Recall that a tangle assigns the "small" and the "large" side to each separation of bounded size in a graph G. In case G is drawn on a surface other than the sphere, a tempting idea is to declare the side which is non-planar to be the large side. Assuming this works (i.e., every separation has a unique non-planar side, and the resulting object satisfies the tangle axioms), the resulting (unique) tangle can be conveniently used to describe properties of the drawing, e.g., it enables us to say whether vertices u and v are "far apart" in the sense that G contains a system of disjoint non-contractible cycles such that any path from u to v intersects many of them. Let us now develop this theory in detail, including also the the case of graphs drawn in the sphere (i.e., planar graphs).

1 Terminology

Let Σ be a surface. A closed curve in Σ is *contractible* if it can be continuously transformed to a point; a simple (i.e., non-self-intersecting) closed curve c is contractible exactly if one of the components of $\Sigma \setminus c$ is an open disk; we denote the closure of this disk by $\mathrm{lake}(c)$. Note that every closed curve drawn in the sphere is contractible.

Consider a graph G drawn in Σ. Each closed walk in G is naturally traced by a closed curve; we naturally extend the terminology above to the closed walks (e.g., a closed walk is contractible if the closed curve tracing it is contractible, for a cycle C traced by a simple closed curve c we define $\mathrm{lake}(C) = \mathrm{lake}(c)$). The *representativity* of (the drawing of) G is the minimum number of intersections of a non-contractible closed curve with G (this notion clearly only makes sense if Σ is not the sphere). Note that we can always "shift" a curve to intersect G only in vertces without increasing the number of intersections, and thus it suffices to consider such closed curves; we say a closed curve is *G-normal* if it intersects G only in vertices.

A drawing of a graph G in a surface Σ is *2-cell* if every face is homeomorphic to an open disk. Note that

- if Σ is the sphere, then the drawing of G is 2-cell if and only if G is connected; and
- if Σ is not the sphere, then it is 2-cell if and only if G is connected and has representativity at least 1.

Suppose the drawing of G is 2-cell. The *radial graph* $R(G)$ of G is has as vertex $V(G)$ together with one vertex v_f drawn in each face f of G. For each face f bounded by the closed walk $v_1 v_2 \ldots v_m$, the edge set of $R(G)$ includes the edges $v_1v_f, \ldots, v_mv_f$ drawn inside f in the natural way. Note that each face of $R(G)$ has length four and contains exactly one edge of G drawn as its diagonal. Observe also that for the dual $G^\star$ of G, we have $R(G) = R(G^\star)$. Note that G-normal closed curves naturally correspond to closed walks in $R(G)$, obtained by "shifting" the curve within each face.

Observation 1. *Let G be a graph with a 2-cell drawing in a surface Σ. For each G-normal closed curve c, there exists a closed walk W in $R(G)$ and a bijection f from c to the closed curve tracing W such that for each component k of $c - G$, k and $f(k)$ are homotopic in $\Sigma \setminus G$.*

2 Slopes

Consider a graph H drawn in a surface Σ (you should imagine H being the radial graph of some other graph). A *slope* of order θ in H is a function ins that to every cycle $C \subseteq H$ of length less than 2θ in H assigns a closed disk $\mathrm{ins}(C) \subseteq \Sigma$ bounded by C, such that

(S1) If C_1 and C_2 are cycles of length less than 2θ in H and $C_1 \subseteq \mathrm{ins}(C_2)$, then $\mathrm{ins}(C_1) \subseteq \mathrm{ins}(C_2)$.

(S2) If $F \subseteq H$ is a theta graph (union of three paths intersecting exactly in their common endpoints) and all three cycles in F have length less than 2θ, then there exists a cycle $C \subseteq F$ such that every other cycle $C' \subseteq F$ satisfies $\mathrm{ins}(C') \subseteq \mathrm{ins}(C)$.

Note that if Σ is not the sphere, then there exists a slope of order θ in H if and only if every cycle of length less than 2θ is contractible, and in this case $\mathrm{ins}(C) = \mathrm{lake}(C)$ for every such cycle C. In case that Σ is the sphere, there are more ways how to choose ins. One is to drill a hole into one of the faces, transforming the sphere into an open disk (or, up to a homeomorphism, the plane), in which case each cycle encloses a unique closed disk. However, this construction gives slopes which are "degenerate" and we will be more interested in slopes which correspond to tangles in a sense we explore later. Let us now establish some basic properties of slopes.

We say $F \subseteq H$ is *confined* if every cycle in F has length less than 2θ. For a confined subgraph, we define $\mathrm{ins}(F)$ as the union of F and the disks $\mathrm{ins}(C)$ for every cycle in C. Thus, (S2) can be restated as saying that for every confined theta-subgraph F, $\mathrm{ins}(F) = \mathrm{ins}(C)$ for some cycle C in F. The following observation easily follows by considering the cycles C for which $\mathrm{ins}(C)$ is inclusionwise-maximal. A *cactus* is a graph in which any two cycles intersect in at most one vertex (equivalently, a cactus is a graph where every 2-connected block is either an edge or a cycle).

Lemma 2. *Let ins be a slope of order θ and let F be a confined graph drawn in a surface Σ. There exists a cactus $F' \subseteq F$ such that $ins(F) = ins(F')$, and for any distinct 2-connected blocks B_1 and B_2 of F', $ins(B_1)$ and $ins(B_2)$ intersect in at most one vertex. Consequently, there exists a face f of F such that $ins(F) = \Sigma \setminus f$.*

Suppose that H is a bipartite graph with a 2-cell drawing in a surface, and let X be one of the parts of its bipartition. For a set Z of faces of H, let $N(Z)$ denote the subgraph of H consisting of vertices and edges incident both with a face in Z and in $\overline{Z}$. Note that $N(Z) = N(\overline{Z})$. For a slope ins of order θ, we say that Z is X-*small* if $|N(Z)| \cap X| < \theta$ and $Z \subset \text{ins}(N(Z))$; note the former condition implies that $N(Z)$ is confined. Also, by Lemma 2, if $|N(Z)| \cap X| < \theta$, then exactly one of the sets Z and $\overline{Z}$ is X-small. The proof of the following key lemma is quite technical and we skip it.

Lemma 3. *In the situation described in the previous paragraphs, if $Z_1, Z_2, Z_3 \subset F(H)$ are X-small, then $Z_1 \cup Z_2 \cup Z_3 \neq F(H)$.*

3 Respectful pre-tangles

Consider a system $\mathcal{T}$ of separations of order less than θ in a graph G with a 2-cell drawing in a surface Σ. We say that $\mathcal{T}$ is *respectful* if for every cycle C in $R(G)$ of length less than 2θ, there exists a closed disk $\Delta \subseteq \Sigma$ bounded by C such that

$$(G \cap \Delta, G \cap \overline{\Sigma \setminus \Delta}) \in \mathcal{T}.$$

In that case, we define $\text{ins}_{\mathcal{T}}(C) = \Delta$. Let us remark that if Σ is not the sphere, we necessarily have $\text{ins}_{\mathcal{T}}(C) = \text{lake}(C)$. We say that $\mathcal{T}$ is a *pre-tangle* if it satisfies the tangle axioms (T1) and (T2) (see the first lecture).

Lemma 4. *Let G be a graph with a 2-cell drawing in a surface Σ. If $\mathcal{T}$ is a respectful pre-tangle of order θ in G, then $ins_{\mathcal{T}}(C)$ is a slope of order θ in $R(G)$.*

Proof. We need to verify the slope axioms. Note we can assume that Σ is the sphere, as otherwise $\text{ins}_{\mathcal{T}} = \text{lake}$ is a slope. For any cycle C_i in $R(G)$ of length less than 2θ, let $(A_i, B_i) = (G \cap \text{ins}_{\mathcal{T}}(C_i), G \cap \overline{\Sigma \setminus \text{ins}_{\mathcal{T}}(C_i)})$ be the corresponding separation belonging to $\mathcal{T}$.

(S1) Let C_1 and C_2 be cycles of length less than 2θ in $R(G)$ such that C_1 is drawn in the disk $\mathrm{ins}_{\mathcal{T}}(C_2)$. If $\mathrm{ins}_{\mathcal{T}}(C_1) \not\subseteq \mathrm{ins}_{\mathcal{T}}(C_2)$, then the union of the disks $\mathrm{ins}_{\mathcal{T}}(C_1)$ and $\mathrm{ins}_{\mathcal{T}}(C_2)$ is the whole sphere, and thus $A_1 \cup A_2 = G$. But this contradicts (T2).

(S2) Let F be a confined theta-subgraph of $R(G)$, let C_1, C_2, and C_2 be the cycles in F, and assume without loss of generality that $\mathrm{ins}_{\mathcal{T}}(C_1)$ is inclusionwise-maximal among $\mathrm{ins}_{\mathcal{T}}(C_1)$, $\mathrm{ins}_{\mathcal{T}}(C_2)$ and $\mathrm{ins}_{\mathcal{T}}(C_3)$. If $F \subseteq \mathrm{ins}_{\mathcal{T}}(C_1)$, then (S2) holds. Otherwise, the maximality of $\mathrm{ins}_{\mathcal{T}}(C_1)$ implies that $\mathrm{ins}_{\mathcal{T}}(C_1)$, $\mathrm{ins}_{\mathcal{T}}(C_2)$, and $\mathrm{ins}_{\mathcal{T}}(C_3)$ are the closures of the faces of F bounded by C_1, C_2, and C_3. Consequently, $\mathrm{ins}_{\mathcal{T}}(C_1) \cup \mathrm{ins}_{\mathcal{T}}(C_2) \cup \mathrm{ins}_{\mathcal{T}}(C_3)$ is the whole sphere and $A_1 \cup A_2 \cup A_3 = G$, contradicting (T2).

□

We say that the slope $\mathrm{ins}_{\mathcal{T}}$ is *derived* from $\mathcal{T}$. We can also go from a slope ins in $R(G)$ to a pre-tangle, as follows. Recall that $R(G)$ is bipartite, $V(G)$ is one of the parts of the bipartition, and faces of $R(G)$ correspond to edges of G. For a separation (A, B) of G of order less than θ, let Z_A be the set of faces of $R(G)$ corresponding to the edges of A. Note that a vertex v of G belongs to $N(Z_A)$ if and only if v is incident with edges of both A and B, and thus $v \in V(A \cap B)$; hence, $|V(N(Z_A)) \cap V(G)| \le |V(A \cap B)| < \theta$. We define $\mathcal{T}_{\mathrm{ins}}$ as the set of separations (A, B) of G of order less than θ such that Z_A is $V(G)$-small.

Lemma 5. *Let G be a graph with a 2-cell drawing in a surface Σ, and let ins be a slope in $R(G)$ of order θ. Then $\mathcal{T}_{ins}$ is a respectful pre-tangle in G of order θ, and ins is the slope derived from $\mathcal{T}_{ins}$.*

Proof. We need to verify the pre-tangle axioms for $\mathcal{T}_{\mathrm{ins}}$:

(T1) Consider any separation (A, B) of G of order less than θ. As we have argued, $|V(N(Z_A)) \cap V(G)| \le |V(A \cap B)| < \theta$. As we have seen before, this means that either Z_A or $\overline{Z_A} = Z_B$ is small, and thus either $(A, B) \in \mathcal{T}_{\mathrm{ins}}$ or $(B, A) \in \mathcal{T}_{\mathrm{ins}}$.

(T2) Consider any separations $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}_{\mathrm{ins}}$, meaning that Z_{A_1}, Z_{A_2}, Z_{A_3} are small. By Lemma 3, $R(G)$ has a face f not belonging to $Z_{A_1} \cup A_{A_2} \cup A_{A_3}$. This face corresponds to an edge e of G not belonging to $A_1 \cup A_2 \cup A_3$, and thus $A_1 \cup A_2 \cup A_3 \ne G$.

Next, let us argue $\mathcal{T}_{\mathrm{ins}}$ is respectful and ins is derived from it. Consider any cycle C in $R(G)$ of length less than 2θ, and let $(A, B) = (G \cap \mathrm{ins}(C), G \cap$

$\overline{\Sigma \setminus \text{ins}(C)})$ be the corresponding separation of G. It suffices to argue that $(A, B) \in \mathcal{T}$. This is clearly the case, since $N(Z_A) = C$ and $Z_A \subset \text{ins}(C)$, implying that Z_A is small. $\square$

We say that $\mathcal{T}_{\text{ins}}$ is the pre-tangle *induced* by the slope. So far, we have shown that if $\mathcal{T}$ is the pre-tangle induced by the slope ins, then ins is derived from $\mathcal{T}$. We want to establish a $(1 : 1)$-correspondence, and thus we also need the converse.

Lemma 6. *Let G be a graph with a 2-cell drawing in a surface Σ, and let $\mathcal{T}$ be a respectful pre-tangle in G of order θ. Let $ins = ins_{\mathcal{T}}$ be the slope derived from $\mathcal{T}$. Then $\mathcal{T}$ is induced by ins.*

Proof. Consider any separation $(A, B) \in \mathcal{T}_{\text{ins}}$, of order less than θ. We need to argue that $\mathcal{T}$ agrees with $\mathcal{T}_{\text{ins}}$ on this separation, i.e., that $(A, B) \in \mathcal{T}$. Since $(A, B) \in \mathcal{T}_{\text{ins}}$, Z_A is small, and thus for each edge $e \in E(A)$, there exists a cycle $C_e \subseteq N(Z_A)$ such that the face of $R(G)$ corresponding to e, and thus also e, is contained in $\text{ins}(C_e)$. Let $(A_e, B_e) = (G \cap \text{ins}(C_e), G \cap \overline{\Sigma \setminus \text{ins}(C_e)})$. We have $e \in E(A_e)$, and since ins is derived from $\mathcal{T}$, we have $(A_e, B_e) \in \mathcal{T}$. Moreover, $V(A_e \cap B_e) = V(C_e \cap G) \subseteq V(N(Z_A)) \subseteq V(A \cap B)$. Let (A_0, G) be the separation of G with $V(A_0) = V(A \cap B)$ and $E(A_0) = \emptyset$; by (T1) and (T2), we have $(A_0, G) \in \mathcal{T}$. By Lemma 1 from the first lecture notes (whose proof only uses (T1) and (T2), and thus also applies to pre-tangles), we have

$$(A, B) = \left(A_0 \cup \bigcup_{e \in E(A)} A_e, G \cap \bigcap_{e \in E(A)} B_e\right) \in \mathcal{T},$$

as required. $\square$

4 Respectful tangles

As we have seen, in the sphere there are slopes which have nothing to do with tangles, namely those where we choose any face f of $R(G)$ and define as $\text{ins}(C)$ the disk that does not contain f. These are actually the only "degenerate" slope irrelevant with respect to tangles.

Lemma 7. *Let G be a graph with a 2-cell drawing in a surface Σ, and let ins be a slope in $R(G)$ of order $\theta \geq 3$. For an edge $e \in E(G)$, let Δ_e be the closure of the 4-face of $R(G)$ containing e, and let C_e be the cycle bounding it. The pre-tangle $\mathcal{T}_{ins}$ is a tangle if and only if $ins(C_e) = \Delta_e$ for every $e \in E(G)$.*

Proof. If there exists $e \in E(G)$ such that $\mathrm{ins}(C_e) = \overline{\Sigma \setminus \Delta_a}$, then we have $(G - e, e) \in \mathcal{T}_{\mathrm{ins}}$ by the definition of $\mathcal{T}_{\mathrm{ins}}$, and since $V(G-e) = V(G)$, this means $\mathcal{T}_{\mathrm{ins}}$ does not satisfy (T3), and thus it is not a tangle.

Conversely, suppose that $\mathrm{ins}(C_e) = \Delta_e$ for every $e \in E(G)$. It suffices to argue that $\mathcal{T}_{\mathrm{ins}}$ satisfies (T3). Suppose for a contradiction $(A, B) \in \mathcal{T}_{\mathrm{ins}}$ and $V(A) = V(G)$, and thus $V(B) = V(A \cap B)$. For any edge $e \in E(B)$, we have $\mathrm{ins}(C_e) = \Delta_e$, and thus $(e, G - e) \in \mathcal{T}_{\mathrm{ins}}$. By Lemma 1 from the first lecture notes, we have

$$(G, V(B)) = \Big(A \cup \bigcup_{e \in E(A)} e, B \cap \bigcap_{e \in E(B)} (G - e)\Big) \in \mathcal{T}_{\mathrm{ins}},$$

which contradicts (T2) for the pre-tangle $\mathcal{T}_{\mathrm{ins}}$. □

For surfaces other than the sphere, this makes the situation regarding respectful tangles is quite simple.

Theorem 8. *Let G be a graph with a 2-cell drawing in a surface Σ other than the sphere. Then G contains a respectful tangle of order $\theta \geq 3$ if and only if the representativity of G is at least θ. Moreover, this respectful tangle is unique.*

Proof. If G has representativity less than θ, then by definition of the representativity and Observation 1, $R(G)$ contains a non-contractible cycle of length less than 2θ. This cycle does not bound any disk in Σ, and thus G cannot contain a respectful tangle.

Suppose now that G has representativity at least θ. Then there exists a unique slope, namely the slope lake, of order θ in G, and since we have established $(1:1)$-correspondence between slopes and respectful pre-tangles, it follows that G contains a unique pre-tangle $\mathcal{T}$ induced by lake. Moreover, for any edge $e \in E(G)$, we have $\mathrm{lake}(e) = \Delta_e$, and thus $\mathcal{T}$ is a tangle by Lemma 7. □

On the other hand, for graphs drawn in the sphere, every tangle is respectful. Note this has the following interesting corollary.

Theorem 9. *Let G be a plane graph and let $G^\star$ be its dual. Then there is a $(1:1)$-correspondence between tangles in G and $G^\star$, and in particular, G and $G^\star$ have the same tangle number (and branchwidth).*

Proof. Tangles in G are in $(1:1)$-correspondence with slopes in $R(G)$ that satisfy the condition of Lemma 7. The same is true for the tangles in $G^\star$. Since $R(G) = R(G^\star)$, the claim of the theorem follows. □

In particular, this implies that the treewidth of G and $G^\star$ differ by factor of at most 3/2.

5 The metric from respectful tangles

Let G be a graph with a 2-cell drawing in a surface Σ and let $\mathcal{T}$ be a respectful tangle of order θ in G. The *atoms* of G are its vertices, edges, and faces, and we let $A(G)$ denote their set. Note that vertices and faces correspond to vertices of $R(G)$, while the edges correspond to faces of $R(G)$; for each atom a, let $R(a)$ denote the corresponding object in $R(G)$. For a closed walk W in $R(G)$, we let $U(W)$ be the subgraph of $R(G)$ consisting of vertices and edges of W. For $a, b \in A(G)$, let us define $d(a, b)$ as follows:

- if $a = b$, then $d(a, b) = 0$,
- if there exists a closed walk W in $R(G)$ of length less than 2θ such that $R(a), R(b) \subseteq \mathrm{ins}_{\mathcal{T}}(U(W))$ and ℓ is the length of the shortest such walk, then $d(a, b) = \ell/2$, and
- otherwise, $d(a, b) = \theta$.

Then d is a metric and the following claims hold.

Lemma 10. *Let G be a graph with a 2-cell drawing in a surface Σ and let $\mathcal{T}$ be a respectful tangle of order θ in G. For every $a \in A(G)$, there exists $e \in E(G)$ such that $d_{\mathcal{T}}(a, e) = \theta$.*

Lemma 11. *Let G be a graph with a 2-cell drawing in a surface Σ and let $\mathcal{T}$ be a respectful tangle of order θ in G. For any $a \in A(G)$ and any integer $4 \leq k < \theta$, there exists a connected subgraph H of $R(G)$ containing exacly one cycle C and a closed disk Δ bounded by C such that $H - (V(H) \cap \Delta)$ is an independent set, and for each $b \in A(G)$, $d(a, b) \leq k$ if and only if $b \cap (H \cup \Delta) \neq \emptyset$.*

For more details, see the homework assignment.