1 Goals

Towards the structure theorem:

1. Tangles (specifying a highly connected part), structure relative to a tangle $\to$ global structure.
2. Grid theorem (forbidding a planar graph).
3. Minors in graphs with large representativity.
4. Tangles and metric in graphs on surfaces.
5. Flat grid theorem (weak structure theorem), testing presence of a minor.
6. Basic ideas of the structure theorem proof.

Applications of the structure theorem:

1. Decomposition to bounded treewidth graphs.
2. Existence of bipartite minors in graphs of large connectivity.

WQO:

1. bounded treewidth
2. graphs on surfaces

2 Tangles

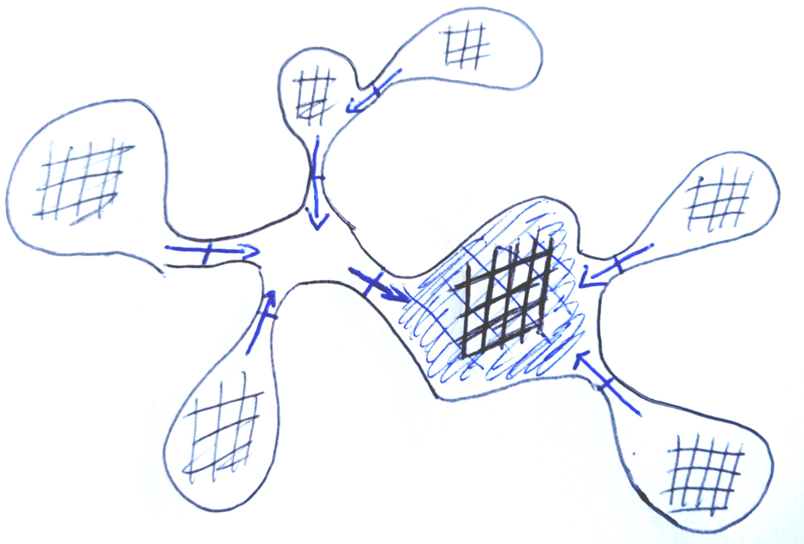
For a graph H, let $|H|$ denote the number of vertices of H. *Separation* in a graph G is a pair (A, B) of edge-disjoint subgraphs such that $A \cup B = G$; its *order* $o(A, B)$ is $|A \cap B|$. Suppose (T, β) is a tree decomposition of G. For $uv \in T$, let T_{uv} be the subtree of $T - uv$ containing u, and let $G_{uv} = G\left[\bigcup_{x \in V(T_{uv})} \beta(x)\right]$ and let G^{uv} be the graph with vertex set $G\left[\bigcup_{x \in V(T_{vu})} \beta(x)\right]$ and edge set $E(G) \setminus E(G_{uv})$. Then (G_{uv}, G^{uv}) is a separation of G of order $|\beta(u) \cap \beta(v)|$.

Tangle of order θ is a set $\mathcal{T}$ of separations of G of order less than θ such that

(T1) for every separation (A, B) of order less than θ, either $(A, B) \in \mathcal{T}$ or $(B, A) \in \mathcal{T}$, (T2) $G \neq A_1 \cup A_2 \cup A_3$ for all $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$, and

(T3) $V(A) \neq V(G)$ for all $(A, B) \in \mathcal{T}$.

$(A, B) \in \mathcal{T}$: "A is the small side of the separation (A, B)"; the tangle $\mathcal{T}$ "points towards a single well-connected piece of the graph".



Lemma 1. *Suppose $\mathcal{T}$ is a tangle of order θ in a graph G.*

1. *If $(A, B) \in \mathcal{T}$, then $(B, A) \notin \mathcal{T}$.*
2. *If $(A, B) \in \mathcal{T}$ and (A', B') is a separation of G of order less than θ such that $A' \subseteq A$, then $(A', B') \in \mathcal{T}$.*
3. *If $(A_1, B_1) \in \mathcal{T}$, $(A_2, B_2) \in \mathcal{T}$, and $(A_1 \cup A_2, B_1 \cap B_2)$ has order less than θ, then $(A_1 \cup A_2, B_1 \cap B_2) \in \mathcal{T}$.*

Proof. 1. By (T2) applied to (A, B), (B, A), (B, A).

2. By (T2), we have $(B', A') \notin \mathcal{T}$, and thus $(A', B') \in \mathcal{T}$ by (T1).

3. Since $A_1 \cup A_2 \cup (B_1 \cap B_2) = (A_1 \cup A_2 \cup B_1) \cap (A_1 \cup A_2 \cup B_2) = G \cap G = G$, the claim follows by (T2) and (T1).

□

Corollary 2. *If G has a tangle $\mathcal{T}$ of order θ, then $tw(G) \geq \theta - 1$.*

Proof. For a contradiction, suppose (T, β) is a tree decomposition of G of width less than $\theta - 1$. By (T1) and Lemma 1, T has an orientation $\vec{T}$ such that $uv \in E(\vec{T})$ if and only if $(G_{uv}, G^{uv}) \in \Theta$. Since T is a tree, $\vec{T}$ has a sink x. By (T3) and (T1), we have $(G[\beta(x)], G - E(G[\beta(x)])) \in \mathcal{T}$, and iteratively applying the last item of Lemma 1 to this separation and separations $(G_{yx}, G^{yx}) \in \mathcal{T}$ for all edges xy of T incident with x, we conclude $(G, B) \in \mathcal{T}$ for some $B \subseteq G[\beta(x)]$, contradicting (T3). □

Examples of tangles:

- In K_n, $(A, B) \in \mathcal{T}$ iff $|V(A)| < \frac{2}{3}n$ is a tangle of order $\frac{2}{3}n$—for (T2), note there exists a vertex v belonging to at most one of A_1, A_2, and A_3, but not all edges incident with v can appear in this subgraph.
- In an $n \times n$ grid, $(A, B) \in \mathcal{T}$ iff its order is less than n and A does not contain any of the rows is a tangle of order n. For (T1), if A and B both contained a row, then in each column there would have to be a vertex of $A \cap B$, and (A, B) would have order at least n. (T2) is not easy to see (a 1-page proof). We call this tangle *canonical*.

We will need the following simple observation.

Lemma 3. *Suppose $\theta \geq 3$. Let $\mathcal{T}$ be a set of separations in a graph G of order less than θ that satisfies (T1), (T2), and such that $(e, G - e) \in \mathcal{T}$ for every $e \in E(G)$. Then $\mathcal{T}$ is a tangle of order θ.*

Proof. It suffices to prove that (T3) holds. We did not use (T3) in the proof of Lemma 1, and thus this lemma holds for $\mathcal{T}$. Consider any separation (A, B) of G of order less than θ such that $V(A) = V(G)$, and suppose for a contradiction that $(A, B) \in \mathcal{T}$. Since the separation has order less than θ, we have $|B| < \theta$. Applying Lemma 1 for all separations $(e, G - e)$ with $e \in E(B)$, we can assume $E(B) = \emptyset$, and thus $A = G$. This contradicts (T2). □

H is a *minor* of G with *model* μ if μ assigns to vertices of H pairwise vertex-disjoint connected subgraphs of G, and for each edge $e = uv$ of H, $\mu(e)$ is a distinct edge of G not contained in any of these subgraphs and with one end in $\mu(u)$ and the other end in $\mu(v)$.

Lemma 4. *Suppose $\mathcal{T}'$ is a tangle of order $\theta \geq 3$ in H, and μ is a model of a minor of H in a graph G. Let $\mathcal{T}$ be the set of separations (A, B) of order less than θ such that $\mathcal{T}'$ contains a separation (A', B') such that $E(A') = \mu^{-1}(E(A))$. Then $\mathcal{T}$ is a tangle of order θ in G.*

Proof. For a separation (A, B) of G of order less than θ, consider the separation (A', B') of H, where $E(A') = \mu^{-1}(E(A))$, $E(B') = \mu^{-1}(E(B))$, $V(A') = \{v \in V(H) : \mu(v) \cap A \neq \emptyset\}$ and $V(B') = \{v \in V(H) : \mu(v) \cap B \neq \emptyset\}$. Since the subgraphs in $\mu(V(H))$ are pairwise vertex-disjoint and connected, (A', B') has order at most the order of (A, B). By (T1) we have $(A', B') \in \mathcal{T}'$ or $(B', A') \in \mathcal{T}'$, and thus $(A, B) \in \mathcal{T}$ or $(B, A) \in \mathcal{T}$, implying (T1) for $\mathcal{T}$.

If $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$ and $A_1 \cup A_2 \cup A_3 = G$, then there would exist $(A'_1, B'_1), (A'_2, B'_2), (A'_3, B'_3) \in \mathcal{T}'$ with $E(A'_1 \cup A'_2 \cup A'_3) = E(H)$. Since $\theta > 0$, $(v, H - v) \in \mathcal{T}'$ for every isolated vertex $v \in H$ by (T1) and (T3), and thus using Lemma 1, we can assume all isolated vertices of H are in $A'_1 - V(B'_1)$. Hence, $A'_1 \cup A'_2 \cup A'_3 = H$, contradicting (T2). This implies (T2) holds for $\mathcal{T}'$.

By Lemma 3, it suffices to prove that for every $e \in E(G)$, we have $(e, G - e) \in \mathcal{T}$. Indeed, consider the separation (A', B') of H, where A' is the subgraph consisting of at most one edge $\mu^{-1}(e)$ and $V(B') = V(H)$. By (T1) and (T3), we have $(A', B') \in \mathcal{T}'$, and thus $(e, G - e) \in \mathcal{T}$, $\square$

We say $\mathcal{T}$ is *induced by $\mathcal{T}'$ and μ* in G. We say that a tangle $\mathcal{T}_1$ of order at least θ in G is *conformal* with $\mathcal{T}'$ and μ if $\mathcal{T} \subseteq \mathcal{T}_1$, i.e., $\mathcal{T}$ consists exactly of separations from $\mathcal{T}_1$ of order less than θ.

3 Grids, brambles and unbreakable sets

A set $\mathcal{B}$ of non-empty subsets of $V(G)$ is a *bramble* in G if

- for every $B \in \mathcal{B}$, $G[B]$ is connected, and
- for every $B_1, B_2 \in \mathcal{B}$, $G[B_1 \cup B_2]$ is connected.

A set $X \subseteq V(G)$ is the *hitting set* of the bramble if $X \cap B \neq \emptyset$ for every $B \in \mathcal{B}$. The *order* of the bramble is the size of the smallest hitting set.

A set $W \subseteq V(G)$ is *s-breakable* if there exists a separation (A, B) of order less than s such that $|W \setminus V(A)| \leq \frac{2}{3}|W|$ and $|W \setminus V(B)| \leq \frac{2}{3}|W|$, and *$s$-unbreakable* otherwise.

Lemma 5. *For a graph G:*

- *$n \times n$ grid minor $\Rightarrow$ a tangle of order n*
- *a tangle of order $\theta \Rightarrow$ a $(\theta/3 - 1)$-unbreakable set of size $\theta - 1$*
- *an s-unbreakable set $\Rightarrow$ a bramble of order at least s*

- *a bramble of order at least $s \Rightarrow$ a tangle of order $s/3$*

Proof. The first part is by Lemma 4.

For the second part, let (A, B) be a separation of order less than θ in the tangle $\mathcal{T}$ with B minimal and subject to that with A maximal, and let $W = A \cap B$. The maximality of A implies $|W| = \theta - 1$, as otherwise we can add a vertex from $V(B) \setminus V(A)$ to A (as an isolated vertex) by Lemma 1. Suppose W is $(\theta/3 - 1)$-breakable, as shown by a separation (A', B'), which by (T1) we can assume to belong to $\mathcal{T}$. Then $(A \cup A', B \cap B')$ has order at most $o(A', B') + \frac{2}{3}o(A, B) < \theta$, and thus $(A \cup A', B \cap B') \in \mathcal{T}$ by Lemma 1. The minimality of B implies $B \subseteq B'$. But then $W \subseteq V(B')$, and thus $|W \setminus V(A')| = |W \setminus V(A' \cap B')| > \frac{2}{3}\theta > \frac{2}{3}|W|$, a contradiction.

For the third part, let W be an s-unbreakable set and let $\mathcal{B}$ consists of all $X \subseteq V(G)$ such that $G[X]$ is connected and $|X \cap W| > |W|/2$. Clearly $\mathcal{B}$ is a bramble, as any two elements of $\mathcal{B}$ intersect. Suppose for a contradiction S is a hitting set for $\mathcal{B}$ of size less than s. Each component of $G - S$ contains at most $|W|/2$ vertices of W, as otherwise it would belong to $\mathcal{B}$ and be disjoint from S. We can distribute the components into two parts such that each of them contains at most $\frac{2}{3}|W|$ elements of W, thus obtaining a separation (A, B) with $S = V(A \cap B)$ that shows that W is s-breakable. This is a contradiction.

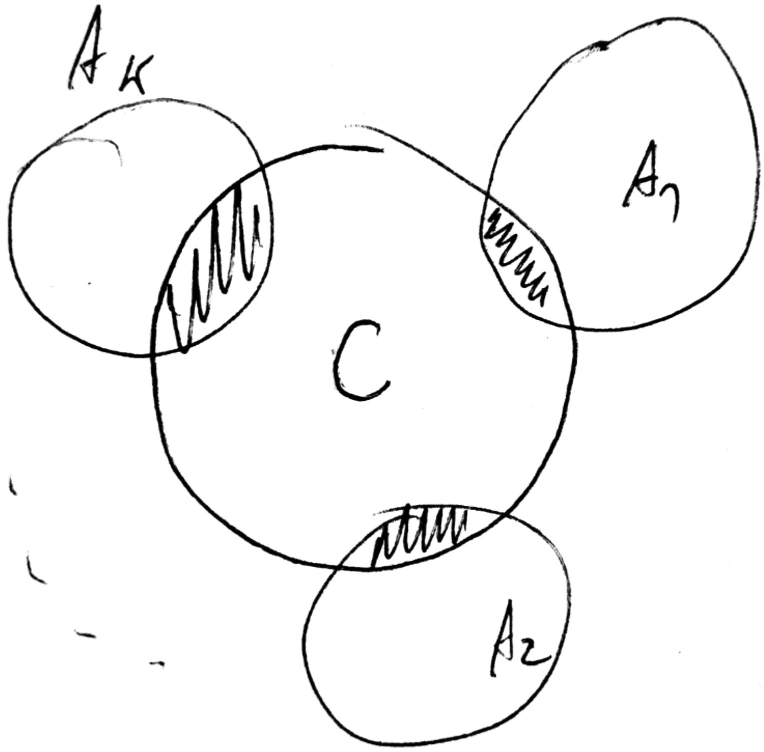
For the last part, let $\mathcal{T}$ consist of all separations (A, B) of order less than $s/3$ such that some element $X \in \mathcal{B}$ is a subset of $V(B) \setminus V(A)$. For (T2), note that $\sum_{i=1}^{3} o(A_i, B_i) < s$, and thus and thus some element $X \in \mathcal{B}$ is disjoint from $V(A_i \cap B_i)$ for all i. Since $V(B_i) \setminus V(A_i)$ contains some $X_i \in \mathcal{B}$ and $G[X \cup X_i]$ is connected, we have $X \subseteq B_i \setminus A_i$ for all i. But then $A_1 \cup A_2 \cup A_3$ is disjoint from X, and thus it is not equal to G. (T3) is easy to prove. □

In the next lecture, we will consider the important Grid Theorem.

Theorem 6. *There exists a function f as follows. If $\mathcal{T}$ is a tangle of order at least $f(n)$ in a graph G, then G contains an $n \times n$ grid H as a minor with model μ such that $\mathcal{T}$ is conformal with the canonical tangle in H and μ.*

4 From a local structure to a global one

A set $\mathcal{L}$ of separations in a graph G is a *location* if for all distinct separations $(A_1, B_1), (A_2, B_2) \in \mathcal{L}$, we have $A_1 \subseteq B_2$. The *center* of the location is the graph C obtained from $\bigcap_{(A,B)\in\mathcal{L}} B$ by adding all edges of cliques with vertex sets $V(A \cap B)$ for $(A, B) \in \mathcal{L}$.



Let $\mathcal{G}$ be a class of graphs (e.g., graphs embeddable in a surface of bounded genus up to a bounded number of apex vertices and vortices). We say a graph G is *$\mathcal{G}$-decomposable* with respect to a tangle $\mathcal{T}$ if there exists a location $\mathcal{L} \subseteq \mathcal{T}$ with center belonging to $\mathcal{G}$. An *a-extension* of a graph G is a graph $G' \supseteq G$ such that for some $A \subseteq V(G')$ of size at most a, $V(G') \setminus V(G) \subseteq A$ and every edge of $E(G') \setminus E(G)$ is incident with a vertex of A.

For a tree decomposition (T, β) of a graph G, the *torso* of $x \in V(T)$ is the graph obtained from $G[\beta(x)]$ by adding cliques on $\beta(x) \cap \beta(y)$ for all $xy \in E(T)$.

Theorem 7. *Suppose every subgraph of a graph G is $\mathcal{G}$-decomposable with respect to every tangle of order θ. Then G has a tree decomposition (T, β) such that for every $x \in V(T)$, either $|\beta(x)| \leq 4\theta$ or the torso of x is a 3θ-extension of a graph from $\mathcal{G}$. Equivalently, G can be obtained from graphs of size at most 4θ and from 3θ-extensions of graphs from $\mathcal{G}$ by clique-sums.*

Proof. We prove the claim in the following stronger form: For every $W \subseteq V(G)$ of size at most 3θ, there exists such a tree decomposition with root r such that $W \subseteq \beta(r)$ and either $|\beta(r)| \leq 4\theta$ or the torso of r together with a clique on W is a 3θ-extension of a graph from $\mathcal{G}$. We prove this claim by induction on the number of vertices of G.

If $|V(G)| \leq 4\theta$, we can let T be a tree with the single vertex r and $\beta(r) = V(G)$. Hence, we can assume $|V(G)| > 4\theta$. Without loss of generality, we can assume $|W| = 3\theta$, as otherwise we can add more vertices to W.

Suppose first W is θ-breakable, via a separation (A, B) of G. Let $W_A = (W \cup V(B) \cap V(A)$ and $W_B = (W \cup V(A)) \cap V(B)$. Note that $|W_A| \leq \theta - 1 + \frac{2}{3}|W| < |W|$, and thus $V(A) \neq V(G)$. Hence, we can apply the induction to A and W_A and obtain a tree decomposition (T_A, β_A). Analogously, we obtain a tree decomposition (T_B, β_B) for B and W_B. Let T be obtained from T_A and T_B by adding a vertex r adjacent to their root, and let β match β_A on T_A, β_B on T_B, and $\beta(r) = W \cup V(A \cap B)$. We have $|\beta(r)| < 4\theta$, and thus (T, β) satisfies the conditions.

Suppose now that W is not θ-breakable. Let $\mathcal{T}$ consist of separations (A, B) of G of order less than θ such that $|W \setminus V(A)| > \frac{2}{3}|W|$. Observe this defines a tangle of order θ. By assumptions, there exists a location $\mathcal{L} \subseteq \mathcal{T}$ with center C belonging to $\mathcal{G}$. For $(A, B) \in \mathcal{L}$, let $W_A = (W \cup V(B)) \cap V(A)$; we have $W_A < \theta - 1 + |W|/3 < |W|$; apply the induction to A and W_A and obtain a tree decomposition (T_A, β_A). Let (T, β) be obtained from these decompositions by adding a new vertex r adjacent to their roots and setting $\beta(r) = W \cup \bigcap_{(A,B)\in\mathcal{L}} B$. Note that the torso of r is a 3θ-extension of the center of $\mathcal{L}$ (which belongs to $\mathcal{G}$) even if we add a clique on W. □

In particular, in case that $\mathcal{G}$ does not contain any tangle of order θ, the assumptions of Theorem 7 are satisfied even with $\mathcal{G} = \emptyset$. Hence, we have the following conclusion.

Corollary 8. *If G does not contain any tangle of order θ, then G has treewidth less than 4θ.*

Actually, in this situation $\mathrm{tw}(G) \leq \frac{3}{2}\theta$; but we will not need this stronger bound.