1 The grid theorem

Let W_n denote the $n \times n$ grid. On one hand, $\text{tw}(W_n) = n$, and thus every graph that contains W_n as a minor has treewidth at least n. The grid theorem states that an approximate converse holds.

Theorem 1. *There exists a function f_1 as follows. For every integer n, if a graph G has treewidth at least $f_1(n)$, then G contains W_n as a minor.*

The first proof of this statement by Robertson and Seymour did not even explicitly state a bound on this function f_1. Later results gave explicit exponential bounds. In a breakthrough result, Chekuri and Chuzhoy (2014) gave a polynomial bound on this function. The current best result is $f_1(n) = O(n^9 \text{polylog } n)$. In the next lecture, we will prove a weaker polynomial bound; in this lecture, we do some preparatory work on obtaining grids in graphs containing certain substructures.

Note that K_m has treewidth $m-1$, but cannot contain a grid W_n for any $n > \sqrt{m}$, since $|V(W_n)| = n^2$. Hence, the function f_1 cannot be better than quadratic. By considering suitably chosen random graphs, one can improve this lower bound by a logarithmic factor.

The following claim is natural and not hard to prove.

Lemma 2. *For every planar graph H, there exists n_H such that H is a minor of W_{n_H}.*

In fact, it suffices to take n_H linear in $|V(H)|$. Combining this lemma with Theorem 1, we obtain a structure theorem for graphs avoiding a planar graph as a minor.

Corollary 3. *For every planar graph H, there exists c_H such that every graph not containing H as a minor has treewidth at most c_H.*

Note that for the application in the proof of the structure theorem, we need the following stronger form of Theorem 1 that shows that any tangle of large order "points towards" a grid minor. In the proof, we will ignore this detail (which can be dealt with by being careful in the proofs and making sure we always operate in the part of the graph specified by the tangle).

Theorem 4. *There exists a function f_4 as follows. If $\mathcal{T}$ is a tangle of order at least $f_4(n)$ in a graph G, then G contains W_n as a minor with model μ such that $\mathcal{T}$ is conformal with the canonical tangle in W_n and μ.*

2 Cleaning up towards a grid

In this section, we show how to obtain a grid minor from a path system in a nearly planar graph. A *linkage* is a set of pairwise vertex-disjoint paths; the linkage is an $(A - B)$-linkage if each of these paths starts in A and ends in B. The $(A - B)$-linkage $\mathcal{L}$ is *total* if $|A| = |B| = |\mathcal{L}|$, i.e., each vertex in $A \cup B$ is incident with one of its paths. For a linkage $\mathcal{L}$ in a graph G, let $G_{\mathcal{L}}$ be the graph with vertex set $\mathcal{L}$, where two paths $L, L' \in \mathcal{L}$ are adjacent if and only if G contains a path with one end in L, the other end in L', and otherwise disjoint from all paths in $\mathcal{L}$.

Let A and B be disjoint sets of vertices in a graph G such that $|A| = |B|$, and let U and D be vertex-disjoint paths from A to B in G. We say that (G, A, B, U, D) is a *loom of size* $|A|$ if for every total $(A - B)$-linkage $\mathcal{L}$ such that $U, D \in \mathcal{L}$, the graph $G_{\mathcal{L}}$ is a path from U to D. Observe this in particular is always the case if G is a plane graph and A, U, B, and D are contained in the boundary of the outer face of G in order.

Lemma 5. *Let (G, A, B, U, D) be a loom of size $n+2$. If G contains a total $(A - B)$-linkage containing U and D and a $(V(U) - V(D))$-linkage of size n, then $W_n \preceq G$.*

Proof. Let $\mathcal{L}$ be a total $(A - B)$-linkage in G containing U and D and $\mathcal{Q}$ a $(V(U) - V(D))$-linkage of size $n + 2$ chosen so that the subgraph $H = \bigcup \mathcal{L} \cup \bigcup \mathcal{Q}$ has the smallest number of edges. Without loss of generality, we can assume each path in $\mathcal{Q}$ intersects U and D in exactly one vertex.

Consider a path $Q \in \mathcal{Q}$. An *$\mathcal{L}$-segment* of Q is a subpath Q' of length at least 1 with both ends in $\bigcup \mathcal{L}$ and otherwise disjoint from paths in $\mathcal{L}$. Suppose Q' would have both ends in the same path $P \in \mathcal{L}$. By assumptions, $P \notin \{U, D\}$. We now could replace P by a path in $P \cup Q'$ passing through Q', obtaining another total $(A - B)$-linkage in G containing U and D and contradicting the minimality of $|E(H)|$. Hence, every $\mathcal{L}$-segment has ends in different paths (adjacent in the path $G_{\mathcal{L}}$).

Consider now two $\mathcal{L}$-segments Q and Q' between paths $P, P' \in \mathcal{L}$ different from U and D. Suppose the ends of Q and Q' appear in different order on P and P'. Then we can replace P and P' by paths in $P \cup P' \cup Q \cup Q'$ passing through Q and Q', obtaining another total $(A - B)$-linkage in G containing U and D and contradicting the minimality of $|E(H)|$. Hence, no such "crossing" $\mathcal{L}$-segments exist.

Let the path $G_{\mathcal{L}}$ be $UP_1P_2 \ldots P_nD$. Let G' be the subgraph of G consisting of $P_1, \ldots, P_n$ and the $\mathcal{L}$-segments between these paths. Let $\mathcal{Q}'$ consist of the paths obtained from those in $\mathcal{Q}$ by taking a minimal subpath with one end in P_1 and the other end in P_n. Let $A' = A \setminus V(U \cup C)$ and $B' = B \setminus V(U \cup D)$, and let $\mathcal{L}' = \{P_1, \ldots, P_n\}$. Then $\mathcal{L}'$ is a total $(A' - B')$-linkage in G' and $\mathcal{Q}'$ is a $(V(P_1) - V(P_n))$-linkage in G', each path in $\mathcal{L}'$ intersects P_1 and P_n in exactly one vertex, and G' can be drawn in the plane so that A', P_1, B, and P_n appear on the boundary of the outer face in order.

Choose a total $(A' - B')$-linkage $\mathcal{L}''$ with $P_1, P_n \in \mathcal{L}''$ and a $(V(P_1) - V(P_n))$-linkage $\mathcal{Q}''$ of size n in G such that the graph $H'' = \bigcup \mathcal{L}'' \cup \bigcup \mathcal{Q}''$ has the smallest number of edges. Without loss of generality, we can assume each path in $\mathcal{Q}''$ intersects P_1 and P_n in exactly one vertex. For $i \in \{1, \ldots, n\}$, let P_i'' be the path in $\mathcal{L}''$ with the same endpoints as P_i.

We claim that the intersection of every path in $\mathcal{L}''$ with every path in $\mathcal{Q}''$ is connected. If not, choose i minimum such that some path in $\mathcal{Q}''$ does not intersect P_i in a connected subpath. Consider a subpath Q' of some path $Q \in \mathcal{Q}''$ of length at least 1 with both ends in P_i and otherwise disjoint from P_i. Note Q' is disjoint from P_{i-1}, since $P_{i-1} \cap Q$ is connected by the minimality of i and Q intersected P_{i-1} before intersecting P_i. If Q' were drawn between P_i and P_{i-1}, then except for its endpoints, Q' would be disjoint from all paths of $\mathcal{L}''$. But then we could replace a part of P_i by Q', contradicting the minimality of $|E(H'')|$. Hence, every such subpath Q' is drawn between P_i and P_n (and it is disjoint from P_n, since Q intersects P_n only in its last vertex).

By the choice of i, there exists $Q \in \mathcal{Q}'$ with a subpath Q' of length at least 1 with both ends in P_i and otherwise disjoint from P_i. Consider the subpath P' of P_i between the ends of Q'. We claim the interior of P' does not intersect any path $Q_1 \in \mathcal{Q}'$; indeed, Q_1 would have to leave the area bounded by $P' \cup Q'$ by passing through the area between P_i and P_{i-1}, and in the previous paragraph, we argued this is not possible. Hence, we can replace Q' by P' in Q, contradicting the minimality of $|E(H'')|$.

Hence, we can contract each path $P_i \cap Q$ for $i \in \{1, \ldots, n\}$ and $Q \in \mathcal{Q}'$ to a single vertex, obtaining a minor of W_n in G. □

In particular, Lemma 5 is sufficient to prove the grid theorem in planar graphs, in a very strong form.

Theorem 6. *for every integer n, every planar graph of treewidth at least $6n + 14$ contains W_n as a minor.*

Proof. Let G be a plane graph of treewidth at least $6n + 14$, without loss of generality connected. Then G contains a tangle $\mathcal{T}$ of order $4n+9$. For a closed disk Δ whose boundary intersects G only in vertices, let $S_\Delta = (A_\Delta, B_\Delta)$ be the separation of G where B_Δ is the subgraph of G drawn in Δ and A_Δ is the subgraph of G drawn in the closure of the complement of Δ. Choose Δ so that $S_\Delta \in \mathcal{T}$ and subject to that A_Δ is maximal. Let $Z = V(A_\Delta \cap B_\Delta)$ be the set of vertices of G contained in the boundary of Δ. We have $|Z| = 4n+8$ and Z is an independent set in B_Δ, as otherwise we can shift the boundary of Δ slightly to make A_Δ include one more vertex or edge (this keeps the separation in the tangle, as is easy to see from Lemma 1 from Lesson 1).

Let us divide Z into four sets $Z_1, \ldots, Z_4$ of size $2n+4$ drawn along the boundary of Δ. We claim that B_Δ contains a total $(Z_1 - Z_3)$-linkage $\mathcal{L}$. Indeed, otherwise by a form of Menger's theorem, we could split Δ along a curve c starting and ending in its boundary and intersecting G only in vertices into two disks Δ_1 and Δ_2 such that say $\Delta_1 \cap Z_3 = c \cap Z_3$ and $|\Delta_1 \cap Z_1| > |c \cap V(G)|$. Then S_{Δ_1} has order less than $|Z| - |Z_3| + |c \cap V(G)| = |Z| - |Z_1| + |c \cap V(G)| < |Z|$ and S_{Δ_2} has order less than $|Z| - |\Delta_1 \cap Z_1| + |c \cap V(G)| < |Z|$. By the maximality of A_Δ, we have $S_{\Delta_1}, S_{\Delta_2} \notin \mathcal{T}$, but this contradicts the tangle properties (T1) and (T2).

Symmetrically, B_Δ contains a total $(Z_2 - Z_4)$-linkage $\mathcal{Q}$. We can now apply Lemma 5 to the loom between the topmost and the bottommost path of $\mathcal{L}$. □

Consider a planar graph G and let $n = \lceil \sqrt{|V(G)| + 1} \rceil$. Then G cannot contain W_n as a minor, since W_n has at least $|V(G)| + 1$ vertices. Hence, Theorem 6 has the following consequence.

Corollary 7. *Every planar graph G has treewidth at most* $6\lceil \sqrt{|V(G)| + 1} \rceil + 13$.

3 Flows and linkedness

A *flow* from A to B in a graph G is an assignment of non-negative flow values $f(e)$ to edges of an orientation of G such that

$$\sum_{e \text{ from } v} f(e) = \sum_{e \text{ to } v} f(e)$$

for $v \in V(G) \setminus (A \cup B)$,

$$\sum_{e \text{ from } v} f(e) - \sum_{e \text{ to } v} f(e) \leq 1$$

for $v \in A$ and

$$\sum_{e \text{ from } v} f(e) - \sum_{e \text{ to } v} f(e) \geq -1$$

for $v \in B$. The *size* of the flow is

$$\sum_{v \in A} \Big(\sum_{e \text{ from } v} f(e) - \sum_{e \text{ to } v} f(e) \Big).$$

The *congestion* of an edge e is $f(e)$, the amount of flow passing across e. Similarly, the *congestion* of a vertex v is the amount of flow passing through v (leaving v if $v \notin B$, entering v if $v \notin A$). The *edge/vertex congestion* of a flow is the maximum congestion of its edges/vertices. If G has maximum degree Δ and the flow has edge congestion at most c, then it clearly also has vertex congestion at most $\Delta c + 1$.

Note the following connection between flows and linkages.

Lemma 8. *Suppose G contains a flow of size s from A to B with vertex congestion at most c. Then G contains an $(A - B)$-linkage of size at least s/c.*

Proof. Divide all flow values by c; we obtain a flow of size s/c with vertex congestion at most 1. In other words, we obtain a flow from A to B where each vertex has capacity 1. Since all capacities are integers, there exists a maximum flow in this network whose values are integers, necessary 0 or 1 due to the capacities. This flow consists of at least s/c pairwise vertex-disjoint paths from A to B. □

We say a set $W \subseteq V(G)$ is *a-well-linked* in a graph G if for all disjoint subsets A and B of W of the same size, G contains a flow from A to B of size $|A|$ and edge congestion at most a. We say W is *node-well-linked* if G contains a total $(A - B)$-linkage for any such subsets A and B. In graphs of bounded maximum degree, these concepts are connected by the following lemma.

Lemma 9. *Let G be a graph of maximum degree Δ and let T be an a-well-linked set of its vertices. Then there exists $T' \subseteq T$ such that $|T'| \geq \frac{|T|}{4(\Delta a+1)}$ and T' is node-well-linked.*

Proof. Let $t = |T|$. Let (A, B) be a separation of G of minimum order such that $|V(A) \cap T|, |V(B) \cap T| \geq t/4$. By symmetry, we can assume $|V(A) \cap T| \geq t/2$. Let $W = V(A \cap B)$. We claim W is node-well-linked in A. Indeed, otherwise Menger's theorem implies A has a separation (X, Y) of order less then $\min(|V(X) \cap W|, |V(Y) \cap W|)$. By symmetry, we can assume $|V(X) \cap T| \geq |V(A) \cap T|/2 \geq t/4$. But $(X, Y \cup B)$ has order $o(A, B) - |V(X) \cap W| + o(X, Y) < o(A, B)$, contradicting the choice of (A, B).

Let C and D be disjoint subsets of $V(A) \cap T$ and $V(B) \cap T$ of the same size at least $t/4$. Since T is a-well-linked, G contains a flow from C to D of size $|C|$ and edge congestion at most a. This flow has vertex congestion at most $\Delta a + 1$, and by Lemma 8, G contains a $(C - D)$-linkage $\mathcal{L}$ of size at least $\frac{t}{4(\Delta a+1)}$. Let T' be the set of vertices of this linkage in C. Then T' is node-well-linked—for any subsets of T', we can follow the paths of $\mathcal{L}$ to W, then link them appropriately in A using the fact that W is node-well-linked in A. □

We say that two disjoint sets X and Y in G are *node-linked* if for all $A \subseteq X$ and $B \subseteq Y$ of the same size, G contains a total $(A - B)$-linkage. Again, we can derive this property from a weaker one.

Lemma 10. *Let G be a graph of maximum degree Δ and let L and R be disjoint node-well-linked sets of its vertices of size at least k. If $L \cup R$ is a-well-linked, then any subsets of L and R of size at most $\frac{k}{\Delta a+2}$ are node-linked.*

Proof. Consider any sets $A \subseteq L$ and $B \subseteq R$ of the same size k'. If G did not contain a total $(A - B)$-linkage, we could separate A from B by deleting a set S of less than k' vertices. Let A' be the set of vertices of L separated from A by S; since L is node-well-linked, we have $|A'| \leq |S| < k'$. Similarly, the set B' of vertices of R separated from B by S has size less than k'. Then S separates $L \setminus A'$ from $R \setminus B'$. Since $L \cup R$ is a-well-linked, G contains a flow from a subset of $L \setminus A'$ to a subset of $R \setminus B'$ of size $k - k'$ and with edge congestion at most a, implying the vertex congestion at most $\Delta a + 1$. Such a flow passes through S, and thus its size is less than $k'(\Delta a + 1)$. It follows that $k - k' < k'(\Delta a + 1)$, and $k' > \frac{k}{\Delta a+2}$. □

4 Grids in node-linked bricks

We need the following standard result about spanning trees.

Lemma 11. *Let H be a connected graph with at least $2a(b + 5)$ vertices. Then either H contains a spanning tree with at least a leaves, or a path with b vertices which all have degree two in H.*

Proof. Consider a spanning tree T of H with the largest number of leaves. Let X denote the set of vertices whose degree in T is other than two, and let Y be the set of vertices at distance at most two from X in T. We can assume T has less than a leaves, and consequently $|X| < 2a$ and $|Y| < 10a$. Note that $T - Y$ is a union of less than $2a$ paths, and thus one such path P contains at least b vertices. We claim that every vertex v of P has degree two in H. Indeed, if not, consider an edge $e \in E(H) \setminus E(T)$ incident with v. Let C be the unique cycle in $T + e$, and let e' be the edge of $C - e$ at distance 1 from v. The choice of Y implies that both ends of e' have degree two in T. Hence, $T - e + e'$ is a spanning tree of G with more leaves than T, which is a contradiction. □

A *brick* of height h is a triple (G, A, B), where G is a graph and A and B are disjoint subsets of vertices of G of size h. The brick is *a-well-linked* if $A \cup B$ is a-well-linked in G, and *node-linked* if A and B are node-well-linked in G and A and B are node-linked in G. We need the following claim about node-linked bricks.

Lemma 12. *Let (G, A, B) be a node-linked brick of height $2n(6n+9)$. Then either $W_n \preceq G$, or there exists an $(A-B)$-linkage $\mathcal{L}$ in G of size n and a connected subgraph H of G disjoint from the paths in $\mathcal{L}$ and with a neighbor in each of the paths in $\mathcal{L}$.*

Proof. Let $\mathcal{L}_0$ be a total $(A-B)$-linkage in G chosen so that the graph $G_{\mathcal{L}_0}$ (defined at the beginning of Section 2) has as few vertices of degree two as possible. If $G_{\mathcal{L}_0}$ has a spanning tree T with at least n leaves, then let $\mathcal{L} \subset \mathcal{L}_0$ consist of n paths corresponding to the leaves, and let H be the union of $\bigcup(\mathcal{L}_0 \setminus \mathcal{L})$ with the paths in G corresponding to the edges of T; then H is connected and has a neighbor in each of the paths in $\mathcal{L}$.

Hence, we can assume $G_{\mathcal{L}_0}$ has no such spanning tree, and thus it contains a path P with $6n+4$ vertices, such that all vertices of P have degree two in $G_{\mathcal{L}_0}$. Let $L_1, \ldots, L_{6n+4}$ be the paths from $\mathcal{L}_0$ corresponding to the vertices of P in order, and let a_i and b_i denote the ends of L_i in A and B. Let $\mathcal{L}_1 = \{P_1, \ldots, P_{n+2}\}$, $\mathcal{L}_2 = \{P_{3n+3}, \ldots, P_{4n+4}\}$, $A_1 = \{a_{n+3}, \ldots, a_{3n+2}\}$ and $B_1 = \{b_{4n+5}, \ldots, b_{6n+4}\}$. For $i \in \{1, 2\}$, let F_i be the subgraph of G consisting of $\bigcup \mathcal{L}_i$ and all vertices and edges on paths starting and ending in $\bigcup \mathcal{L}_i$ and otherwise disjoint from $\mathcal{L}_0$.

Since A and B are node-linked, there exists a total $(A_1 - B_1)$-linkage $\mathcal{Q}_0$ in G. Since P is a path of vertices of degree two in $G_{\mathcal{L}_0}$, observe that each path $Q \in \mathcal{Q}_0$ contains a subpath Q' that either is contained in F_1 and joins P_1 with P_{n+2}, or is contained in F_2 and joints P_{3n+3} with P_{4n+4}. Without loss of generality, we can assume that there exists $\mathcal{Q}_1 \subset \mathcal{Q}_0$ of size n such that for every $Q \in \mathcal{Q}_1$, the path Q' is contained in F_1. Let $\mathcal{Q} = \{Q' : Q \in \mathcal{Q}_1\}$, let $L = \{a_1, \ldots, a_{n+2}\}$ and let $R = \{b_1, \ldots, b_{n+2}\}$.

We claim that $(F_1, L, R, P_1, P_{n+2})$ is a loom of size $n+2$. Indeed, consider any total $(L-R)$-linkage $\mathcal{L}'$ in F_1 that contains P_1 and P_{n+2}, and let $\mathcal{L}'_0 = (\mathcal{L}_0 \setminus \mathcal{L}_1) \cup \mathcal{L}'$. Then $\mathcal{L}'_0$ is a total $(A-B)$-linkage in G. Observe that the graph $G_{\mathcal{L}'_0}$ differs from the graph $G_{\mathcal{L}_0}$ only in adjacencies of vertices corresponding to $\mathcal{L}_1$ or $\mathcal{L}'$. Since $G_{\mathcal{L}_0}$ has the smallest number of vertices of degree two, we conclude that all vertices of $\mathcal{L}'$ have degree two in $G_{\mathcal{L}'_0}$ as well, and thus the subgraph induced by $\mathcal{L}'$, which is equal to $(F_1)_{\mathcal{L}'}$, is a path from P_1 to P_{n+2}.

Therefore, F_1 (and thus also G) contains W_n as a minor by Lemma 5. $\square$

5 Path of sets systems

We now combine bricks into a larger structure. A *path-of-sets system* of width w and height h in a graph G is a sequence (H_1, A_1, B_1), $\ldots$, (H_w, A_w, B_w) of vertex-disjoint bricks of height h such that H_i is an induced subgraph of G for $i \in \{1, \ldots, w\}$, and G contains total $(B_i - A_{i+1})$-linkages $\mathcal{L}_i$ for $i \in \{1, \ldots, w-1\}$ such that the paths in $\bigcup_{i=1}^{w-1} \mathcal{L}_i$ are pairwise disjoint and disjoint from $H_1 \cup \ldots \cup H_w$ except for their endpoints; we say $\mathcal{L}_i$ is an *i-connector* of the system. The system is a-well-linked or node-linked if its bricks have these properties.

Lemma 12 easily gives a way to turn a node-linked system into a grid minor.

Lemma 13. *If G contains a node-linked path-of-sets system of width $2n^2$ and height $2n(6n+9)$, then $W_n \preceq G$.*

Proof. Let (H_i, A_i, B_i) be the bricks of the system. We can assume $W_n \not\preceq H_i$ for each i, and thus by Lemma 12, there exists an (A_i, B_i)-linkage $\mathcal{P}_i$ and a connected subgraph F_i in H_i that has a neighbor in each path of $\mathcal{P}_i$. Let $\mathcal{Q}_i$ be the linkage consisting of the paths from the i-connector $\mathcal{L}_i$ starting from the ends of the paths in $\mathcal{P}_i$, the paths from the $(i+1)$-connector $\mathcal{L}_{i+1}$ ending with the starting vertices of the paths in $\mathcal{P}_{i+1}$, and a total linkage in H_{i+1} between the ends of the paths in $\mathcal{L}_i$ and starts of the paths in $\mathcal{L}_{i+1}$.

We join paths from $\mathcal{P}_1$, $\mathcal{Q}_1$, $\mathcal{P}_3$, $\mathcal{Q}_3$, $\ldots$ to n long paths, which will form the rows of the grid minor. We then use paths in F_1, F_3, $\ldots$ to represent the vertical edges in the minor. $\square$

Finally, in a graph of bounded maximum degree, we can turn an a-well-linked path-of-sets system into a node-linked one.

Lemma 14. *Suppose (H_1, A_1, B_1), $\ldots$, (H_w, A_w, B_w) is an a-well-linked path-of-sets system of height at least $16(\Delta a + 1)^2 h$ in a graph G of maximum degree at most Δ. Then there exist sets $A'_i \subseteq A_i$ and $B'_i \subseteq B_i$ of size h such that (H_1, A'_1, B'_1), $\ldots$, (H_w, A'_w, B'_w) is a node-linked path-of-sets system.*

Proof. We apply Lemma 9 to A_1 in H_1 and to B_w in H_w and select node-well-connected subsets A'_1 and B'_w of size h from the resulting sets. For $i = 1, \ldots, n-1$, we perform the following. We apply Lemma 9 in H_i to B_i, obtaining a node-well-linked set B''_i of size at least $4(\Delta a + 1)h$. We let $A''_{i+1} \subseteq A_{i+1}$ be the set of vertices connected to B''_i by the i-connector. We apply Lemma 9 in H_{i+1} to obtain a node-well-linked set $A'_{i+1} \subset A''_{i+1}$ of size h. Then we choose B'_i as the subset of B''_i connected to A'_i by the i-connector. Note that by Lemma 10, A_i and B_i are node-linked in H_i. $\square$

Combining these two results, we obtain the following conclusion.

Corollary 15. *If G has maximum degree Δ and contains an a-well-linked path-of-sets system of width $2n^2$ and height $32(\Delta a+1)^2 n(6n+9)$, then $W_n \preceq G$.*

Hence, to prove the grid theorem, it suffices to find such a system in graph of large treewidth.