Is cube term a prime Mal'cev condition?

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The poset of Mal'cev conditions

A strong Mal'cev condition is a condition of the form

$$\exists t_1, \ldots, t_n \text{ terms, such that } \textit{some equations}$$

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Mal'cev conditions are naturally ordered by implication. This obviously forms a $\vee$-semilatice.

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Interpretation from variety $\mathcal{V}$ to variety $\mathcal{W}$ is

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The resulting order is lattice order, let's denote it $\mathcal{L}$ (Garcia, Taylor, The lattice of interpretability types of varieties, 1984).

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Join of two varieties $\mathcal{V}$ and $\mathcal{W}$ in $\mathcal{L}$ can be described as the variety $\mathcal{V} \vee \mathcal{W}$ whose operations are operations of both varieties (taken as a discrete union of operations of $\mathcal{V}$ and operations $\mathcal{W}$), and whose identities are all identities of both varieties.

In the other worlds, we can describe algebras in $\mathcal{V} \vee \mathcal{W}$ as $(A, F \cup G)$ where $(A, F) \in \mathcal{V}$ and $(A, G) \in \mathcal{W}$.

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Example (Mal'cev $\vee$ majority)

Let $\mathcal{V}$ be variety with single Maltsev operation $q(x, x, y) = q(y, x, x) = y$, and $\mathcal{W}$ be variety $\mathcal{W}$ with majority operation $x = m(x, x, y) = m(x, y, x)$ $= m(y, x, x)$.

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$$p(x, y, z) = q(x, m(x, y, z), z)$$

is a Pixley term ($p(x, x, y) = p(y, x, x) = p(y, x, y) = y$) of $\mathcal{V} \vee \mathcal{W}$. On the other hand, Pixley term implies both majority, and Mal'cev term. So, Pixley is the join of Mal'cev and majority.

Mal’cev conditions and prime filters

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Prime

- Groups, cyclic terms of prime arity (Garcia, Taylor, 1984),
- CP (Tschantz, unpublished).

Cube terms

Definition

An *n-cube term* is $(2^n - 1)$-ary term q satisfying equations

$$q\begin{pmatrix} y & x & y & \dots & y \\ x & y & y & \dots & y \\ x & x & x & \dots & y \\ \vdots & & & \ddots & \vdots \\ x & x & x & \dots & y \end{pmatrix} = \begin{pmatrix} x \\ \vdots \\ x \\ x \\ x \end{pmatrix}$$

where the matrix on the left hand side composes of all columns of x's and y's except the one with all x's.

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Theorem (Tschantz)

The Mal'cev filter of having a Mal'cev operation (2-cube term) is a $\vee$-prime filter.

Coloring of terms by variables

(Sequeira, Barto) Let X be a given set of variables, and $A \subseteq \mathrm{Eq}(X)$. We say that variety $\mathcal{V}$ is *A-colorable* if there is a map $c\colon F_{\mathcal{V}}(X) \to X$ such that

1. $c(x) = x$ for all $x \in X$, and
2. for every $\alpha \in A$ whenever $f \sim_{\hat{\alpha}} g$ then $c(f) \sim_{\alpha} c(g)$

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We say that Mal'cev condition $\mathcal{P}$ satisfies *coloring condition A* if variety $\mathcal{V}$ satisfies $\mathcal{P}$ if and only if $\mathcal{V}$ is *not* A-colorable.

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And, the converse is also true, i.e., variety is CP if and only if it is not A-colorable.

Coloring for cube terms

Let n be fixed, and let $X = \{x_1, \ldots, x_{2^n-1}\}$, and for $i < n$ define α_i as the equivalence on X defined as $x_k \sim_{\alpha_i} x_l$ if and only if k and l has the same digit in the binary expansion at the i-th position.

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If $n = 2$ then $X = \{x_1, x_2, x_3\}$, $\alpha_0 = x_1 x_3 | x_2$, and $\alpha_1 = x_1 | x_2 x_3$.

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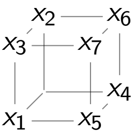
If $n = 3$ then $X = \{x_1, \ldots, x_7\}$, $\alpha_0 = x_1x_3x_5x_7|x_2x_4x_6$, $\alpha_1 = x_1x_4x_5|x_2x_3x_6x_7$, and $\alpha_3 = x_1x_2x_3|x_4x_5x_6x_7$.



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Example

If $n = 2$ then $X = \{x_1, x_2, x_3\}$, $\alpha_0 = x_1 x_3 | x_2$, and $\alpha_1 = x_1 | x_2 x_3$.

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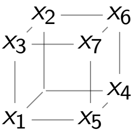

Observation

The n-cube term satisfies coloring condition for $A = \{\alpha_0, \ldots, \alpha_{n-1}\}$.

Linear varieties

Theorem (Sequeira, (Barto); Bentz-Sequeira; O)

Congruence modularity, n-permutability, satisfying non-trivial congruence identity, and n-cube term are prime with respect to varieties axiomatized by linear equations.

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Proof.

Suppose that $\mathcal{V}$ is A-colorable ($A \subseteq \operatorname{Eq} X$). Then there is a structure of $\mathcal{V}$ algebra $\mathbf{X}$ on X such that all $\alpha \in A$ are congruences of $\mathbf{X}$.

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$$f^{\mathbf{X}}(x_0, \dots, x_n) = c(f(x_0, x_1, \dots, x_n))$$

for every basic operation f, and check that

- these operations satisfy all linear identities of $\mathcal{V}$,
- every $\alpha \in A$ is preserved by all f's.

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- these operations satisfy all linear identities of $\mathcal{V}$,
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Now, if neither of $\mathcal{V}$, $\mathcal{W}$ satisfy $\mathcal{P}$ then both are A-colorable. So, in both varieties we have algebra on X with congruences A, hence it is in $\mathcal{V} \vee \mathcal{W}$. And we get a coloring of $\mathcal{V} \vee \mathcal{W}$ as the unique expansion of the identity map on X to homomorphism $f \colon F(X) \to \mathbf{X}$. □

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Theorem (Valeriote, Willard)

n-permutability is prime with respect to idempotent varieties.

Theorem (McGarry, Valeriote)

Congruence modularity is prime with respect to locally finite idempotent varieties.

Theorem (Barto, O)

Congruence permutability is prime with respect to locally finite idempotent varieties.

Cube term blockers

A proper subalgebra **B** of an algebra **A** is called a cube term blocker if for every term t there is i such that

$$t(A, \dots, A, B, A, \dots, A) \subseteq B$$

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Or equivalently, for every n the set $A^n \setminus (A \setminus B)^n$ is a universe of proper subalgebra of $\mathbf{A}^n$.

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Theorem (Marković, Maróti, McKenzie, 2012; Barto, Kozik, Stanovský, 2014)

A finite idempotent algebra has a cube term if and only if no subalgebra of $\mathbf{A}$ *has a cube term blocker.*

The inflating trick

Take κ sufficiently large cardinal (ω is enough in this case), and the free algebra $\mathbf{F}$ over the set $X = \{x_i : i < \kappa\} \cup \{y_i : i < \kappa\}$.

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Conjecture

An n-cube term gives a prime Maltsev filter.

Das ist alles...

Thank you for your attention!