List coloring

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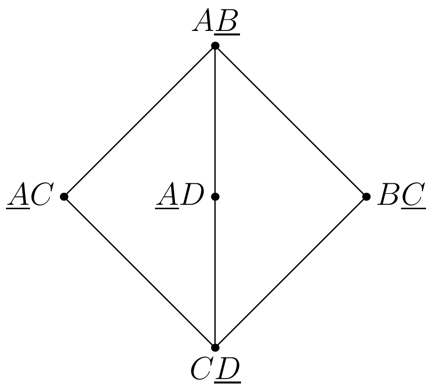
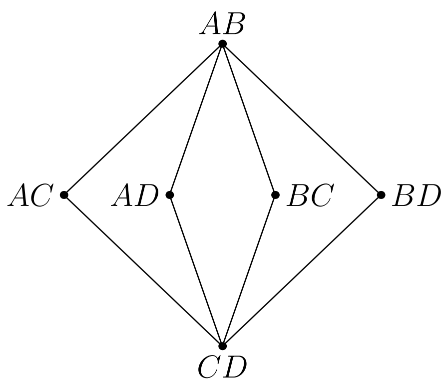
A *list assignment* for a graph G is a function that to each vertex of G assigns a set of colors. Given a list assignment L, an L*-coloring* is a function φ such that

- $\varphi(v) \in L(v)$ for every $v \in V(G)$, and
- $\varphi(u) \neq \varphi(v)$ for every $uv \in E(G)$.





The *choosability* $\chi_l(G)$ of a graph G is the smallest integer k such that G can be L-colored for every assignment L of lists of size at least k to vertices of G.

Observation 1.
$$\chi(G) \le \chi_l(G),$$
and there exist graphs with $\chi(G) < \chi_l(G)$.

A graph G is d*-degenerate* if every subgraph of G has a vertex of degree at most d.

Lemma 2. *The following conditions are equivalent.*

(a) *G is d-degenerate.*

(b) *There exists an ordering $v_1, \ldots, v_n$ of vertices of G such that for $i = 1, \ldots, n$, the vertex v_i has at most d neighbors among $v_{i+1}, v_{i+2}, \ldots, v_n$.*

(c) *There exists an acyclic orientation of G with maximum indegree at most d.*

Proof.

(a) ⇒ (b) For $i = 1, \ldots, n$, let v_i be a vertex of $G - \{v_1, \ldots, v_{i-1}\}$ of degree at most d.

(b) ⇒ (c) For each edge $v_a v_b \in E(G)$, orient the edge towards v_b if and only if $b < a$.

(c) ⇒ (a) Consider any subgraph H of G. Since the orientation is acyclic, H contains a vertex v with no outgoing edges. Hence, the degree of v in H is equal to its indegree, which is at most d.

□

Observation 3. *If G is d-degenerate, then $\chi_l(G) \le d + 1$.*

Observation 3 follows from the following more general result.

Lemma 4. *Let G be a graph with an acyclic orientation. Let $d^+(v)$ denote the degree of v in the orientation. If L is a list assignment for G such that $|L(v)| \ge d^+(v) + 1$ for every $v \in V(G)$, then G is L-colorable.*

Proof. We proceed by induction on the number of vertices of G; hence, assume that the claim holds for every proper subgraph of G. Since the orientation is acyclic, there exists $v \in V(G)$ with no outgoing edges. By the induction hypothesis, $G - v$ has an L-coloring φ. Let N be the set of neighbors of v in G. We have $|N| = d^+(v) < |L(v)|$, and thus there exists a color c in $L(v) \setminus \{\varphi(v) : v \in N\}$. We can set $\varphi(v) = c$. □

In general, small chromatic number does not imply degeneracy (e.g., bipartite graphs may have arbitrary minimum degree). However, every graph with choosability at most k is $2^{O(k)}$-degenerate (Alon).

The condition that the orientation must be acyclic is somewhat restrictive, and sometimes there exists an orientation with smaller maximum indegree that is not acyclic.

Lemma 5. *Let $d \geq 0$ be an integer. A graph G has an orientation (not necessarily acyclic) with maximum indegree at most d if and only if every subgraph H of G satisfies $|E(H)| \leq d|V(H)|$.*

Proof. If G has an orientation with maximum indegree at most d, then so does every subgraph H of G. Each edge of H points towards some vertex, and thus $|E(H)| \leq d|V(H)|$.

Suppose now that every subgraph H of G satisfies $|E(H)| \leq d|V(H)|$. Let G' be the bipartite graph constructed as follows. The vertex set of G' is $(\{1, \ldots, d\} \times V(G)) \cup E(G)$ (that is, G' has a vertex for every edge of G, and d vertices for every vertex of G). For each edge $e = uv$ of G, the graph G' contains the edges $(i, u)e$ and $(i, v)e$ for every $i = 1, \ldots, d$.

Let X be any subset of $E(G)$, and let $N(X)$ be the set of neighbors of vertices of X in G'. Let H be the subgraph of G with vertex set $\{v : (i, v) \in N(X) \text{ for } i = 1, \ldots, d\}$ and edge set X. Observe that $|N(X)| = d|V(H)| \geq |E(H)| = |X|$.

By Hall's theorem, there exists a matching M in G' that covers $E(G)$. If $(i, v)e \in M$ for any $i \in \{1, \ldots, d\}$, orient the edge e of G towards v. Note that every edge of G is oriented in exactly one direction, and that the maximum indegree of G in this orientation is at most d. $\square$

For example, every planar graph on n vertices has at most $3n$ edges, and thus it has an orientation with maximum indegree at most 3; while planar graphs in general are only 5-degenerate. Having an orientation with maximum indegree at most d does not always imply that the choosability is at most $d + 1$; e.g., odd cycles have an orientation with maximum indegree at most 1 and choosability 3. We now study two sufficient conditions guaranteeing the bound on choosability.

1 Kernels

A *kernel* in a directed graph G is a non-empty independent set $S \subseteq V(G)$ such that every vertex $v \in V(G) \setminus S$ has an in-neighbor in S.

- The cyclic orientation of an even cycle has a kernel (every second vertex).
- The cyclic orientation of an odd cycle does not have a kernel.

Lemma 6. *Let G be a directed graph such that every induced subgraph of G has a kernel. If L is a list assignment for G such that $|L(v)| \geq d^+(v) + 1$ for every $v \in V(G)$, then G is L-colorable.*

Proof. We proceed by induction on the number of vertices of G; hence, assume that the claim holds for every proper induced subgraph of G. Let v be any vertex of G. Since $|L(v)| \geq d^+(v) + 1 \geq 1$, there exists a color $c \in L(v)$. Let H be the subgraph of G induced by vertices whose list contains c. Let S be a kernel of H. Let $G' = G - S$ and let $L'(u) = L(u) \setminus \{c\}$ for every $u \in V(G')$.

Consider a vertex $u \in V(G')$. If $c \notin L(u)$, then $|L'(u)| = |L(u)| \geq d^+_G(u)+1$, and thus $|L'(u)|$ is greater than the indegree of u in G'. If $c \in L(u)$, then $u \in V(H)$, and thus u has an in-neighbor in the kernel S. Therefore, $|L'(u)| = |L(u)|-1 \geq d^+_G(u) > d^+_{G'}(u)$. Therefore, by the induction hypothesis there exists an L'-coloring φ of G'. We can set $\varphi(u) = c$ for every $u \in S$. $\square$

Lemma 7. *Any orientation of a bipartite graph has a kernel.*

Proof. We proceed by induction on the number of vertices of G; hence, assume that the claim holds for every proper induced subgraph of G. Suppose that G contains a vertex v with no in-neighbors. Let N be the set of neighbors of v. By the induction hypothesis, $G - (\{v\} \cup N)$ has a kernel S'. Then, $S' \cup \{v\}$ is a kernel of G.

Hence, assume that every vertex of G has at least one in-neighbor. Let $\psi : V(G) \to \{1, 2\}$ be a proper 2-coloring of G. Then the set of vertices colored by 1 is a kernel, as every vertex colored by 2 has an in-neighbor which is colored by 1. $\square$

By Euler's formula, a bipartite planar graph on n vertices has at most $2n$ edges. Hence, Lemmas 5, 6 and 7 imply that every bipartite planar graph is 3-choosable.

2 Combinatorial Nullstellensatz

We need a basic result from algebra.

Theorem 8. *Every non-zero polynomial of degree at most n has at most n distinct roots.*

We prove the following generalization of the claim.

Lemma 9. *Let $p(x_1, \ldots, x_k)$ be a polynomial and let $n_1, \ldots, n_k$ be integers such that the maximum degree of x_i in p is at most n_i for $i = 1, \ldots, k$. For $i = 1, \ldots, k$, let S_i be a set of complex numbers of size at least $n_i + 1$. If p is non-zero, then there exist $a_1 \in S_1, \ldots, a_k \in S_k$ such that*

$$p(a_1, \ldots, a_k) \neq 0.$$

Proof. We prove the claim by induction on k. For $k=1$, the claim follows from Theorem 8. Suppose that $k \geq 2$. We can write

$$p(x_1,\ldots,x_k) = p_0(x_1,\ldots,x_{k-1}) + p_1(x_1,\ldots,x_{k-1})x_k + p_2(x_1,\ldots,x_{k-1})x_k^2 + \ldots + p_{n_k}(x_1,\ldots,x_{k-1})x_k^{n_k}$$

for some polynomials $p_0, \ldots, p_{n_k}$. Since p is non-zero, there exists $m \in \{0,\ldots,n_k\}$ such that p_m is non-zero. By the induction hypothesis, there exist $a_1 \in S_1, \ldots, a_{k-1} \in S_{k-1}$ such that $p_m(a_1,\ldots,a_{k-1}) \neq 0$. For $i=0,\ldots,n_k$, let $A_i = p_i(a_1,\ldots,k_{k-1})$. Then $q(x) = A_0 + A_1x + \ldots + A_{n_k}x^{n_k}$ is a non-zero polynomial of degree at most n_k, and by Theorem 8, there exists $a_k \in S_k$ such that

$$0 \neq q(a_k) = p(a_1,\ldots,a_k).$$

$\square$

For a directed graph G on vertices $v_1, \ldots, v_n$, let

$$p_G(x_1,\ldots,x_n) = \prod_{(v_i,v_j)\in E(G)} (x_j - x_i).$$

Note that a function φ is a proper coloring of G if and only if $p_G(\varphi(v_1),\ldots,\varphi(v_n)) \neq 0$. The *total degree* of term $x_1^{m_1}x_2^{m_2}\ldots x_n^{m_n}$ is $m_1+m_2+\ldots+m_n$. Note that every term in p_G has total degree $|E(G)|$.

Theorem 10 (Alon, Tarsi)**.** *Let G be a directed graph on vertices $v_1, \ldots, v_n$, and let L be an assignment of lists (of complex number) to vertices of G such that $|L(v_i)| \geq d^+(v_i)+1$ for $i=1,\ldots,n$. If the coefficient of p_G at $\prod_{i=1}^n x_i^{d^+(v_i)}$ is non-zero, then G has an L-coloring.*

Proof. For $i = 1\ldots,n$, let $m_i = d^+(v_i)$. Without loss of generality, we can assume that the set $L(v_i)$ has size exactly m_i+1. Hence, $p_i(x) = \prod_{a\in L(v_i)}(x-a)$ is a polynomial of degree m_i+1. Let $q_i(x) = x^{m_i+1} - p_i(x)$. Observe that $q_i(x)$ is a polynomial of degree at most m_i. Since $p_i(a)=0$ for every $a \in L(v_i)$, we conclude that $q_i(a) = a^{m_i+1}$ for every $a \in L(v_i)$.

Let p be the polynomial obtained from p_G by repeatedly substituting q_i for $x_i^{m_i+1}$, for each $i=1,\ldots,n$. Hence, p is a polynomial in that x_i has degree at most m_i, and $p(a_1,\ldots,a_n) = p_G(a_1,\ldots,a_n)$ for every $a_1 \in L(v_1), \ldots, a_n \in L(v_n)$. Furthermore, p and p_G have the same coefficient at $\prod_{i=1}^n x_i^{m_i}$, since we cannot perform any substitution in this term, all terms in p_G have total degree $|E(G)|$, and all terms created by the substitutions have strictly smaller total degree. Therefore, p is a non-zero polynomial.

By Lemma 9, there exist $a_1 \in L(v_1), \ldots, a_n \in L(v_n)$ such that $p(a_1,\ldots,a_n) \neq 0$. Therefore, $p_G(a_1,\ldots,a_n) \neq 0$, and thus the function φ defined by $\varphi(v_i) = a_i$ for $i=1,\ldots,n$ is an L-coloring of G. $\square$

To apply Theorem 10, we need a way to determine the value of the coefficient of p_G at $\prod_{i=1}^{n} x_i^{d^+(v_i)}$. A subgraph H of a directed graph G is *eulerian* if the indegree of each vertex of H is equal to its outdegree. It is *spanning* if $V(H) = V(G)$. We say that H is *even* if $|E(H)|$ is even and *odd* if $|E(H)|$ is odd.

Lemma 11. *Let G be a directed graph on vertices $v_1, \ldots, v_n$. The absolute value of the coefficient of p_G at $\prod_{i=1}^{n} x_i^{d^+(v_i)}$ is equal to the difference between the number of even and odd spanning eulerian subgraphs of G.*

Proof. Let D be any orientation of G. Let $|D \triangle G|$ denote the number of edges that have opposite orientations in D and G. Let $x_D = \prod_{(v_i,v_j)\in E(G)} x_j$. We interpret D as encoding a choice from the terms in the product defining p_G. Thus,

$$p(G) = \sum_{D \text{ orientation of } G} (-1)^{|D \triangle G|} x_D.$$

Let $\mathcal{D}$ denote the set of orientations D of G such that v_i has indegree $d^+(v_i)$ for $i = 1, \ldots, n$. Let $\mathcal{D}_o = \{D \in \mathcal{D} : |D \triangle G| \text{ is odd}\}$ and $\mathcal{D}_e = \{D \in \mathcal{D} : |D \triangle G| \text{ is even}\}$. Observe that the absolute value of the coefficient of p_G at $\prod_{i=1}^{n} x_i^{d^+(v_i)}$ is $||\mathcal{D}_e| - |\mathcal{D}_o||$. For each $D \in \mathcal{D}$, the subgraph of G with edge set consisting of the edges with opposite orientations in D and G is eulerian, and conversely, reversing the orientation of edges in an eulerian subgraph of G results in an orientation belonging to $\mathcal{D}$. Therefore, $|\mathcal{D}_e|$ is equal to the number of spanning even eulerian subgraphs of G, and $|\mathcal{D}_o|$ is equal to the number of spanning odd eulerian subgraphs of G. $\square$

Corollary 12. *Let G be a directed graph on vertices $v_1, \ldots, v_n$, and let L be an assignment of lists (of complex number) to vertices of G such that $|L(v_i)| \geq d^+(v_i) + 1$ for $i = 1, \ldots, n$. If G has different number of even and odd spanning eulerian subgraphs, then G has an L-coloring.*

As a special case, if G is bipartite, then it has no odd eulerian subgraphs, and at least one even eulerian subgraph (with no edges), and thus Corollary 12 applies. This gives another proof that every bipartite planar graph is 3-choosable.

3 Exercises

1. ($\star\star$) Prove that any acyclic orientation has a kernel.

2. ($\star$) Find an example of a graph with orientation for that we can apply Corollary 12, but not Lemma 6 (i.e., has different number of even and odd spanning eulerian subgraphs, but some induced subgraph does not have a kernel).

3. ($\star\star$) Find an example of a graph with orientation for that we can apply Lemma 6, but not Corollary 12 (i.e., every induced subgraph has a kernel, but the number of even and odd spanning eulerian subgraphs is the same).

4. ($\star\star$) Let G be a connected graph with minimum degree at least two. Prove that G is 2-choosable if and only if G is either an even cycle, or the union of three paths of even length with shared endpoints such that two of the paths have length two.