Our goal in this lecture is, for any fixed graph H and an integer k, to obtain the following algorithm.

Algorithm 1. ***Input:*** *A graph G, an assignment r of k roots in G to vertices in H.*

Output: *A model of H in G rooted in r, or correctly decides no such model exists.*

Time complexity: $O(|G|^3)$.

Note that the fact that H and k are fixed is necessary; for example, if k is part of the input, deciding for k pairs of vertices in G whether they are joined by pairwise disjoint paths is NP-complete.

One of the main results of the Graph Minors series is that every minor-closed class $\mathcal{G}$ is characterized by a finite number of forbidden minors H_1, ..., H_m. Hence, we can decide whether $G \in \mathcal{G}$ in time $O(|G|^3)$ by using Algorithm 1 for H_1, ..., H_m (for many minor-closed classes, the list of forbidden minors is not explicitly known, though—in that case we know such an algorithm exists, but we cannot actually construct it).

For fixed H and r, we say that a vertex $v \in V(G)$ distinct from all the roots is *irrelevant* if the following holds: If H has a minor in G rooted in r, then H also has a minor in $G - v$ rooted in r. Algorithm 1 easily follows from the following algorithm for some function f.

Algorithm 2. ***Input:*** *A graph G, an assignment r of k roots in G to vertices in H.*

Output:

- *A model of H in G rooted in r, or*
- *a tree decomposition of G of width at most $f(|H|, k)$, or*
- *an irrelevant vertex $v \in V(G)$.*

Time complexity: $O(|G|^2)$.

To obtain Algorithm 1, we repeatedly run Algorithm 2, deleting the irrelevant vertices as long as it returns them (this does not change the presence of the minor of H rooted in r). Eventually, we either obtain a model of H in G rooted in r, or a tree decomposition of bounded width; in the latter case, we decide the presence of H in G rooted in r by using dynamic programming (or Courcelle's result, since the problem can be expressed in MSOL).

1 Irrelevant vertices assuming a clique minor

Let r be an assignment of k roots in G to vertices of a graph H. Suppose μ is a model of K_m in G. We say that μ is *separated from the roots* if there exists a separation (A, B) of G of order less than k and $x \in V(K_m)$ such that

- $r(u) \subseteq V(A)$ for each $u \in V(H)$ and
- $\mu(x) \subseteq B - V(A)$.

We use the following basic result, whose proof is analogous to the proof of Theorem 2 in Combinatorics and Graph Theory III lecture notes.

Theorem 3. *Let G and H be graphs and let r be an assignment of at most k roots in G to vertices of a graph H. Let $m = 2k + |H|$, and suppose we are given a model μ of K_m in G. If μ is not separated from the roots, then we can in time $O(|G|^2)$ find a model of H in G rooted in r.*

The basic intuition is that since μ cannot be separated from the roots, we can link the roots to μ by disjoint paths and find the minor of H inside the large clique minor.

This in particular gives us the following algorithm.

Algorithm 4. ***Input:*** *A graph G, an assignment r of k roots in G to vertices in H, a model of K_m in G for $m = 3k + |H|$.*

Output:

- *A model of H in G rooted in r, or*
- *an irrelevant vertex $v \in V(G)$.*

Time complexity: *$O(|G|^2)$.*

This algorithm works as follows. Using Dinitz algorithm, we either find a separation (A, B) of G of order less than k such that all roots are in A, there exists a vertex $x \in V(K_m)$ such that $\mu(x) \subseteq B - V(A)$, and subject to these conditions B is minimal, or prove that no such separation exists. In the latter case, we can find H as a minor in G rooted in r by Theorem 3. Otherwise, we claim that any vertex $v \in V(\mu(x))$ is irrelevant.

Indeed, suppose G contains H as a minor rooted in r, and let ν be its model. The intersection of ν with B gives us a minor of some graph H' rooted in r', where $r'(u) \subseteq V(A \cap B)$ for each $u \in V(H')$. Let $k' = |V(A \cap B)|$ and note that $k' \leq k - 1$. Let K be the subclique of K_m consisting of the vertices y such that $V(\mu(y)) \cap (V(A \cap B) \cup \{v\}) = \emptyset$; clearly $|V(K)| \geq$

$|H| + 2k > |H'| + 2k'$. Furthermore, since $\mu(x)$ is contained in $B - V(A)$ and G contains an edge between $\mu(x)$ and $\mu(y)$ for each $y \in V(K_m) \setminus \{x\}$, we have $\mu(y) \subseteq B$ for each $y \in V(K)$. The minimality of B implies that there is no separation (C, D) of B of order at most k' such that $D \neq B$, $A \cap B \subseteq C$ and $\mu(y) \subseteq C - V(D)$ for some $y \in V(K)$. Therefore, the restriction of μ to K cannot be separated from the roots in $B - v$, and by Theorem 3, H' is a minor of $B - v$ rooted in r'; let ν' be its model. Replacing $\nu \cap B$ by ν' in ν gives us a minor of H rooted in r in $G - v$, as required.

Let us now give the first application of Algorithm 4. We use the following fact: There exists a function b such that for each m, each graph G of average degree at least $b(m)$ contains a minor of K_m; moreover, the model of this minor can be found in time $O(|G|^2)$. For a proof, see Combinatorics and Graph Theory III lecture notes. Combining this with Algorithm 4, we obtain the following result.

Algorithm 5. ***Input:*** *A graph G of average degree more than $b(3k + |H|)$, an assignment r of k roots in G to vertices in H.*

Output: *A model of H in G rooted in r, or an irrelevant vertex $v \in V(G)$.*

Time complexity: *$O(|G|^2)$.*

Hence, we can restrict ourselves to graphs of bounded average degree (and in particular, this is why we can ignore the number of edges in the time complexity of the algorithm).

2 Walls

A *elementary wall* W_n is obtained from an $n \times n$ grid G_n by deleting every even vertical edge in the first, third, fifth, ... row and every odd vertical edge in the second, fourth, ... row. A *wall* of height n is a subdivision of W_n.

Imagine a wall W drawn in the plane in a natural way. For two vertices v_1 and v_2 of W, let $d(v_1, v_2)$ denote the minimum number of intersections of a curve from v_1 to v_2 with W. Note that we can define a respectful tangle $\mathcal{T}$ in W in the natural way (a side of the separation is small if it does not contain any row path of W), and then $d(v_1, v_2) = \Theta(d_{\mathcal{T}}(v_1, v_2))$. Hence, using the results from the last lecture and homework assignment, we have the following. If $W \subset G$, a *W-path* in G is a path in G intersecting W exactly in its endpoints.

Lemma 6. *For every m, there exists d_m as follows. Suppose $W \subset G$ is a wall and G contains $\binom{m}{2}$ disjoint W-paths such that any two endpoints x and y of these paths satisfy $d(x, y) \geq d_m$. Then G contains K_m as a minor, and the model of this minor can be found in time $O(|G|^2)$.*

The *perimeter* of a wall W' is the cycle bounding its outer face, and the *interior* is everything not on the perimeter. Consider a subwall W' of W with perimeter C. A *cross over* W' is a pair of disjoint paths P_1 and P_2 whose ends are branch vertices belonging to C, the ends of P_1 are in different components of $C - V(P_2)$, and $(P_1 \cup P_2) \cap W \subseteq W'$. Another result from the homework assignment implies the following.

Lemma 7. *For every m, there exists d'_m as follows. Suppose $W \subset G$ is a wall and $W_1, \ldots, W_{m^4}$ are subwalls of W such that $d(W_i, W_j) \geq d'_m$ for all distinct i and j. If G contains pairwise disjoint crosses over $W_1, \ldots, W_{m^4}$, then G contains K_m as a minor, and the model of this minor can be found in time $O(|G|^2)$.*

A vertex $v \in V(G)$ is (l,s)-*central* over W if there exist vertices $w_1, \ldots, w_l \in V(W)$ and paths from v to $w_1, \ldots, w_l$ intersecting only in v and disjoint from W except for their endpoints such that $d(w_i, w_j) \geq s$ for $1 \leq i < j \leq l$.

Lemma 8. *For every m, there exist l and s as follows. Suppose $W \subset G$ is a wall. If G contains $\binom{m}{2}$ vertices that are (l,s)-central over W, then G contains K_m as a minor, and the model of this minor can be found in time $O(|G|^2)$.*

Proof. Let $q = \binom{m}{2}$, and suppose $v_1, \ldots, v_q$ are distinct (l,s)-central vertices over W. We can assume $v_1, \ldots, v_q \notin V(W)$: Otherwise, instead of W, take a subwall of height $n - 2q$ avoiding $v_1, \ldots, v_q$, extending the paths from the definition of centrality along the deleted parts of the wall if necessary, and noting that this decreases the distance between the endpoints by at most $4q$.

We will inductively construct disjoint W-paths $P_1, \ldots, P_q$ with endpoints pairwise at distance at least $s/2$. If we succeed, we then conclude the argument by Lemma 6. Suppose we have already found the paths $P_1, \ldots, P_{t-1}$. We will maintain the invariant that $V(P_1 \cap \ldots \cap P_t) \cap \{v_{t+1}, \ldots, v_q\} = \emptyset$. Let $Q_1, \ldots, Q_{l-q}$ be paths from v_t to $w_1, \ldots, w_{l-q} \in V(W)$ intersecting only in v_t and disjoint from W except for their endpoints such that $d(w_i, w_j) \geq s$ for $1 \leq i < j \leq l - q$, and not containing the central vertices other than v_t; such paths exist by the (l,s)-centrality of v_t.

Suppose first there exists $i \leq t-1$ such that P_i intersects at least $2t$ of the paths $Q_1, \ldots, Q_{l-q}$. For these paths, let $L_1, \ldots, L_{2t}$ be their segment from the last intersection with P_i to W, in the order of their intersections with P_i. For $j = 1, \ldots, t$, let P'_j be the path consisting of L_{2j-1}, L_{2j}, and a path between them in P_i. The distances between the ends of these paths in W are at least s, and thus we can use $P'_1, \ldots, P'_t$ as the t chosen paths.

Hence, we can assume that each of the paths $P_1, \ldots, P_{t-1}$ intersects less than $2t$ of the paths $Q_1, \ldots, Q_{l-q}$. Since $l \gg t, q$, we can assume the paths

$Q_1, \ldots, Q_{2t}$ are disjoint from $P_1, \ldots, P_{t-1}$. Since the distance between w_1, $\ldots$, w_{2t} is at least s, for $j = 1, \ldots, t-1$, there is at most one of these points at distance less than $s/2$ from each end of P_j. Hence, we can assume w_1 and w_2 are at distance at least $s/2$ from all these ends, and we can set $P_t = Q_1 \cup Q_2$. □

A subwall W' of W is *non-dividing* if there exists a W-path with one end in the interior of W' and the other end not in W'. For $F \supseteq W$, a W*-bridge* of F is a subgraph of F consisting either of an edge of $E(F) \setminus E(W)$ with both ends in W, or of a connected component of $F - V(W)$ and the edges from this component to W.

Lemma 9. *For every l, m, and s, there exists k and d''_m as follows. Suppose $W \subset G$ is a wall and $W_1, \ldots, W_k$ are subwalls of W such that $d(W_i, W_j) \geq d''_m$ for all distinct i and j. If all the subwalls are non-dividing, then either G contains an (l, s)-central vertex or K_m as a minor. Moreover, the vertex or the model of such a minor can be found in time $O(|G|^2)$.*

Proof. We choose $k, d''_m \gg a \gg l, m, s$.

Let F be a minimal subgraph of $G - E(W)$ such that for $i = 1, \ldots, k$, F contains a W-path from the interior of W_i to $W - V(W_i)$. Consider a W-bridge F' of F. The minimality of F implies F' is a tree. Moreover, if F' has at least three leaves, then each of them is in a distinct subwall among $W_1, \ldots, W_k$, and F' is a subdivision of a star. Consequently, if a vertex $v \in V(F) \setminus V(W)$ has degree at least l in F, then v is (l, s)-central over W. Consider a vertex $v \in V(F \cap W)$. If v is contained in at least two W-bridges of F, then the minimality of F implies all of them are paths with the other end in a distinct subwall among $W_1, \ldots, W_k$, and thus again, if v has degree at least l in F, then it is (l, s)-central over W.

Hence, we can assume F has maximum degree less than l, and every W-bridge of F intersects W in less than l vertices. Therefore, every W-bridge intersects less than l^2 other W-bridges, and thus F has disjoint W-bridges $F_1, \ldots, F_a$. By the minimality of F, we can assume that for $i = 1, \ldots, a$, F_i is the only W-bridge containing a path Q_i from the interior of W_i to a vertex of $V(W) \setminus V(W_i)$. Note this implies Q_i is the only path among $Q_1, \ldots, Q_a$ intersecting the interior of W_i. Let s_i denote the end of Q_i in W_i, and t_i the other end. Let us distinguish several cases.

- For at least $a_1 = 3\binom{m}{2}$ of the paths, say for $Q_1, \ldots, Q_{a_1}$, we have $d(s_i, t_i) \geq 2d_m$ and $d(t_j, t_j) \geq 2d_m$ for $i \neq j$. For each i, there exists at most two indices j such that $d(s_i, t_j) < d_m$ or $d(s_j, t_i) < d_m$. Thus, in the auxiliary graph where we join such indices i and j, we have an independent set of size at least $a_1/3$; hence, we can assume $d(s_i, t_j) \geq d_m$ and $d(s_j, t_i) \geq d_m$ for $1 \leq i \leq j \leq a_1/3$. By Lemma 6, we obtain a minor of K_m in G.

- For at least $a_2 = m^4$ of the paths, say for $Q_1, \ldots, Q_{a_2}$, we have $d(s_i, t_i) \leq d''_m/10$. Enlarging $W_1, \ldots, W_{a_2}$, we obtain subwalls of W at distance at least d'_m from one another and with a cross over each of them. Lemma 7 then gives a minor of K_m in G.

- For $a_3 = 3\binom{m}{2}$, there exists an index i_0 (say $i_0 = 1$) and at least a_3 of the paths (say $Q_1, \ldots, Q_{a_3}$) such that $d(t_1, t_i) \leq 2d_m$ for $1 \leq i \leq a_3$. Let C be the part of W at distance at most $2d_m$ from t_1. There exists a subwall W' of W avoiding C such that the distances in W' are decreased by at most $16d_m$ compared to the distances in W'. Since C has maximum degree at most three, we can find $a_3/3$ disjoint paths in C joining vertices among $\{t_1, \ldots, t_{a_3}\}$. Combining these paths with some of $Q_1, \ldots, Q_{a_3}$ and applying Lemma 6, we obtain a minor of K_m in G.

For $a \geq a_2 + a_1 a_3$, at least one of these cases happens. □

We now iterate Lemma 9: If an (l, s)-central vertex v is returned, we replace G by $(G - v) \cup W$ and repeat. Note that v is not (l, s)-central in $(G - v) \cup W$, since v has degree at most three in this graph and $l \geq 4$. If we perform $\binom{m}{2}$ iterations, we have $\binom{m}{2}$ (l, s)-central vertices, and we obtain a minor of K_m in G by Lemma 8.

Corollary 10. *For every m, there exists k_0 and d''_m as follows. Suppose $W \subset G$ is a wall for $k \geq k'$, $W_1, \ldots, W_k$ are subwalls of W such that $d(W_i, W_j) \geq d''_m$ for all distinct i and j. In time $O(|G|^2)$, we can either find a model of K_m in G, or a set X of less than $\binom{m}{2}$ vertices of G such that all but less than k_0 of the walls $W_1, \ldots, W_k$ are dividing in $(G - X) \cup W$.*

Let W' be a wall with perimeter C in G. The *compass* of W' is C together with the C-bridge of G containing the interior of W. We say that a wall W' in G is *flat* if the compass of W' does not contain a cross over W'.

Since W_n has maximum degree three, if G contains W_n as a minor, then it contains a wall of height n as a subgraph. By the grid theorem, every graph of sufficiently large treewidth contains a wall of large height. In this wall, we can find many subwalls that are far apart, and by Corollary 10, we can find a set X of less than $\binom{m}{2}$ vertices such that many of the subwalls are dividing. Note that compasses of disjoint dividing walls are disjoint. Hence, applying again Lemma 12, we either obtain a large clique minor or at least $\binom{m}{2}$ flat subwalls, and at least one of them is disjoint from X. This gives the following important Flat Grid Theorem (also called the Weak Structure Theorem).

Theorem 11. *For all m and h, there exists t as follows. If G has treewidth at least t, then in time $O(|G|^2)$, we can either find a model of K_m in G, or a set X of less than $\binom{m}{2}$ vertices of G such that $G - X$ contains a flat wall of height h.*

3 Algorithm

Algorithm 2 now works as follows. If G has treewidth at most $f(|H|, k)$, we are done (the corresponding tree decomposition can be found efficiently). Otherwise, we apply Theorem 11 to find either a model of K_m in G, or a set X of less than $\binom{m}{2}$ vertices and a large flat wall in $G-X$. In the former case, we finish by Algorithm 4. In the latter case, one can prove that a "sufficiently generic" part of the flat wall contains an irrelevant vertex (however, the proof is very complicated!).

For the intuition, flat walls are indeed "almost planar" in the following sense. Let G be a graph and let Ω be a cyclic sequence of vertices of G. Then (G, Ω) is a *society*. A *cross* in a society consists of two disjoint paths P_1 and P_2 in G such that the labels of the ends x_1 and y_1 of P_1 and x_2 and y_2 of P_2 can be chosen so that they appear in Ω in order x_1, x_2, y_1, and y_2. A society is a *cell* if $|\Omega| \leq 3$.

A society (G_1, Ω_1) is a *subsociety* of (G, Ω) if G_1 is a subgraph of G, every edge of G_1 incident with $V(G_1) \setminus \Omega_1$ belongs to G_1, and $G_1 \cap \Omega \subseteq \Omega_1$. Two subsocieties (G_1, Ω_1) and (G_2, Ω_2) are *disjoint* if $G_1 \cap G_2 = \Omega_1 \cap \Omega_2$. A *segregation* of (G, Ω) is a set $\{(G_i, \Omega_i) : i = 1, \ldots, n\}$ of its disjoint subsocieties such that $G = G_1 \cup \ldots \cup G_n$. An *arrangement* of the segregation in an disk Δ is a function α such that $\alpha(G_i, \Omega_i)$ is a disk $\Delta_i \subseteq \Delta$ and for each $v \in \Omega_i$, $\alpha(v)$ is a distinct point in Δ contained in the boundary of Δ_i, such that

- for each i, the order of the points $\alpha(v)$ for $v \in \Omega_i$ in the boundary of Δ_i matches the order of the vertices v in Ω_i.
- for distinct i and j, the disks Δ_i and Δ_j intersect exactly in the points $\alpha(v)$ for $v \in \Omega_1 \cap \Omega_2$, and
- for each $v \in \Omega$, the point $\alpha(v)$ is contained in the boundary of Δ and their order in the boundary matches the order of the vertices v in Ω.

A society is *rural* if it has a segregation into cells with an arrangement in a disk. We need the following result, from the next homework assignment.

Lemma 12. *A non-rural society* (G, Ω) *contains a cross. Moreover, either the cross or the segregation and the arrangement witnessing the rurality of* (G, Ω) *can be found in time* $|G|^2$.

With regards to the flat walls, Lemma 12 is applied with G being the compass of the wall and Ω consisting of the branch vertices of the perimeter of the wall.