Recall H is a *minor* of G with *model* μ if

- μ assigns to vertices of H pairwise vertex-disjoint connected subgraphs of G, and
- for each edge $e = uv$ of H, $\mu(e)$ is a distinct edge of G not contained in any of these subgraphs and with one end in $\mu(u)$ and the other end in $\mu(v)$.

A *root function* is a function $r : V(H) \to 2^{V(G)}$ such that $r(v) \cap r(w) = \emptyset$ for distinct $v, w \in V(H)$. We say H is *rooted* in r if $r(v) \subseteq V(\mu(v))$ for every $v \in V(H)$. We most commonly deal with the case $|r(v)| \le 1$ for each $v \in V(H)$ (and indeed, rooted minors are usually defined in this way, as a partial injective function from $V(H)$ to $V(G)$); we will call such roots *simple*.

Let us note one important case, when H has no edges and $|r(v)| = 2$ for each $v \in V(H)$ (or, almost equivalently, the case H is a matching and r is a simple root function). Then, we are looking of pairwise vertex-disjoint paths with prescribed endpoints.

Our aim for the next two lectures is to give an algorithm to determine whether a graph G embedded in a surface contains a fixed graph H rooted in given r. Furthermore, we will prove that such a minor always exists when G contains a respectful tangle of sufficiently large order θ_H and the distance between any two root vertices is θ_H in the corresponding metric.

We say drawing of a graph in a surface with holes is *normal* if it intersects the boundary of the surface only in vertices. We say a root function r is *normal* if for each v, all vertices in $r(v)$ are contained in the boundary of the surface.

1 The disk case

Consider a graph G drawn normally in the disk, and let $v_1, v_2, \ldots, v_m$ be the vertices of G drawn in the cyclic order around the boundary of the disk. Let H be an edgeless graph and let r be a normal root function. We say r is *topologically infeasible* if there exist distinct $u, v \in V(H)$ and indices $i_1 < i_2 < i_3 < i_4$ such that $v_{i_1}, v_{i_3} \in r(u)$ and $v_{i_2}, v_{i_4} \in r(v)$, and *topologically feasible* otherwise. Note that if H has a minor in G rooted in r, then r is topologically feasible.

A *G-slice* is a simple G-normal curve c joining distinct points in the boundary of the disk and otherwise disjoint from the boundary. The endpoints of c divide the boundary of the disk into two arcs, let A_c and B_c denote the sets of vertices of G drawn in these two arcs (if the endpoints of c are root vertices, they are included both in A_c and B_c). We define r/c to be the set of vertices $v \in V(H)$ such that $r(v) \cap A_c \neq \emptyset \neq r(v) \cap B_c$. We say that r is *connectivity-wise feasible* if $|G \cap c| \geq |r/c|$ for every G-slice c. Note that if H has a minor in G rooted in r, then r is connectivity-wise feasible.

Our first result is a converse to these necessary conditions (which can be viewed as a disk version of Menger's theorem).

Theorem 1. *Let G be a graph G drawn normally in a disk Σ, let H be an edgeless graph and let r be a normal root function assigning to each vertex of H a non-empty set. If r is topologically and connectivity-wise feasible, then H is a minor of G rooted in r.*

Proof. We proceed by induction on $|V(G)|$. We can assume only root vertices are contained in the boundary, as otherwise we can shift the non-root vertices slightly away from the boundary without violating the topologic and connectivity-wise feasibility.

Suppose first there exists a G-slice disjoint from G such that both disks Σ_1 and Σ_2 into which it splits Σ intersect G. We have $|G \cap c| = 0$, and thus the connectivity-wise feasibility implies that for each v, $r(v)$ is contained in Σ_1 or Σ_2. Hence, we can find a minor in $\Sigma_1 \cap G_1$ and $\Sigma_2 \cap G_2$ by the induction hypothesis. Therefore, we can assume no such G-slice exists.

Next, consider the case that there exists a G-slice c intersecting G in exactly one vertex x such that $x \in r(v)$ for some $v \in V(H)$, and both disks Σ_1 and Σ_2 into which c splits Σ intersect $G - v$. We have $|G \cap c| = 0$ and $v \in r/c$, and thus by the connectivity-wise feasibility, for each $w \in V(H) \setminus \{v\}$, $r(w)$ is contained in Σ_1 or Σ_2. For $i \in \{1, 2\}$, let $G_i = \Sigma_i \cap G$, let H_i consist of v and vertices $w \in V(H) \setminus \{v\}$ such that $r(w) \subset \Sigma_i$, and let $r_i(v) = r(v) \cap \Sigma_i$ and $r_i(w) = r(w)$ for $w \in V(H_i) \setminus \{v\}$. Observe that r_i is topologically and connectivity-wise feasible in G_i, and thus by the induction hypothesis, H_i is a minor of G_i rooted in r_i. Connecting the two models, we obtain a minor of G rooted in r. Therefore, we can assume no such G-slice exists.

Suppose c is a simple closed curve in Σ intersecting G in exactly one vertex x, and at least one vertex of G is drawn in the open disk Λ bounded by c. Let G' be the subgraph of G obtained by deleting vertices and edges in Λ. Note that r is topologically and connectivity-wise feasible in G', and thus G' (and G) contains a minor of H rooted in r by the induction hypothesis. Therefore, we can assume no such closed curve exists. It follows that the boundary of each face of G is either a cycle or a path with both ends in the boundary.

Let v_1, v_2, …, v_m be the vertices of G drawn in order in the boundary of Σ. For $v \in V(H)$, let $I(v)$ be the minimal interval $\{v_i, v_{i+1}, \ldots, v_j\}$ containing $r(v)$. Let y be the vertex of H such that $I(y)$ is minimal among all vertices of

H. The minimality of $I(y)$ and the topological feasibility implies $I(y) \cap r(v) = \emptyset$ for every $v \in V(H - y)$.

If $|r(y)| = 1$, then let r' be the restriction of r to $H - y$, and note that r' is topologically and connectivity-wise feasible in $G - r(y)$. The claim then follows by the induction hypothesis, using the vertex in $r(y)$ as the model of y. Hence, we can assume $|r(y)| \geq 2$. It follows $r(y)$ contains two vertices x_1 and x_2 consecutive in the boundary of Σ. Let P be the path forming the boundary of the face containg the the arc of the boundary of Σ between x_1 and x_2. Note that P intersects the boundary of Σ only in x_1 and x_2, as otherwise there would exist a G-slice intersecting G in exactly one internal vertex of P contained in the boundary of Σ; we dealt with this case before. Let G/P be the graph obtained from G by contracting P to a single vertex p drawn in the boundary of Σ, and let r/P be obtained by replacing x_1 and x_2 by p in $r(y)$. Observe that r/P is topologically and connectivity-wise feasible in G/P, and thus H has a minor rooted in r/P in G/P. Replacing p by P in this model gives a minor of H in G rooted in r. □

2 Highly linked case

Let G be a graph with a normal drawing in a surface Σ which is neither the sphere nor the disk. The components of the boundary of Σ are called *cuffs*. For an integer p, we say the drawing is *p-generic* if

- every G-normal curve with ends in different cuffs intersects G in at least p points, and
- if a simple closed G-normal non-contractible curve c intersects G in less than p points, then there exists a cuff k such that $G \cap k \subseteq G \cap c$ and c is homotopic to k.

Let H be an edgeless graph and let r be a normal root function in G. We say r is *topologically feasible* if there exists a forest F drawn without crossings in Σ such that for each $v \in V(H)$, the forest F has a component F_v with $r(v) \subseteq V(F_v)$, and $F_v \neq F_w$ for distinct $v, w \in V(H)$. Note that the drawing of F in this definition is independent of the drawing of G, they can intersect arbitrarily.

Theorem 2. *For every surface Σ and integer k, there exists p such that the following holds. Let G be a graph with a normal drawing in a surface Σ with at least two holes, such that at most k vertices of G are drawn in the boundary of Σ, and each cuff contains at least one vertex of G. Let H be an edgeless graph and let r be a normal root function assigning to each vertex* *of H a non-empty set. If r is topologically feasible and the drawing of G is p-generic, then H is a minor of G rooted in r.*

Proof. Let g be the genus of Σ and h the number of holes in Σ. We will choose $p \gg s \gg g, h, p$ suitably.

A *G-net* is a graph N drawn in Σ so that

- $N \cap G = V(N) \cap V(G)$, i.e., N and G intersect only in vertices,
- each cuff traces a cycle in N, and
- N has exactly one face and this face is homeomorphic to an open disk Λ.

Choose such a G-net N with the smallest number of intersectins with G, and subject to that with the smallest number of vertices. Clearly N is connected and has minimum degree at least two. Moreover, since N has only one face, every cycle in N is non-separating, and thus non-contractible.

Let N' be the multigraph obtained from N by suppressing all vertices of degree two; note that N' can contain loops and parallel edges, but has minimum degree at least three. Let g be the genus of Σ and h the number of holes in Σ. Since N' has only one face, by Euler's formula we have $|E(N')| = |V(N')| + (h+1) + g - 2$, and since $|E(N')| \ge \frac{3}{2}|V(N')|$, this implies $|V(N')| \le 2(h+g-1)$ and $|E(N')| \le 3(h+g-1)$. Hence, N has at most $2(h+g-1)$ vertices of degree at least three, joined by at most $3(h+g-1)$ paths.

Let X be the set of vertices of N of degree at least three or belonging to the boundary of Σ. Let S be the subgraph of N induced by X, vertices at distance at most s from X, and paths of length at most $3s$ between the vertices of X. Note that S has at most $k + 9s(h+g) \ll p$ vertices. Since every cycle in N non-contractible and the drawing of G is p-generic, we conclude that this cycle must trace a cuff. Moreover, any path in S has length less than p, and thus each component of S contains at most one cuff. Hence, each component of S is either a tree, or a unicyclic graph containing a cuff. In particular, the surface with interior $\Sigma - S$ is connected and homeomorphic to the surface Σ with a bounded number of new holes.

Consider any vertex v drawn in the boundary of Σ, and let z be an arbitrary vertex drawn in a different cuff. Note that if $Z \subseteq V(G) \setminus \{v, z\}$ has size less than p, then no simple non-contractible curve can intersect G only in vertices of Z, since the drawing of G is p-generic. Consequently, Z does not separate v from z, and thus by Menger's theorem, G contains p internally vertex-disjoint paths from v to z. Out of these, all but $|V(S)|$ intersect S only in their endpoints, and from those internally disjoint from S, we can choose a set $\mathcal{P}_v$ of size at least $(p - |V(S)|)/|V(S)| \gg s$ that leave v through the same angle a_v among the incident edges of S.

Consider a drawing of the forest F in Σ certifying that r is topologically feasible. Up to homeomorphism there are (for fixed Σ and k) only finitely many options for the graph N' and the forest F, and for each combination of N' and F, we can fix a drawing where they intersect a finite number of times. Hence, there is a constant γ depending only on Σ and k such that F and N' intersect at most γ times; consequently, we have $s \gg \gamma$. Note that $N - S$ is a union of paths of length at least S, and thus we can shift F slightly so that it is disjoint from S except for the vertices in the cuffs, edges of F only leave each vertex v in the boundary of Σ through the angle a_v, and F intersects N only in vertices.

Let G' be the graph obtained from G by cutting along N, drawn in a disk Δ with interior homeomorpic to the face Λ. Cutting along N splits F into a number of components, let H' be the edgeless graph whose vertices are these components. For each component $Q \in V(H')$, let $r'(Q)$ consist of the vertices in which Q intersects the boundary of G'. Note that to obtain a minor of H in G rooted in r, it suffices to obtain a minor of H' in G' rooted in r' and combine parts of the model corresponding to $Q \in V(H')$ contained in the same component of F. Due to the way r' arises from the drawing of F, it is topologically feasible. Hence, by Theorem 1, we only need to argue it is connectivity-wise feasible.

Consider any G'-slice c, intersecting G' in t vertices. Suppose for a contradiction that $t < |r'/c|$. Note that $|r'/c| \le |V(H')| \le k + 2\gamma \ll s$. Let N_1 be the graph obtained from N by adding c, with vertices at intersections with G' and possibly at ends of c. Then N_1 has two faces, and thus it contains a cycle C (necessarily containing c) separating them.

If $C \not\subseteq S \cup c$ and $C \cap X \neq \emptyset$, then by the construction of S, C contains a path R of length s consisting of vertices of N of degree two not belonging to the boundary of Σ. Note that $t < s$ and that $N - R \cup c$ is a G-net, contradicting the choice of N intersecting G in the smallest number of vertices.

If $C \cap X = \emptyset$, then C consists of a path R of vertices of degree two in N and of c, and $|r'/c| \le |V(R)|$. Therefore, again $N - R \cup c$ is a G-net contradicting the minimality of G.

Therefore, $C \subseteq S \cup c$. Note that the only root vertices in C must belong to some cuff k intersecting C. Since $0 \le t < |r'/c|$, such a cuff must exist. Note that $C \cup k$ intersects G in less than $V(S) + s + k < p$ vertices. Since the drawing of G is G-generic, $C \cup k$ contains a contractible cycle K. Let f be the open disk bounded by K. The minimality of the G-net N implies f contains no vertices and edges of N, and thus f is a face of $N \cup c$ bounded by K. Since $r'/c \neq \emptyset$, the angle a_v for some $v \in V(G) \cap k$ must be contained in f. However, then every path in $\mathcal{P}_v$ must intersect c and $t \geq s$, which is a contradiction. □

Theorem 2 has a number of problematic assumptions. Excluding surfaces with less than two holes and requiring a vertex on each cuff is annoying, but relatively easy to work around. More substantial problem is that we forbid non-contractible curves with less than p intersections with G that are homotopic to a cuff k (but do not contain it); for applications, we will need to relax this assumption and only forbid such curves with less than $|G \cap k|$ intersections. We will do this (as well as obtaining the connection to the respectful tangles) in the next lecture.