphi_B, c\varphi_B)$ (c is the coequalizer of $F\varphi_B Fb$ and $\varphi_F BFb = b\varphi_B$) is isomorphism. Surely, $b\varphi_B = 1_B$, because (B, b) is F-algebra. In the diagram below c is the coequalizers of two path in the left square but b equalizes them too because $F\varphi_B Fb = 1_{FB}$ and $b\varphi_B b = b1_B$; hence there is unique f such that $b = fc$.



Finally, $(1_B, f)$ is inverse to $\tau_{(B,b,B)} = (1_B, c\varphi_B)$ and it is also well-defined morphism of comma-category because $b1_B = fc$.

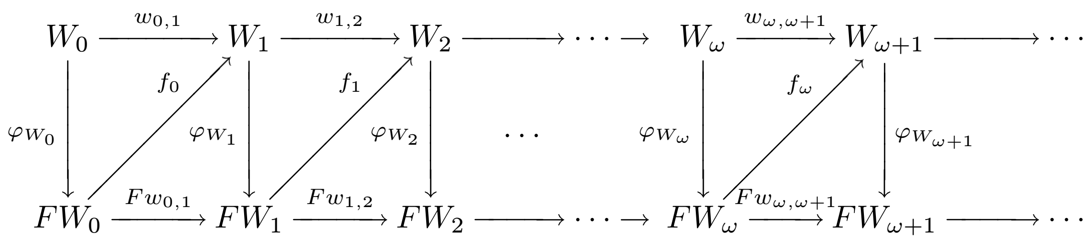
To prove the converse suppose that (A, a, B) is T-algebra. We'll construct an F-algebra which is isomorphic to (A, a, B). Let $T(A, a, B) = (B, c, C)$ and (α, β) be inverse of $\tau_{(A,a,B)}$. Consider the diagram below and define F-algebra $(B, \beta b)$.



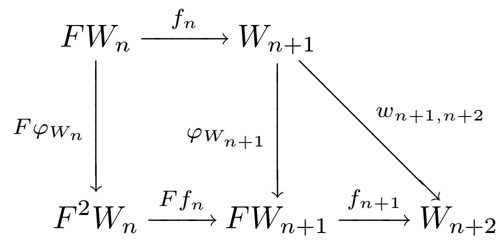
Consider morphisms $(a\varphi_A, 1_B)$ and $(\alpha, 1_B)$—they are morphism in $F \downarrow \mathcal{K}$ because both squares in the upper diagram commutes ($\tau_{(A,a,B)}$ and $(\alpha.\beta$ are morphisms in this category) and so does both triangles (β is inverse to $b\varphi_B$). It's easy to see that these two morphisms are mutually inverse. □

(Free algebra construction) In this paragraph we'll describe the free algebra construction for a pointed functor (F, φ). Functor (T, τ) is same as above.

Let $A \in \mathcal{K}$ be an object. The construction will be a chain $W\colon \mathbf{Ord} \to \mathcal{K}$ together with a natural transformation $f\colon FW \to W+$ (we denote $+\colon \mathbf{Ord} \to \mathbf{Ord}$ s.t. $+(\lambda) = \lambda + 1$, i.e. $f_n\colon FW_n \to W_{n+1}$) which satisfies $f_n \varphi_{W_n} = w_{n,n+1}$.



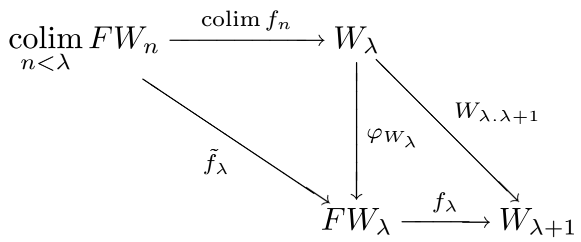
The construction starts with $W_0 = A$, $W_1 = FA$ and $f_0 = 1_{FA}$, i.e. with object $(A, 1_{FA}, FA)$ of $F \downarrow \mathcal{K}$.



For ordinal successor $n+1$, we define a pair W_{n+2}, f_{n+1} to be a coequalizer of $F(f_n\varphi_{W_n}) = Ff_nF\varphi_{W_n}$ and $\varphi_{W_{n+1}}f_n = Ff_n\varphi_{FW_n}$ and $w_{n+1,n+2} = f_{n+1}\varphi_{W_{n+1}}$. Note that this is similar to the construction for well-pointed functor T, i.e.

$$(W_{n+1}, f_{n+1}, W_{n+2}) = T(W_n, f_n, W_{n+1}),$$
$$(w_{n,n+1}, w_{n+1,n+2}) = \tau_{(W_{n+1}, f_{n+1}, W_{n+2})}.$$

For limit ordinal λ, we define W_λ as a colimit of so far constructed chain $W|_{n<\lambda}$. The object $W_{\lambda+1}$ and morphism f_λ are defined as coequalizer of $\tilde{f}_\lambda$ and $\varphi_{W_\lambda} \operatorname{colim} f_n$ in the following diagram



where $\tilde{f}_\lambda$ is factoring morphism for cocone $(Fw_{n,\lambda})$. This is precisely the construction of colimit in category $F \downarrow \mathcal{K}$.

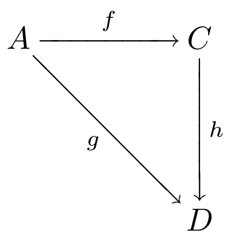
We say that the free algebra construction *converges* after λ steps if $w_{\lambda,\lambda+1}$ is isomorphism. In that case $(W_\lambda, w^{-1}_{\lambda,\lambda+1}f_\lambda)$ is the free algebra. There is no need to prove the last claim because from the construction we know that $(W_\lambda, f_\lambda, W_{\lambda+1})$ is the free T-algebra and it is obvious that $(W_\lambda, w^{-1}_{\lambda,\lambda+1}f_\lambda) \simeq (W_\lambda, f_\lambda, W_{\lambda+1})$ via isomorphism $(1_{W_\lambda}, w_{\lambda,\lambda+1})$. □

9. A note on relatively initial algebras

Relatively terminal coalgebras and the construction of relatively terminal coalgebras have been recently described in [AT$_3$]. Here we show connection between the dual case, relatively initial algebras, and free algebra construction for suitable pointed functor F. (We use the dual case just for better comparision with the rest of this thesis.)

Let F be any endofunctor of a category $\mathcal{K}$ and (A, a) an (unpointed) F-coalgebra. We define a *relatively initial algebra* with respect to (A, a) as an F-algebra (B, b) and a coalgebra to algebra homorphism $r\colon A \to B$ such that for any other algebra (C, c) and a coalgebra-algebra homorphism $f\colon A \to C$, there is a unique algebra homorphism $\tilde{f}\colon B \to C$ such that $\tilde{f}r = f$.

The relatively initial algebra to (A, a) can be viewed as the initial object of the category where objects are pairs (C, f) with C is an algebra and $f\colon A \to C$ is coalgebra-algebra morphism. Morphisms from (C, f) to (D, g) in this category are such algebra homomorphisms h that the triangle



commutes.

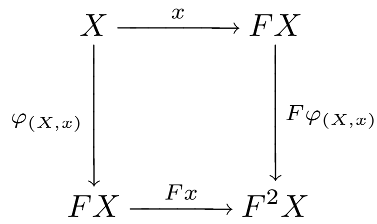
(Relatively initial algebra construction) Suppose that F is an endofunctor of $\mathcal{K}$ and (A, a) an F-coalgebra. The relatively initial chain is the chain W:

$$A \xrightarrow{a} FA \xrightarrow{Fa} F^2A \xrightarrow{F^2a} \cdots \to W_\omega \xrightarrow{w_{\omega,\omega+1}} W_{\omega+1} \longrightarrow \cdots$$

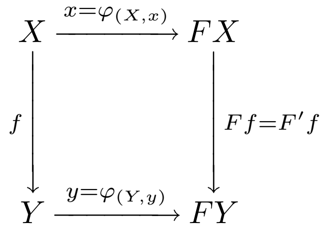
We start with $W_0 = A$ and $w_{0,1} = a$. For ordinal successors $\lambda + 1$ we define $W_{\lambda+1} = FW_\lambda$ and for limit λ, W_λ is the colimit of the chain up to λ and $w_{\lambda,\lambda+1}$ is the factoring morphism of cocone $Fw_{n,\lambda}w_{n,n+1}$ (this is the same definition as in the initial algebra construction).

The same chain can be obtained as the free chain for a functor $F'\colon F\text{-}\mathbf{Coalg} \to F\text{-}\mathbf{Coalg}$ and a point $\varphi\colon 1 \to F'$, where F' is defined by $F'(X, x) = (FX, Fx)$ on obejects, $F'f = Ff$ on morphisms, and $\varphi_{(X,x)} = x$. The pair (F', φ) is well-pointed. First, we need to prove that φ is well-defined.

(φ_X is coalgebra homomorphism) The following square trivially commutes because $\varphi_{(X,x)} = x$.



(φ is natural) If $f\colon (X,x) \to (Y,y)$ is coalgebra homomorphism. Then the square



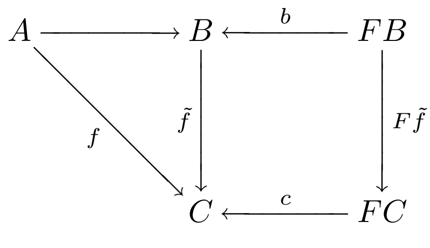
commutes and φ is natural.

((F', φ) is well-pointed) It holds that $F'\varphi_{(X,x)} = Fx = \varphi_{(FX,Fx)} = \varphi_{F'(X,x)}$.

Proposition 9.1. *If F and F' are functors as above. Then the relatively initial F-algebra with respect to (A,a) is the free F'-algebra above (A,a).*

*Furthermore, the construction of relatively initial F-algebra coincides with that of free F'-algebra. Precisely, if W is the relatively initial chain with respect to (A,a), W' free chain for (A,a), and U, the forgetful functor from F-**Coalg** to $\mathcal{K}$ then $W \simeq UW'$.*

Proof. To prove the first part suppose that (B,b) is relatively initial F-algebra. We know that b is isomorphism; hence (B, b^{-1}) is a F-coalgebra. Moreover it is also a F'-algebra, since $\varphi_{(B,b^{-1})} = b^{-1}$ is isomorphism. We'll show that it is free over (A,a). Suppose we have any F'-algebra $(C, c^{-1}) \in F$-**Coalg** and a morphism $f\colon A \to C$ in F-**Coalg**. Then f is F-coalgebra to F-algebra homomorphism; hence there is unique $\tilde{f}$ such that the following diagram commutes.



It is obvious that in that case $\tilde{f}$ is also a homomorphism of F-coalgebras (B, b^{-1}) and (C, c^{-1}). The uniqueness of $\tilde{f}$ can be shown similar way.

The second part is easy. We'll prove the statement by step by step comparision of both constructions—let W denote the relatively initial chain for functor F and W' the free chain for F'. They both start with coalgebra (A,a). If

$n+1$ is ordinal successor and $(W_n, w_{n,n+1}) = W'_n$, then $W_{n+1} = FW_n$ and $w_{n+1,n+2} = Fw_{n,n+1}$, i.e. by the chain W we get coalgebra $(FW_n, Fw_{n,n+1})$, which is exactly $F'W'_n$ by definition of F'. And note that $w'_{n+1,n+2} = \varphi_{F'W'_n} = Fw_{n,n+1}$.

For a limit ordinal λ, observe that $(W_\lambda, w_{\lambda,\lambda+1})$ is exactly the colimit of $(W_n, w_{n,n+1})$, $n < \lambda$ in the category of F-coalgebras because $W_\lambda = \operatorname{colim} W_n$ and $w_{\lambda,\lambda+1} = \operatorname{colim} w_{n,n+1}$; hence $W'_\lambda = (W_\lambda, w_{\lambda,\lambda+1})$. $\square$

10. Well-pointed functors in sets and many-sorted sets

In this chapter we'll describe all well pointed functors in the category of sets and the opposite category. Further, we'll investigate well-pointed functors of many-sorted sets and describe all possible classes of well-pointed algebras.

Theorem 10.1. *Let (F, φ) be a well-pointed endofunctor of* **Set** *then one of the following holds*

(1) *$F \simeq 1_{\mathbf{Set}}$ and φ is natural isomorphism.*
(2) *φ is a distinguished point and for each constant map $f: X \to Y$ and $x \in FX$ holds $Ff(x) = \varphi_Y$.*

Moreover, any pointed endofunctor (F, φ) that satisfies any of the two conditions is well pointed.

Proof. Let F be well-pointed. First we'll prove that F is connected. For contradiction suppose that it is not then there is a natural epitransformation $\varepsilon: F \to C_{F0,2}$; hence also $(C_{F0,2}, \varepsilon\varphi)$ is well-pointed but that's not true. The point $\mu = \varepsilon\varphi$ is distinguished point of $C_{F0,2}$; hence $\mu_{C_{F0,2}1}$ is distinguished point but $C_{F0,2}\mu_1 = 1_2$.

F is connected—so there is only one possibility for a natural transformation φ and φ is either distinguished, or monotransformation.

(1) φ is mono. We'll prove that φ is natural isomorphism, i.e. each component of φ is an isomorphism. Let X be an arbitrary set, and define $f_X: FX \to X$ such that $f_X\varphi_X = 1_X$—it's possible since φ_X is monomorphism with non-empty domain. Hence (X, f_X) is an F-algebra which implies that φ_X is isomorphism and f_X is its inverse.

(2) If φ is distinguished the only thing which remains to prove is the second condition. Let f be any constant map from X to Y. It factors through φ_X, i.e. $f = g\varphi_X$. From the well-pointedness of F we get that $Ff = \varphi_Y g$, hence Ff is constantly the distinguished point φ_Y.

At last, it is easy to see that the functor (F, φ) which satisfies (1) is well-pointed and for the functor that satisfies (2) we need to prove that $F\varphi = \varphi F$, which is guaranteed by the second condition of (2). □

Theorem 10.2. *Let (F, φ) be a well-copointed endofunctor of* **Set***, then one of the following conditions holds*

(1) *φ is natural isomorphism.*
(2) *F is constantly* 0.

Proof. Suppose that F is not C_0 then there is some natural transformation

$\mu\colon 1_{\mathbf{Set}} \to F$. The composition $\varphi\mu$ is endotransformation of the identity functor which is identity; hence every set X with operation φ_X is a coalgebra and φ is a natural isomorphism. $\square$

In the several following paragraphs we'll describe some properties of well-pointed functors in many sorted sets and all possible subcategories of many-sorted sets which may be a category of all algebras for some well-pointed functor. For each of these classes we'll give an example of such functor.

The following general lemma will be useful for describing classes of well-pointed algebras in many-sorted sets.

Lemma 10.3. *Let $\{A_i : i \in I\}$ be a set of (some) algebras for a well pointed functor (F, φ) then also $A = \prod_{i\in I} A_i$ is an algebra.*

Proof. For any $A_i \in \mathcal{A}$ we have that φ_{A_i} is an isomorphism. We need to prove that φ_A is an isomorphism. For any A_i the following square commutes

$$\begin{array}{ccc} A_i & \xrightarrow{\varphi_{A_i}} & FA_i \\ {\scriptstyle \pi_i}\uparrow & & \uparrow{\scriptstyle F\pi_i} \\ A & \xrightarrow{\varphi_A} & FA \end{array}$$

Note that FA together with morphisms $\varphi_{A_i}^{-1} F\pi_i$, $i \in I$ is a cone for the product A; hence there is a (unique) factoring morphism $a\colon FA \to A$ s.t. the following commutes

$$\begin{array}{ccc} A_i & \xrightarrow{\varphi_{A_i}} & FA_i \\ {\scriptstyle \pi_i}\uparrow & & \uparrow{\scriptstyle F\pi_i} \\ A & \underset{a}{\overset{\varphi_A}{\rightleftarrows}} & FA \end{array}$$

We'll prove that a is an operation on A, i.e. $a\varphi_A = 1_A$. Holds

$$\varphi_{A_i}(\pi_i a)\varphi_A = \varphi_{A_i}(\varphi_{A_i}^{-1} F\pi_i)\varphi_A = F\pi_i\varphi_A = \varphi_{A_i}\pi_i,$$

and since φ_{A_i}'s are isomorphisms then

$$\pi_i(a\varphi_A) = \pi_i, \quad \forall i \in I.$$

Which implies that $a\varphi_A = 1_A$ because both are factoring morphisms for the product $\prod A_i$. $\square$

A category of *I-sorted sets* is a category of functors from I to **Set** where I is viewed as a category with no nontrivial morphisms. I.e. each object X of I-sorted sets is an I-tuple $(X^i : i \in I)$ where $X^i \in \mathbf{Set}$ for each $i \in I$ and a morphism f of I-sorted sets is an I-tuple $(f^i : i \in I)$. We denote this category

$\mathbf{Set}^I$. If X is I-sorted set and $i \in I$ we'll denote X^i the i-th sort of X. Similarly, for a morphism f of I-sorted sets we denote i-th sort of f by f^i.

In **Set** every functor which is not constantly 0 restricts to category $\mathbf{Set}_*$ of non-empty sets. The similar works in many sorted sets but the situation is more complicated.

For $\overline{A} = \{A_i : i < I\}$ an I-sorted set. Denote $n(A) = \{i < I : A_i = 0\}$. We'll call $n(A)$ *a nullity* of A. The full subcategory consisting of all I-sorted sets with the same nullity is called a *component* of $\mathbf{Set}^I$. By a symbol $n(\mathcal{C})$ we mean the set of all non-empty sorts of $\mathcal{C}$.

Observe that a component of $A \in \mathbf{Set}^I$ is precisely class of all objects B s.t. there exists morphism $f: A \to B$ and $f: B \to A$. We'll write $\mathcal{C} \preceq \mathcal{D}$ if there exists morphism $f: C \to D$ for some $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Similarly, for $X \in \mathbf{Set}^I$ we write $X \preceq \mathcal{C}$, if there is a morphism from X to some object $Y \in \mathcal{C}$.

The above observation means that if both $\mathcal{C} \preceq \mathcal{D}$ and $\mathcal{D} \preceq \mathcal{C}$ then $\mathcal{C} = \mathcal{D}$. Therefore, the class of all components forms a partially ordered class (ordered by $\preceq$). With the description by nullity, it's easy to show that this partially ordered class is—in fact—isomorphic to boolean algebra $\mathcal{P}(I)$ (ordered by inclusion).

Based on above observations we can define a following factorcategory of $\mathbf{Set}^I$. Objects of the category $\mathcal{S}^I$ are same as $\mathbf{Set}^I$. Given two objects $C, D \in \mathcal{S}^I$ we define

$$\mathrm{Hom}_{\mathcal{S}^I}(C, D) = \mathrm{Hom}_{\mathbf{Set}^I}(C, D)/\sim$$

where $\sim$ is equivalence s.t. $f \sim g$ for each pair $f, g \in \mathrm{Hom}_{\mathbf{Set}^I}(C, D)$. Hence $\mathcal{S}^I$ is narrow category. Components of $\mathbf{Set}^I$ are exactly classes of isomorphic objects in $\mathcal{S}^I$. Let's denote $s: \mathbf{Set}^I \to \mathcal{S}^I$ the defining epifunctor.

Lemma 10.4. *If F is an endofunctor of $\mathbf{Set}^I$ then there is a functor $\tilde{F}$ of $\mathcal{S}^I$ s.t. $sF = \tilde{F}s$.*

Proof. The functor $\tilde{F}$ is defined same as F on objects, and on morphisms we define $\tilde{F}f = [Ff]_\sim$. It's obvious that $\tilde{F}$ has the required property. $\square$

In next few paragraphs we will describe well-pointed functors in $\mathbf{Set}^I$. First we'll describe factorfunctors of identity because if $\varphi: 1 \to F$ is natural then we can do image factorization of each component of φ. Let $\varphi_X = \varepsilon_X \mu_X$ is such factorization then both ε and μ are natural transformation, i.e. There is a factor functor of identity H which is subfunctor of F. Moreover (as said in lemma 7.6) any factorfunctor of identity is well-pointed.

Lemma 10.5. *If F is an endofunctor of $\mathbf{Set}^I$ and $\mu: 1 \to F$ then for any component $\mathcal{C}$ of $\mathbf{Set}^I$ holds either for each $X \in \mathcal{C}$, μ^i_X is injective, or for each $X \in \mathcal{C}$, μ^i_X is constant. Moreover, if μ^i_X is constant and $|X_i| > 1$ then for each $Y \in \mathbf{Set}^I$ such that $n(Y) \subseteq n(X)$ also μ^i_Y is constant.*

Proof. Suppose that μ^i_X is not injective and Y is arbitrary I-sorted set satisfying $n(Y) \subseteq n(X)$. Since μ^i_X is not injective, there exist x, y such that $\mu^i_X(x) = \mu^i_X(y)$. Let a and b be two arbitrary points in Y. Then there is a map

$f^i\colon X^i \to Y^i$ such that $f^i(x) = a$ and $f^i(y) = b$. Extend map f^i to a homomorphism $f\colon X \to Y$ (it is possible since $n(Y) \subseteq n(X)$). The following square commutes

$$\begin{array}{ccc} X^i & \xrightarrow{f^i} & Y^i \\ \Big\downarrow{\scriptstyle \mu^i_X} & & \Big\downarrow{\scriptstyle \mu^i_Y} \\ (FX)^i & \xrightarrow{(Ff)^i} & (FY)^i \end{array}$$

Hence

$$\mu^i_Y(a) = \mu^i_Y(f^i(x)) = (Ff)^i(\mu^i_X(x)) = (Ff)^i(\mu^i_X(y)) = \mu^i_Y(f^i(y)) = \mu^i_Y(b)$$

and μ^i_Y is constant. If we set $Y = X$ then this proves the first part. □

Example 10.6. (Factorfunctors of identity in $\mathbf{Set}^2$) We'll explicitly describe all factorfunctors of identity in 2-sorted sets to give an idea how factorfunctors of identity looks like in general many-sorted sets.

First of all, identity is always a trivial factor functor of identity. Secondly, any factor functor of identity satisfies $n(X) = n(FX)$ because each sort of FX is a factor of corresponding sort of X. Specially, $F(0,0) = (0,0)$. Further, we can define $F(X,0)$ two different ways: (1) $F(X,0) = (1,0)$, (2) $F(X,0) = (X,0)$ (there are no other possibilities which is proved by the previous lemma). It's obvious how is F defined on morphisms in these two cases. The similar holds for $F(0,Y)$.

Finally, let's look at $F(X,Y)$ for both X and Y non-empty. In general, we have four possibilities $F(X,Y) = (1,1)$, $(X,1)$, $(1,Y)$, and (X,Y). But not all these cases are always possible—for example if $F(X,0) = (1,0)$ then $F(X,Y)$ cannot be (X,Y) or $(X,1)$ (that's the second part of previous lemma). Depending on the choose of $F(X,0)$ and $F(0,Y)$ we have one, two, or four possibilities. There are exactly 9 factorfunctors of identity. One of the less trivial examples is the following one

$$F(X,Y) = \begin{cases} (X,0) & \text{if } Y = 0, \\ (0,1) & \text{if } X = 0 \text{ and } Y \neq 0, \\ (1,1) & \text{if both } X \text{ and } Y \text{ are non-empty.} \end{cases}$$

Example 10.7. Following example shows a little more complicated factor of identity functor in 3-sorted sets

$$F(X,Y,Z) = \begin{cases} (X,Y,0) & \text{if } Z \text{ is empty,} \\ (0,Y,1) & \text{if } Z \neq 0 \text{ and } X = 0, \\ (1,Y,1) & \text{if both } X \text{ and } Z \text{ are non-empty.} \end{cases}$$

The natural equivalence $\varphi\colon 1 \to F$ is defined such that $\varphi^i_{(X,Y,Z)}$ is isomorphism where possible, otherwise $\varphi^i_{(X,Y,Z)}$ is the unique morphism from $(X,Y,Z)^i$ to $F(X,Y,Z)^i$.

Corollary 10.8. *Let F be a factor of identity functor on I-sorted sets then for each component $\mathcal{C}$ of $\mathbf{Set}^I$ there exists a set of sorts $C(\mathcal{C})$ such that $C(\mathcal{C})$ doesn't contain any empty sorts of $\mathcal{C}$, and*

$$(FX)^i = \begin{cases} 1 & \text{if } i \in C_{\mathcal{C}}, \\ X^i & \text{otherwise.} \end{cases}$$

Furthermore, if $\mathcal{C}$ and $\mathcal{D}$ are two components such that $\mathcal{C} \preceq \mathcal{D}$ then $C_{\mathcal{C}} \subseteq C_{\mathcal{D}}$. □

Given an endofunctor F of I-sorted sets we say that a component $\mathcal{C}$ is *fixed component* of F, if for each $C \in \mathcal{C}$, $FC \in \mathcal{C}$.

Lemma 10.9. *Let (F, φ) be a well-pointed endofunctor of $\mathbf{Set}^I$ and $\mathcal{C}$ a component of $\mathbf{Set}^I$. Denote $O \in \mathcal{C}$ the terminal object of $\mathcal{C}$ (i.e. the many-sorted set such that $O^i = 1$ for each non-empty component of $\mathcal{C}$ and $O^i = 0$ for $i \in n(\mathcal{C})$). Then $\mathcal{C}$ is fixed component of F if and only if O is (F, φ)-algebra.*

Furthermore, if $\mathcal{C}$ is a component containing at least one (F, φ)-algebra then $\mathcal{C}$ is fixed component of F.

Proof. First, suppose that $\mathcal{C}$ is a fixed component of F. To prove that O is F-algebra define $o\colon FO \to O$. Note that $FO \in \mathcal{C}$, so there exists such morphism o and it is unique because O is terminal in $\mathcal{C}$. It's easy to see that $o\varphi_O = 1_O$ because there is no non-trivial endomorphism of O.

The converse is corollary of the second part—if $A \in \mathcal{C}$ is (F, φ) algebra then $A \simeq FA$ and both are in component $\mathcal{C}$. Hence F preserves that component. □

Corollary 10.10. *If (F, φ) is well-pointed then the set of all fixed components of F is closed to all (even infinite) infima with respect to order $\preceq$.*

Furthermore given any set M of components which is closed to all infima then there is a well-pointed functor (F, φ) such that M is the set of all fixed points of F.

Proof. Suppose that F is well-pointed and N is a set of (some) fixed components of F. Let $O_{\mathcal{C}} \in \mathcal{C}$ denote the terminal object of $\mathcal{C}$. Then all $O_{\mathcal{C}}$'s for $\mathcal{C} \in N$ are (F, φ)-algebras, and from lemma 10.3 so is $\prod_{\mathcal{C} \in N} O_{\mathcal{C}}$. But $\prod O_{\mathcal{C}}$ lies in component $\inf N$; hence $\inf N$ is fixed.

If we're given M satisfying the condition in the statement. We define

$$FX = O_{\mathcal{C}} \quad \text{where } \mathcal{C} = \inf\{\mathcal{D} \in M : X \preceq \mathcal{D}\}.$$

The functor F on morphisms is determined by above because for $O_{\mathcal{C}}$ and every $Y \in \mathbf{Set}^I$ there is at most one morphism $f\colon Y \to O_{\mathcal{C}}$. The same holds for natural transformation φ. Consequently, (F, φ) is well-pointed.

Finally, observe that all components in M are fixed components of F because for each $\mathcal{C} \in M$ holds $\inf\{\mathcal{D} \in M : \mathcal{C} \prec \mathcal{D}\} = \mathcal{C}$; hence $FO_{\mathcal{C}} = O_{\mathcal{C}}$ and $\mathcal{C}$ is fixed. □

Finally, next theorem is the one which describes all possible classes of well-pointed algebras in many-sorted sets. It also characterize the key properties of well-pointed functors and gives examples of some but not all of them.

Theorem 10.11. *Let I be a set of sorts and (F, φ) a well-pointed functor in $\mathbf{Set}^I$. Define M as a set of all fixed components of F and for $\mathcal{C} \in M$ define $C(\mathcal{C})$ as a set of all sorts i such that i is non-empty sort of $\mathcal{C}$ and φ^i_X is constant for all $X \in \mathcal{C}$. Then M and $C: M \to \mathcal{P}(I)$ satisfies the following conditions*

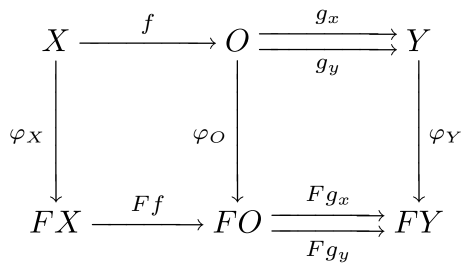
(1) *M is closed to infima.*
(2) *C is non-increasing, i.e. $C(\mathcal{C}) \supseteq C(\mathcal{D})$ holds for any two components $\mathcal{C} \preceq \mathcal{D}$ in M.*
(3) *Whenever i is non-empty sort of $\mathcal{C}$ such that $i \notin C(\mathcal{C})$ then a component $\hat{\mathcal{C}}_i$ such that $n(\hat{\mathcal{C}}_i) = n(\mathcal{C}) \cup \{i\}$ is in M.*

Furthermore, (F, φ)-algebras can be described as precisely those objects of some component $\mathcal{C} \in M$ such that $|X^i| = 1$ for every $i \in C(\mathcal{C})$.

Conversely, if M and $C: M \to \mathcal{P}(I)$ satisfies conditions (1)–(3) then there exists such well-pointed endofunctor of $\mathbf{Set}^I$ for which M is a set of all fixed components and $C(\mathcal{C})$ a set of all non-empty sorts i of $\mathcal{C}$ for which φ^i_X is constant for any $X \in \mathcal{C}$.

Proof. Let (F, φ) be well-pointed and M, C as above. The property (1) is just corollary of 10.10 and the condition (2) is direct consequence of lemma 10.5.
(3) Suppose that $X \in \hat{\mathcal{C}}_i$ for some i but $(FX)^i \neq 0$. From $X \preceq \mathcal{C}$ we know that $FX \preceq \mathcal{C}$ since $\mathcal{C}$ is fixed component of F. We'll prove that φ^i_Y is constant for all $Y \in \mathcal{C}$ and $i \in C(\mathcal{C})$.

Let $x, y \in Y$. Denote O the terminal object of $\mathcal{C}$. O is (F, φ)-algebra because $\mathcal{C}$ is fixed component of F. Define morphisms $g_x: O \to Y$ and $g_y: O \to Y$ such that $g^i_x(0) = x$, $g^i_y(0) = y$ and $g^j_x = g^j_y$ for any $j \neq i$. Consider following commutative diagram



Since X^i is empty and $g^j_x = g^j_y$ for any $j \neq i$ then $g_x f = g_y f$ and also $Fg_x Ff = Fg_y Ff$. Let $a \in (FO)^i$ be the unique element of FO^i and $w \in (FX)^i$. The following holds

$$\varphi^i_Y(x) = \varphi^i_Y g^i_x(0) = (Fg_x)^i \varphi^i_O(0) = (Fg_x)^i(a) = Fg^i_x(Ff^i(w)) = F(g_x f)^i(w).$$

Similarly, we can get $\varphi^i_Y(y) = F(g_y f)^i(w)$ but since $F(g_x f) = F(g_y f)$ then $\varphi^i_Y(x) = \varphi^i_Y(y)$ and φ^i_Y is constant.

Surely, for each sort j other then i holds if X^j is empty then also $(FX)^j$ is empty because $\mathcal{C}$ is fixed component and $\hat{\mathcal{C}}_i$ differs from $\mathcal{C}$ only at the sort i; hence we know that if $\hat{\mathcal{C}}_i$ is not fixed then $i \notin C(\mathcal{C})$.

Next we describe all (F, φ)-algebras. Let X be an (F, φ)-algebra then surely $X \in \mathcal{C}$ for some fixed component $\mathcal{C}$ of F (lemma 10.9). Moreover, φ^i_X is epimorphism for each i, hence $|X^i| = |\operatorname{im} \varphi^i_X| = 1$ whenever $i \in C(\mathcal{C})$.

Conversely, let $X \in \mathcal{C}$ for some fixed component $\mathcal{C}$ and $|X^i| = 1$ for each $i \in C(\mathcal{C})$. Then we define an operation x on X such that $x_i \varphi^i_X = 1_{X^i}$ whenever $X^i \neq 0$ and $i \notin C(\mathcal{C})$ (it's possible since φ^i_X is monomorphism with non-empty domain) and x_i is constant whenever $|X^i| = 1$ ($i \in C(\mathcal{C})$). In other cases both X^i and FX^i are empty. It's obvious that $x\varphi_X = 1_X$ and (X, x) is an (F, φ)-algebra.

To show the last part, we'll construct an example of endofunctor F with prescribed M and C. Let M and $C\colon M \to \mathcal{P}(I)$ satisfy conditions (1) to (3). Let $X \in \mathit{Set}^I$ and $\mathcal{C}$ be the least component of M such that $X \preceq \mathcal{C}$ (such component is infimum of all components $\mathcal{D}$ in M satisfying $X \preceq \mathcal{D}$).

$$(FX)^i = \begin{cases} 0 & \text{if } i \in n(\mathcal{C}), \\ 1 & \text{if } i \in C(\mathcal{C}), \\ X^i & \text{otherwise.} \end{cases}$$

The natural transformation φ is defined the obvious way; hence for $x \in X^i$

$$\varphi^i_X(x) = \begin{cases} x & \text{if } i \notin C(\mathcal{C}), \\ 0 & \text{if } i \in C(\mathcal{C}). \end{cases}$$

The definition of F on morphisms is obvious—$(Ff)^i$ is identity whenever possible, otherwise we have only one option.

It's easy to see that M is a set of fixed components of F as well as (given i non-empty sort of $\mathcal{C}$) φ^i_X is constant if and only if $i \in C(\mathcal{C})$. $\square$

The list of well-pointed functors in the previous proof is not complete—the following two examples shows some functors which are not listed above.

Example 10.12. We define endofunctor of $\mathbf{Set}^\lambda$ where λ is some ordinal on objects as

$$(FX)^i = \begin{cases} 1 & \text{if } X^i \text{ is non-empty,} \\ 1 & \text{if } i \text{ is the least ordinal, such that } X^i \text{ is empty,} \\ 0 & \text{otherwise.} \end{cases}$$

The natural transformation $\varphi\colon 1 \to F$ is defined the only possible way as well as images of maps. It's easy to see that this functor is well-pointed (because there is always unique morphism from FX to FY, if there is some morphism $f\colon X \to Y$).

Furthermore, it's not listed in the proof because the only fixed component of F is the greatest one. Especially, F doesn't satisfy the condition that for all X, $FX \in \mathcal{C}$ for some fixed component $\mathcal{C}$.

Example 10.13. The second class of examples shows some freedom of choosing the functor F. If we have M and $C\colon M \to \mathcal{P}(I)$ we can define F same as in the proof with the difference that

$$(FX)^i = \{A \subseteq X : |A| = 3\} \cup \{*\} \quad \text{if } i \in \mathcal{C}(C).$$

and F is defined on maps $f\colon X \to Y$ as (let $C(X)$ denote $C(\mathcal{C})$ for component $\mathcal{C} \ni X$).

$$(Ff)^i(x) = \begin{cases} f^i(x) & \text{if } i \notin C(Y), \\ f^i[x] & \text{if } i \in C(X),\ X^i \neq 0,\ x \neq *, \text{ and } f^i \text{ is monomorphism}, \\ * & \text{otherwise.} \end{cases}$$

The transformation φ^i_X is defined as

$$\varphi^i_X(x) = \begin{cases} x & \text{if } i \notin C(\mathcal{C}), \\ * & \text{if } i \in C(\mathcal{C}). \end{cases}$$

It's easy to see that (F, φ) is well-pointed and M is class of fixed components of F as well as $C(\mathcal{C})$ is set of all sorts such that φ^i_X is constant for every $X \in \mathcal{C}$.

Used symbols

0	Empty set.
$1, 2, 3, \dots$	Natural numbers as ordinals, i.e. $1 = \{0\}$, $n + 1 = n \cup \{n\}$, etc. (n is an n-element set)
ω	The set of all natural numbers (including 0)
Set	A category of all (small) sets.
$\mathbf{Set}_*$	A category of all non-empty sets.
Ord	The set of all ordinals or a category of all ordinals with inclusions.
Card	A class of all cardinals.
$\mathcal{C}^{op}$	The dual category to $\mathcal{C}$.
1_X	Identity on X.
1	The identity functor.
$\mathcal{P}(X)$	The set of all subsets of X.
C_X	Constant functor such that $C_X A = X$ for each A.
$C_{M,N}$	Constant set functor which maps empty set to M and each non-empty set to N.
F-**Alg**	The category of all F-algebras.
F-**Coalg**	The category of all F-coalgebras.
(F, φ)-**Alg**	The category of all pointed (F, φ)-algebras.
(F, φ)-**Coalg**	The category of all pointed (F, φ)-coalgebras.
$F \downarrow \mathcal{K}$	The comma category. If $F\colon \mathcal{H} \to \mathcal{K}$ then the objects of $F \downarrow \mathcal{K}$ are triples (A, a, B), where $A \in \mathcal{H}$, $B \in \mathcal{K}$ and $a\colon FA \to B$.
$\lvert X \rvert$	The cardinality of set X.
$\operatorname{cf} \kappa$	The cofinality of ordinal κ.
$f[X]$	Image of the set X under map f, i.e. the set $\{f(x) : x \in X\}$.
$f^{-1}[X]$	Preimage of the set X under map f, i.e. the set $\{x : f(x) \in X\}$.
$\operatorname{im} f$	Image of the map $f\colon A \to B$, i.e. the set $\{f(x) : x \in A\} \subseteq B$.
lim	Limit of a diagram, a chain, or a sequence of ordinals.
colim	Colimit of a diagram, or a chain.
$\coprod$	Coproduct.
$X + Y$	The coproduct of X and Y.
$G \dashv U$	G is left adjoint to U and U is right adjoint to G.
X^i	The i-th sort of many-sorted set X. (Otherwise, set power or cardinal power.)
f^i	The i-th sort of morphism f of many-sorted sets.

References

[AKo] Adámek, J. and Koubek, V., *On the greatest fixed point of a set functor*, Theoret. Comput. Sci. **15** (1995), 57–75.

[AT_1] Adámek, J. and Trnková, V., *Automata and algebras in categories* (Hazewinkel, M., ed.), Kluwer Acad. Publ., Dordrecht, 1990.

[AT_2] ———, *Initial algebras and terminal coalgebras in many-sorted sets*, Math. Struct. in Comp. Science **21/2** (2011), 481–509.

[AT_3] ———, *Relatively terminal coalgebras* (sent for publication).

[BŠ] Balcar, B. and Štěpánek, P., *Teorie Množin*, Academia, Praha, 2001 (in czech).

[K] Kelly, G. M., *A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on*, Bull. Austral. Math. Soc. **22** (1980), 1–83.

[Kn] Knaster, B., *Un théorème sur les fonctions d'ensembles*, Ann. Soc. Polon. Math. **6** (1928), 133–134.

[Ko] Koubek, V., *Set functors*, Comment. Math. Univ. Carolinae **12** (1971), 175–195.

[R] Rutten, J. J. M. M., *Universal coalgebra: a theory of systems*, Theoret. Comp. Sci. **249** (2000), 3–80.

[Ta] Tarski, A., *A lattice-theoretical fixpoint theorem and its applications*, Pacific J. Math. **5** (1955), 285–309.

[TAKR] Trnková, V., Adámek, J., Koubek, V., Reiterman, J., *Free algebras, input process and free monads*, Comment. Math. Univ. Carolinae **16** (1975), 339–351.

[W] Worrell, J., *On the final sequence of a finitary set functor*, Theoret. Comp. Sci. **338** (2005), 184–199.