Reminders

Let $\mathbf{V}$ be a vector space.

Definition

A set $S \subseteq \mathbf{V}$ is a basis if

- S generates $\mathbf{V}$, i.e., $\mathrm{span}(S) = \mathbf{V}$, and
- S is linearly independent.

Assuming that $\mathbf{V}$ has a finite basis:

- All bases of $\mathbf{V}$ have the same size, the dimension $\dim(\mathbf{V})$.
- Every linearly independent set can be extended to a basis.
- If $\mathbf{U} \Subset \mathbf{V}$, then $\dim(\mathbf{U}) \leq \dim(\mathbf{V})$.

A square matrix is regular if it has a multiplicative inverse.

Uniqueness of basis linear combinations

Lemma

Let **V** *be a vector space over a field* **F**. *If* $v_1, \ldots, v_n$ *is a basis of* **V**, *then for every* $v \in \mathbf{V}$, *there exist* *unique* $\alpha_1, \ldots, \alpha_n \in \mathbf{F}$ *such that*

$$v = \alpha_1 v_1 + \ldots + \alpha_n v_n.$$

Proof.

- $\alpha_1, \ldots, \alpha_n$ exist, since $\mathbf{V} = \text{span}(v_1, \ldots, v_n)$.
- If $v = \alpha_1' v_1 + \ldots + \alpha_n' v_n$, then

$$(\alpha_1 - \alpha_1')v_1 + \ldots + (\alpha_n - \alpha_n')v_n = v - v = o, \text{ and}$$

- since $v_1, \ldots, v_n$ are linearly independent, $\alpha_i' = \alpha_i$ for $i = 1, \ldots, n$.

□

Coordinates

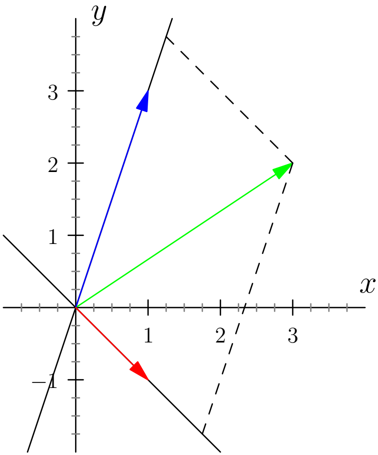
Let **V** be a vector space over a field **F**. Let $B = v_1, \ldots, v_n$ be a basis of **V**.

Definition

The coordinates of a vector $v \in \mathbf{V}$ with respect to the basis B are the vector $[v]_B = (\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n$ such that

$$v = \alpha_1 v_1 + \ldots + \alpha_n v_n.$$

Example



- the standard basis $(0, 1), (1, 0)$: coordinates $(3, 2)$
- the basis $(1, 3), (1, -1)$: coordinates $(5/4, 7/4)$

Example

Problem

Let $B = 1, x + 1, x^2 + x + 1$ *be a basis of* $\mathcal{P}_2$*. Determine the coordinates of* $x^2 + 3x + 6$ *with respect to B.*

We need

$$\alpha_1 \cdot 1 + \alpha_2(x + 1) + \alpha_3(x^2 + x + 1) = x^2 + 3x + 6.$$

By comparing the coefficients

$$\begin{aligned} \alpha_3 &= 1 && \text{at } x^2 \\ \alpha_2 + \alpha_3 &= 3 && \text{at } x \\ \alpha_1 + \alpha_2 + \alpha_3 &= 6 && \text{constant term} \end{aligned} \qquad \begin{aligned} \alpha_3 &= 1 \\ \alpha_2 &= 2 \\ \alpha_1 &= 3 \end{aligned}$$

$$[x^2 + 3x + 6]_B = (3, 2, 1)$$

Size of vector spaces over finite fields

Corollary

If $\mathbf{V}$ *is a vector space over a finite field* $\mathbf{F}_n$*, then*

$$|\mathbf{V}| = n^{\dim(\mathbf{V})}.$$

Proof.

Each element of $\mathbf{V}$ bijectively corresponds to a $\dim(\mathbf{V})$-tuple of its coordinates. □

Sum of subspaces

Let $\mathbf{V}_1, \mathbf{V}_2$ be subspaces of the same space.

- Union of two subspaces typically is not a subspace.

Definition

$$\mathbf{V}_1 + \mathbf{V}_2 = \{v_1 + v_2 : v_1 \in \mathbf{V}_1, v_2 \in \mathbf{V}_2\}$$

Sum and union

Lemma

$\mathbf{V}_1 + \mathbf{V}_2$ is the smallest subspace containing $\mathbf{V}_1$ and $\mathbf{V}_2$, i.e.,

$$\mathbf{V}_1 + \mathbf{V}_2 = span(\mathbf{V}_1 \cup \mathbf{V}_2).$$

Proof.

$\subseteq$ If $v \in \mathbf{V}_1 + \mathbf{V}_2$, then

- $v = v_1 + v_2$ for some $v_1 \in \mathbf{V}_1$ and $v_2 \in \mathbf{V}_2$, and
- $v_1 + v_2 \in \text{span}(\mathbf{V}_1 \cup \mathbf{V}_2)$.

□

Sum and union

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Proof.

$\supseteq$ If $v \in \text{span}(\mathbf{V}_1 \cup \mathbf{V}_2)$, then

- $v = \alpha_1 u_1 + \ldots + \alpha_n u_n$, where $u_1, \ldots, u_k \in \mathbf{V}_1$ and $u_{k+1}, \ldots, u_n \in \mathbf{V}_2$
- Let $v_1 = \alpha_1 u_1 + \ldots + \alpha_k u_k \in \text{span}(\mathbf{V}_1) = \mathbf{V}_1$
- Let $v_2 = \alpha_{k+1} u_{k+1} + \ldots + \alpha_n u_n \in \text{span}(\mathbf{V}_2) = \mathbf{V}_2$
- $v = v_1 + v_2 \in \mathbf{V}_1 + \mathbf{V}_2$.

□

Dimension of intersection and sum

Lemma

Let $\mathbf{V}$ have a finite dimension, let $\mathbf{V}_1, \mathbf{V}_2 \Subset \mathbf{V}$.

$$\dim(\mathbf{V}_1 \cap \mathbf{V}_2) + \dim(\mathbf{V}_1 + \mathbf{V}_2) = \dim(\mathbf{V}_1) + \dim(\mathbf{V}_2)$$

Proof.

- Let B be a basis of $\mathbf{V}_1 \cap \mathbf{V}_2$.
- For $i = 1, 2$, B extends to a basis $B_i \supseteq B$ of $\mathbf{V}_i$.
- $B_1 \cup B_2$ is a basis of $\mathbf{V}_1 + \mathbf{V}_2$

Dimension of intersection and sum

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Proof.

- $B_1 \cup B_2$ is a basis of $\mathbf{V}_1 + \mathbf{V}_2$

$B_1 \cup B_2$ generates $\mathbf{V}_1 + \mathbf{V}_2$:

$$\mathbf{V}_1 + \mathbf{V}_2 = \operatorname{span}(\mathbf{V}_1 \cup \mathbf{V}_2) = \operatorname{span}(B_1 \cup B_2)$$

□

Dimension of intersection and sum

Lemma

Let $\mathbf{V}$ *have a finite dimension, let* $\mathbf{V}_1, \mathbf{V}_2 \Subset \mathbf{V}$*.*

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Proof.

- $B_1 \cup B_2$ is a basis of $\mathbf{V}_1 + \mathbf{V}_2$

Suppose $\alpha_1 v_1 + \ldots + \alpha_n v_n = o$ with $v_1, \ldots, v_k \in B$, $v_{k+1}, \ldots, v_t \in B_1 \setminus B$ and $v_{t+1}, \ldots, v_n \in B_2 \setminus B$. Then
$x = \alpha_1 v_1 + \ldots + \alpha_t v_t = -\alpha_{t+1} v_{t+1} - \ldots - \alpha_n v_n \in \mathbf{V}_1 \cap \mathbf{V}_2 = \operatorname{span}(B)$.

x is a unique linear combination of $B_1 \supseteq B \Rightarrow \alpha_i = 0$ for $i = k+1, \ldots, t$. Symmetrically for $i = t+1, \ldots, n$.
Thus, $\alpha_1 v_1 + \ldots + \alpha_k v_k = o$ and $\alpha_i = 0$ for $i = 1, \ldots, k$ by linear independence of B.

Dimension of intersection and sum

Lemma

Let $\mathbf{V}$ *have a finite dimension, let* $\mathbf{V}_1, \mathbf{V}_2 \Subset \mathbf{V}$.

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- Let B be a basis of $\mathbf{V}_1 \cap \mathbf{V}_2$.
- For $i = 1, 2$, B extends to a basis $B_i \supseteq B$ of $\mathbf{V}_i$.
- $B_1 \cup B_2$ is a basis of $\mathbf{V}_1 + \mathbf{V}_2$

$$\begin{aligned}\dim(\mathbf{V}_1 \cap \mathbf{V}_2) + \dim(\mathbf{V}_1 + \mathbf{V}_2) &= |B| + |B_1 \cup B_2| \\ &= |B_1 \cap B_2| + |B_1 \cup B_2| \\ &= |B_1| + |B_2| \\ &= \dim(\mathbf{V}_1) + \dim(\mathbf{V}_2)\end{aligned}$$

Example

Let $\mathbf{V}_1, \mathbf{V}_2$ be 2-dimensional planes in $\mathbf{R}^4$. One of the following holds:

- $\mathbf{V}_1 = \mathbf{V}_2$, $\dim(\mathbf{V}_1 \cap \mathbf{V}_2) = \dim(\mathbf{V}_1 + \mathbf{V}_2) = 2$, or
- $\mathbf{V}_1 \cap \mathbf{V}_1$ is a line (dimension 1), $\dim(\mathbf{V}_1 + \mathbf{V}_2) = 3$, or
- $\mathbf{V}_1 \cap \mathbf{V}_1$ is $\{o\}$ (dimension 0) , $\mathbf{V}_1 + \mathbf{V}_2 = \mathbf{R}^4$.

Minimum intersection size

Corollary

If $\mathbf{V}_1$ *and* $\mathbf{V}_2$ *are subspaces of a space of dimension n, then*

$$\dim(\mathbf{V}_1 \cap \mathbf{V}_2) \geq \dim(\mathbf{V}_1) + \dim(\mathbf{V}_2) - n.$$

Example: two planes in $\mathbf{R}^3$ cannot intersect in exactly one point

- w.l.o.g. the point would be $(0, 0, 0)$, hence
- the planes are subspaces
- their intersection has dimension at least $2 + 2 - 3 = 1$

Matrix spaces

Let A be an $n \times m$ matrix with entries from field $\mathbf{F}$.

Definition

The row space of A is the linear span of its rows.

$$\mathrm{Row}(A) = \mathrm{span}(A_{1,\star}, A_{2,\star}, \ldots, A_{n,\star}) \Subset \mathbf{F}^{1\times m}$$

Definition

The column space of A is the linear span of its columns.

$$\mathrm{Col}(A) = \mathrm{span}(A_{\star,1}, A_{\star,2}, \ldots, A_{\star,m}) \Subset \mathbf{F}^{n\times 1}$$

Equivalently:

$$\mathrm{Row}(A) = \{xA : x \in \mathbf{F}^{1\times n}\}$$

$$\mathrm{Col}(A) = \{Ax : x \in \mathbf{F}^{m\times 1}\}$$

Example

Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}.$$

$$\mathrm{Row}(A) = \mathrm{span}((1,1,1),(1,2,3),(2,3,4)) = \mathrm{span}((1,1,1),(1,2,3))$$

$$\begin{aligned} \mathrm{Col}(A) &= \mathrm{span}\left(\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}\right) \\ &= \mathrm{span}\left(\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right) \end{aligned}$$

Matrix spaces and multiplication

Lemma

Let A be an $m \times n$ matrix, B an $n \times p$ matrix, C an $q \times m$ matrix.

$$Row(CA) \Subset Row(A)$$
$$Col(AB) \Subset Col(A)$$

Proof.

$$\begin{aligned}
\text{Row}(CA) &= \{x(CA) : x \in \mathbf{F}^{1\times q}\} \\
&= \{(xC)A : x \in \mathbf{F}^{1\times q}\} \subseteq \{yA : y \in \mathbf{F}^{1\times n}\} = \text{Row}(A) \\
\text{Col}(AB) &= \{(AB)x : x \in \mathbf{F}^{p\times 1}\} \\
&= \{A(Bx) : x \in \mathbf{F}^{p\times 1}\} \subseteq \{Ay : y \in \mathbf{F}^{m\times 1}\} = \text{Col}(A)
\end{aligned}$$

Multiplication by regular matrices

Let A be an $m \times n$ matrix, B an $n \times n$ matrix, C an $m \times m$ matrix.

Corollary

If C is regular, then $Row(CA) = Row(A)$.
If B is regular, then $Col(AB) = Col(A)$.

Proof.

$$\text{Row}(CA) \Subset \text{Row}(A)$$

$$\text{Row}(A) = \text{Row}(C^{-1}(CA)) \Subset \text{Row}(CA)$$

$$\text{Col}(AB) \Subset \text{Col}(A)$$

$$\text{Col}(AB) = \text{Col}((AB)B^{-1}) \Subset \text{Col}(AB)$$

□

Multiplication from the other side

…may change the space. But preserves linear dependences.

Lemma

If $\alpha_1 A_{1,\star} + \ldots + \alpha_n A_{n,\star} = o$*, then*

$$\alpha_1 (AB)_{1,\star} + \ldots + \alpha_n (AB)_{n,\star} = o.$$

If $\alpha'_1 A_{\star,1} + \ldots + \alpha'_m A_{\star,m} = o$*, then*

$$\alpha'_1 (CA)_{\star,1} + \ldots + \alpha'_m (CA)_{\star,m} = o.$$

Proof.

$$\begin{aligned}
\alpha_1 (AB)_{1,\star} + \ldots + \alpha_n (AB)_{n,\star} &= (\alpha_1 A_{1,\star} + \ldots + \alpha_n A_{n,\star}) B \\
&= oB = 0 \\
\alpha'_1 (CA)_{\star,1} + \ldots + \alpha'_m (CA)_{\star,m} &= C(\alpha'_1 A_{\star,1} + \ldots + \alpha'_m A_{\star,m}) \\
&= Co = o
\end{aligned}$$

Multiplication by regular matrix

Let A be an $m \times n$ matrix, B an $n \times n$ matrix, C an $m \times m$ matrix.

Corollary

If B is regular, then

$$\alpha_1 A_{1,\star} + \ldots + \alpha_n A_{n,\star} = o$$

if and only if

$$\alpha_1 (AB)_{1,\star} + \ldots + \alpha_n (AB)_{n,\star} = o.$$

If C is regular, then

$$\alpha'_1 A_{\star,1} + \ldots + \alpha'_m A_{\star,m} = o$$

if and only if

$$\alpha'_1 (CA)_{\star,1} + \ldots + \alpha'_m (CA)_{\star,m} = o.$$

Multiplication by regular matrix and bases

Corollary

If B is regular and

$$A_{i_1,\star}, \dots, A_{i_k,\star}$$

is a basis of $Row(A)$*, then*

$$(AB)_{i_1,\star}, \dots, (AB)_{i_k,\star}$$

is a basis of $Row(AB)$*. Hence,* $\dim(Row(A)) = \dim(Row(AB))$*.*

If C is regular and

$$A_{\star,j_1}, \dots, A_{\star,j_t}$$

is a basis of $Col(A)$*, then*

$$(CA)_{\star,j_1}, \dots, (CA)_{\star,j_t}$$

is a basis of $Col(CA)$*. Hence,* $\dim(Col(A)) = \dim(Col(CA))$*.*

RREF and matrix spaces

Recall: for every A, there exists a reqular matrix C such that

$$\mathrm{RREF}(A) = QA.$$

Corollary

Let A be an $n \times m$ matrix.

$$Row(A) = Row(RREF(A)),$$

and

$$\alpha_1 A_{\star,1} + \ldots + \alpha_m A_{\star,m} = o$$

if and only if

$$\alpha_1 (RREF(A))_{\star,1} + \ldots + \alpha_m (RREF(A))_{\star,m}.$$

Dimensions of matrix spaces

- Non-zero rows of RREF(A) form a basis of Row(A).
- If $p_1, \ldots, p_k$ are the basis colum indices of RREF(A), then $A_{\star,p_1}, \ldots, A_{\star,p_k}$ is a basis of Col(A).

Corollary

$$rank(A) = \dim(Col(A)) = \dim(Row(A)) = rank(A^T)$$

- The “proper” definition of rank:

Definition

Rank of a matrix A is the maximum number of its linearly independent rows.

- A square matrix is regular if and only if its rows (and columns) are linearly independent.

Description of RREF

Lemma

Let A be an $n \times m$ matrix and let $A' \sim A$ be in RREF.

- *For $p = 1, \ldots, m$, the column p is a basis column of A' if and only if $A_{\star,p} \notin span(A_{\star,1}, \ldots, A_{\star,p-1})$.*

Let $p_1 < p_2 < \ldots < p_k$ be the basis column indices of A'.

- *For every $i = 1, \ldots, m$, $A'_{1,i}, \ldots, A'_{k,i}$ are the unique coefficients such that*

$$A'_{1,i}A_{\star,p_1} + A'_{2,i}A_{\star,p_2} + \ldots + A'_{k,i}A_{\star,p_k} = A_{\star,i},$$

and $A'_{j,i} = 0$ for $j = k+1, \ldots, n$.

Corollary (as promissed in the 2nd lecture)

There exists exactly one matrix A' in RREF such that $A \sim A'$.

Example

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 3 & 6 \\ 1 & 3 & 5 & 2 & 6 \end{pmatrix} \sim \begin{pmatrix} \underline{1} & 0 & -1 & 0 & 1 \\ 0 & \underline{1} & 2 & 0 & 1 \\ 0 & 0 & 0 & \underline{1} & 1 \end{pmatrix}$$

$$\begin{aligned} \text{Row}(A) &= \text{span}((1,0,-1,0,1),(0,1,2,0,1),(0,0,0,1,1)) \\ &= \{(x,y,2y-x,z,x+y+z) : x,y,z \in \mathbf{R}\} \\ \text{Col}(A) &= \text{span}((1,1,1)^T,(1,2,3)^T,(1,3,2)^T) = \mathbf{R}^{3\times 1} \\ (1,3,5)^T &= -(1,1,1)^T + 2(1,2,3)^T \\ (3,6,6)^T &= (1,1,1)^T + (1,2,3)^T + (1,3,2)^T \end{aligned}$$