Properties that are not characterizable by linear identities

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Identities of height 1 and height at most 1

Given a signature Σ, a term t over Σ is of height 1 if it contains exactly one operation symbol from Σ. E.g.

$$t(x_0, x_1, x_2, x_3) = f(x_1, x_0, x_0, x_2)$$

where f is a basic symbol. A projection is said to have height 0.

An identity is of height 1 (height at most 1, resp.) if both its sides are of height 1 (height at most 1, resp.).

Example

$$f(x_{i_1}, \ldots, x_{i_n}) \approx g(x_{j_1}, \ldots, x_{i_m}), \quad \text{or} \quad f(x_{i_1}, \ldots, x_{i_n}) \approx x_j.$$

Identities of height at most 1 are usually called linear.

Linear Mal'cev conditions

A linear Mal'cev condition is a Mal'cev condition which only includes linear identities.

Examples

Mal'cev term, Pixley term, Day terms, Gumm terms, near unanimity, cube term, Jónsson terms, etc.

Not examples

group terms, lattice terms, semilattice term.

Retractions and reflections

Definition (Barto, Pinsker)

An algebra **A** is said to be a reflection of **B** defined by mappings $h_1\colon B \to A$ and $h_2\colon A \to B$, if for every basic operation f we have

$$f_{\mathbf{A}}(a_1, \ldots, a_n) = h_1 f_{\mathbf{B}}(h_2(a_1), \ldots, b(a_n)).$$

It is a retraction if in addition $h_1 h_2 = 1_{\mathbf{A}}$

Observation

If **A** is a retraction of **B** then **A** satisfies all the linear identities that **B** does.

For a clone $\mathcal{B}$, **A** is a retraction (reflection, resp.) of $\mathcal{B}$ if **A** is a retraction (reflection, resp.) of the algebra $(B, (f)_{f \in \mathcal{B}})$.

Linear Birkhoff

The class of all retractions (reflections, resp.) of algebras from $\mathcal{K}$ is denoted $\mathbf{R}\,\mathcal{K}$ ($\mathbf{R}_{\mathrm{ret}}\,\mathcal{K}$).

Theorem (Barto, Pinsker, O)

A class of algebras is definable by linear identities (identities of height 1, resp.) if and only if it is closed under $\mathbf{R}_{ret}$ and $\mathbf{P}$ ($\mathbf{R}$ and $\mathbf{P}$, resp.).

Congruence regular, uniform, and singular varieties

An algebra **A** is said to be

- congruence regular if every two congruences of **A** that share a congruence class are identical;
- congruence uniform if every two classes of a single congruence of **A** are of the same size;
- congruence singular if every two congruences α and β, and every element $a \in A$ satisfy

$$|a/\alpha| \cdot |a/\beta| = |a/\alpha \wedge \beta| \cdot |a/\alpha \vee \beta|.$$

A variety is said to be congruence . . . , if all its algebras are.

A variety of groups is congruence regular, uniform and also singular.

Mal'cev conditions

There are several Mal'cev conditions that characterize **congruence regular** varieties (Csákány, Grätzer, Wille).

Congruence uniformity cannot be characterized by a Mal'cev condition (Taylor), but it is characterized by some identities.

Congruence singularity . . .

Characterization by linear identities

Theorem

Congruence regularity, congruence singularity, and congruence singularity is not characterizable by linear identities. (Even for finitely generated varities.)

Proof.

Goal: Construct $\mathcal{V}$ and $\mathcal{W}$ both finitely generated such that $\mathcal{W}$ satisfies all linear identities that $\mathcal{V}$ does, $\mathcal{V}$ has the property, but $\mathcal{W}$ does not.

Take **A** the clone of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. And $\mathcal{V} = \mathbf{HSP}(\mathbf{A})$.

Define **B** as a retraction of **A**...

Look there →

And finally, let $\mathcal{W} = \mathbf{HSP}(\mathbf{B})$. □

(Except congruence singularity.)

Meet of Mal'cev conditions

Mal'cev conditions are naturally ordered by syntactical consequence. This is actually a lattice ordering!

Meet of two Mal'cev conditions is the strongest Mal'cev condition that which is weaker then both of the original ones.

Meet of Mal'cev and Jónsson terms

There exists ternary terms q, $d_1, \ldots, d_n$, and a binary term $\cdot$ such that

$$q(x,y,y)\cdot w \approx x \cdot w, \text{ and } q(y,y,x)\cdot w \approx x \cdot w,$$

$$\begin{aligned}
w \cdot d_0(x,y,z) &\approx w \cdot x, \text{ and } w \cdot d_n(x,y,z) \approx w \cdot x,\\
w \cdot d_i(x,y,x) &\approx w \cdot x \text{ for every } i,\\
w \cdot d_i(x,x,y) &\approx w \cdot d_{i+1}(x,x,y) \text{ for odd } i,\\
w \cdot d_i(x,y,y) &\approx w \cdot d_{i+1}(x,y,y) \text{ for even } i,
\end{aligned}$$

$$\begin{aligned}
xy \cdot zw &\approx xw\\
f(x_0x_1, y_0y_1, z_0z_1) &\approx f(x_0,y_0,z_0)\cdot f(x_1,y_1,z_1) \text{ for } f \in \{q, d_1, \ldots, d_n\}.\\
w \cdot q(x,y,z) &\approx w \cdot x\\
f(x,y,z)\cdot w &\approx x \text{ for } f \in \{d_1, \ldots, d_n\}.
\end{aligned}$$

Meet of Mal'cev and Jónsson terms (cont.)

Observation

A clone satisfies this Mal'cev condition if and only if it is a product of a clone with Mal'cev operation and a clone with Jónsson terms.

A product of clones $\mathcal{A}$ and $\mathcal{B}$ is the clone $\mathcal{C}$ with $C = A \times B$, and $\mathcal{C}^{[n]} = \{f \times g \colon f \in \mathcal{A}^{[n]}, g \in \mathcal{B}^{[n]}\}$.

Meet of Mal'cev and Jónsson terms is not linear

Theorem

The meet of Mal'cev and Jónsson terms is not charactarizable by linear identities.

Proof.

Let $\mathcal{A}$ be the clone on $\{0, 1\}$ generated by the minority operation, and $\mathcal{B}$ be the clone on $\{0, 1\}$ generated by the majority operation.

And define $\mathcal{C}$ as a retraction of $\mathcal{A} \times \mathcal{B}$...

Look there $\rightarrow$

$\mathcal{C}$ cannot be written non-trivially as a product, and it has neither Mal'cev, nor Jónsson operations. $\square$

Some open problems...

Problem

Find a satisfactory description of linear meet.

Problem

Are Day terms the linear meet of Mal'cev and Jónsson terms?

Thank you for your attention!