

Taylor's modularity conjecture for idempotent varieties

Jakub Opršal

Jagiellonian University, Kraków
Charles University in Prague

Praha, May 28, 2016

Congruence modular varieties

A variety is **congruence modular** if every algebra in it and every a triple of congruences α, β , and γ such that $\alpha \geq \gamma$ we have

$$\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee \gamma.$$

Theorem (A. Day, 1969)

A variety \mathcal{V} is congruence modular if and only if there are terms d_0, \dots, d_n such that

$$d_0(x, y, z, w) \approx x, \quad d_n(x, y, z, w) \approx w,$$

$$d_i(x, y, y, x) \approx x, \text{ for all } i,$$

$$d_i(x, x, y, y) \approx d_{i+1}(x, x, y, y), \text{ for all even } i, \text{ and}$$

$$d_i(x, y, y, z) \approx d_{i+1}(x, y, y, z), \text{ for all odd } i.$$

Conjecture (Taylor)

Suppose that Σ_1 and Σ_2 are two sets of identities in disjoint languages such that neither of them implies existence of Day terms. Then $\Sigma_1 \cup \Sigma_2$ does not imply existence of Day terms either.

O.C. Garcia and Walter Taylor, *The lattice of interpretability types of varieties*. Mem. Amer. Math. Soc., 50:v+125, 1984.

Other examples

Suppose that Σ_1 and Σ_2 are two sets of identities in disjoint languages such that neither of them implies existence of $(*)$. Then $\Sigma_1 \cup \Sigma_2$ does not imply existence of $(*)$ either.

Theorem (Tschantz, 1996)

The above is true for $() = \text{Mal'cev term}$.*

$$q(x, x, y) \approx q(y, x, x) \approx y$$

Example

The above is false for $(*) = \text{Majority term}$.

$$m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx x$$

(For Σ_1 take Mal'cev term identities, and for Σ_2 Jónsson chain of length ≥ 2 .)

The lattice of interpretability types of varieties

An **interpretation** of a variety \mathcal{V} in a variety \mathcal{W} is a mapping I of basic operations of \mathcal{V} to terms of \mathcal{W} such that

$$\mathcal{V} \models t \approx s \rightarrow \mathcal{W} \models I(t) \approx I(s).$$

Interpretability is a partial order of the class of all varieties, and after factoring out the equi-interpretable varieties, we get a (class-size) latticed ordered poset.

Reformulation of the conjecture

An **interpretability join** of two varieties \mathcal{V} and \mathcal{W} is the variety whose signature is the disjoint union of signatures of \mathcal{V} and \mathcal{W} axiomatized by the union of $\text{Eq}(\mathcal{V})$ and $\text{Eq}(\mathcal{W})$.

Conjecture (Taylor)

Suppose that Σ_1 and Σ_2 are two sets of identities in disjoint languages such that neither of them implies existence of Day terms. Then $\Sigma_1 \cup \Sigma_2$ does not imply existence of Day terms either.

Conjecture (Alternative formulation)

For any two varieties \mathcal{V} and \mathcal{W} which are not congruence modular, their interpretability join is not congruence modular either.

Stop this abstract nonsense!

Theorem (Kearnes, Kiss, 2013)

An idempotent variety satisfies a non-trivial congruence identity if and only if it is not interpretable in the variety of semilattices. = the variety generated by the idempotent reduct of $\text{Pol}(\{0, 1\}, \{(a, b, a \vee b) : a, b \in \{0, 1\}\})$.

Theorem (Valeriote, Willard, 2014)

An idempotent variety is n -permutable if and only if it is not interpretable in the variety of distributive lattices. = the variety generated by the idempotent reduct of $\text{Pol}(\{0, 1\}, \leq)$.

Looking for the right relational structure

Definition (Bova, Chen, and Valeriote, 2011)

A *pentagon* is a structure \mathbb{P} over the signature $\{\alpha, \beta, \gamma\}$, all binary relations which are equivalence relations on P that satisfy

- ▶ $\alpha \leq \beta$,
- ▶ $\beta \wedge \gamma = 0_P$,
- ▶ $\beta \circ \gamma = 1_P$, and
- ▶ $\alpha \vee \gamma = 1_P$.

A pentagon is *interesting* if $\alpha < \beta$.

Lemma (Bova, Chen, and Valeriote)

If a locally finite variety is not congruence modular, then there is a finite algebra \mathbf{A} in the variety with three congruences α, β , and γ such that $(A, \alpha, \beta, \gamma)$ is a disjoint union of pentagons of which at least one is interesting.

Looking for the right relational structure

Notation

For a congruence α of a product $\mathbf{A} \times \mathbf{B}$ and $a \in A$ let α^a denotes the equivalence

$$\{(b, b') : ((a, b), (a, b')) \in \alpha\}$$

.

If \mathbf{A} is idempotent then α^a is always a congruence of \mathbf{B} .

Lemma (McGarry, 2009)

A locally finite idempotent variety is not congruence modular if and only if it contains algebras \mathbf{A} and \mathbf{B} with a congruence $\alpha \leq \text{Ker } \pi_{\mathbf{A}}$ of $\mathbf{A} \times \mathbf{B}$ such that

- ▶ $\alpha^a = 1_B$ for some $a \in A$,
- ▶ if $\alpha^a \neq 1_B$ then $\alpha_a = \eta$ for some fixed $\eta < 1_B$.

We say that a congruence $\alpha \leq \text{Ker } \pi_{\mathbf{A}}$ of $\mathbf{A} \times \mathbf{B}$ is a **modularity blocker** if there exists $\eta \in \text{Con } \mathbf{B}$ such that

- ▶ $\alpha^a = 1_B$ for at least one $a \in A$, and
- ▶ $\alpha^a = \eta$ whenever $\alpha^a \neq 1_B$.

Theorem (O, 2016)

An idempotent variety is not congruence modular if and only if $\mathbf{F}(x, y) \times \mathbf{F}(x, y)$ has a modularity blocker.

Definition

A pentagon $(P, \alpha, \beta, \gamma)$ is *special* if

- ▶ $P = A \times B$,
- ▶ $\beta = \text{Ker } \pi_A$, $\gamma = \text{Ker } \pi_B$, and
- ▶ $\alpha^a \in \{0_B, 1_B\}$ for every $a \in A$.

Corollary

An idempotent variety that is not congruence modular is interpretable in the idempotent reduct of a special interesting pentagon.

Back to Taylor's conjecture

For an infinite cardinal κ fix $U_\kappa \subseteq \kappa$ with $|U_\kappa| = |\kappa \setminus U_\kappa| = \kappa$, let \mathbb{P}_κ denotes a special pentagon $(P_\kappa, \alpha, \beta, \gamma)$ with $P_\kappa = \kappa \times \kappa$ and $\alpha^a = 1_\kappa$ if $a \in U_\kappa$, and $\alpha^a = 0_\kappa$, otherwise.

Lemma

For every special interesting pentagon P there exists a clone homomorphism from its polymorphism clone to the polymorphism clone of \mathbb{P}_κ for all infinite cardinals $\kappa \geq |P|$.

Theorem (O., 2016)

Every idempotent variety that is not congruence modular is interpretable in the variety generated by $(P_\kappa, \text{Pol } \mathbb{P}_\kappa)$ for all large enough κ .

Idempotent Taylors conjecture

Conjecture (Taylor)

If \mathcal{V} and \mathcal{W} are two varieties that are not congruence modular then their join is not congruence modular either.

Theorem (O., 2016)

*If \mathcal{V} and \mathcal{W} are two *idempotent* varieties that are not congruence modular then their join is not congruence modular either.*

Thank you for your attention!

Identities satisfied by $\text{Pol } \mathbb{P}_\kappa$ but not by $\text{Pol } \mathbb{P}_\lambda$ for $\lambda < \kappa$

Functions f_i , $i \in \kappa$ are binary and $p_{i,j}$, and $q_{i,j}$, $r_{i,j}$, $i, j \in \kappa$ are ternary.

$$\begin{aligned}x &\approx p_{i,j}(x, f_j(x, y), y), \\p_{i,j}(x, f_i(x, y), y) &\approx q_{i,j}(x, f_j(x, y), y), \\q_{i,j}(x, f_i(x, y), y) &\approx r_{i,j}(x, f_j(x, y), y), \\r_{i,j}(x, f_i(x, y), y) &\approx y\end{aligned}$$

for all $i \neq j$, and $f_i(x, x) \approx x$ for all i .

Corollary

The set of all interpretability classes of idempotent varieties that are not congruence modular does not have a largest element.