Complexity of terms in congruence modular algebras

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Terms

\[ t_1(x, y, z, w) = q(q(x, y), p(y, w, q(x, x))) \]

A length of a term \( t \) is the total number of operation and variable symbols appearing in \( t \).

We will denote the length of \( t \) by \(|t|\).  \( \text{(Example. } |t_1| = 10.\text{)} \)
Fix a finite algebra $A$. Every term $t$ gives a term operation $t^A : A^n \to A$.

A length of a term operation $t^A$ is

$$|t^A| = \min\{|s| : t^A = s^A\}.$$

For $A$, we can measure ‘the complexity of its terms’ by investigating the sequence

$$\gamma_A(n) = \max\{|t^A| : t \text{ is an } n\text{-ary term}\}.$$
Question

For which algebras $A$ is there a polynomial bound in $n$ on $\gamma_A(n)$?

Given that $A$ is:

- finite, and
- of finite signature.
Reality check.
There are at most $|A|^{|A|^n}$ term operations of arity $n$.

We have $n + c$ symbols to distinguish between them ($c$ is the number of basic operations).

We can write altogether no more than $(n + c + 1)^{p(n)}$ operations using at most $p(n)$ symbols.

Therefore, if $p(n)$ is polynomial, we cannot distinguish between all $|A|^{|A|^n}$ operations.
Bound

Theorem (K. Kearnes)

An algebra $A$ in congruence modular variety has a doubly exponential lower bound on the number of term operations unless it is supernilpotent.

An algebra is called supernilpotent if it satisfies $[1_A, \ldots, 1_A] = 0_A$. 
**Supernilpotence**

Any supernilpotent algebra in a congruence modular variety:

- is nilpotent,
- has a Mal’cev term,
- is a product of prime power order nilpotent algebras.

**Lemma**

An algebra $A$ with a Mal’cev term is supernilpotent of degree $n - 1$ if and only if

$$q_n^A(t^A(a_1, a_2, \ldots, a_n), \ldots, t^A(a_1, b_2, \ldots, b_n)) = t^A(b_1, b_2, \ldots, b_n)$$

for every $a_i, b_i \in A^{k_i}$, $i = 1, \ldots, n$, and term $t$ of arity $k_1 + \cdots + k_n$ (where $q_n$ denotes any strong $n$-cube term in $A$).
The depth of a term is defined inductively $d(x) = 0$ and 

$$d(f(t_1, \ldots, t_n)) = 1 + \max\{d(t_1), \ldots, d(t_n)\}.$$ 

(Example. $d(t_1) = 3$.)

As before:

$$d(t^A) = \min\{d(s) : t^A = s^A\},$$

$$\delta_A(n) = \max\{d(t^A) : t \text{ is an } n\text{-ary term}\}.$$
Theorem

There is a logarithmic bound in $n$ on $\delta_A(n)$ in a supernilpotent algebra with a Mal’cev term.

Corollary

For a supernilpotent algebra $A$ in a congruence modular variety, there is a polynomial bound on $\gamma_A$. 
Proof

Let $A$ be a supernilpotent algebra.

$$q_n^{A}(t^{A}(y, y, \ldots, y), \ldots, t^{A}(y, x_2, \ldots, x_n)) = t^{A}(x_1, x_2, \ldots, x_n)$$

Therefore,

$$\delta_{A}(k) \lesssim 1 + \delta_{A}(\frac{n-1}{n}k)$$

$$\delta_{A}(k) \lesssim \log \frac{n}{n-1} k + c$$
Result

Theorem

Let $A$ be a finite algebra of finite signature in a congruence modular variety. Then the following are equivalent:

1. $A$ is supernilpotent, and
2. there is a polynomial $p(n)$, s.t. $\gamma_A(n) \leq p(n)$.  

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