Tree-width

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A *tree decomposition* of a graph G is a pair (T, β), where $\beta : V(T) \to 2^{V(G)}$ assigns a *bag* $\beta(n)$ to each vertex of T, such that

- for every $v \in V(G)$, there exists $n \in V(T)$ such that $v \in \beta(n)$ – "every vertex is in some bag",
- for every $uv \in E(G)$, there exists $n \in V(T)$ such that $u, v \in \beta(n)$ – "every edge is in some bag", and
- for every $v \in V(G)$, the set $\{n \in V(T) : v \in \beta(n)\}$ induces a connected subtree of T – "every vertex appears in a connected subtree of the decomposition".

Lemma 1. *If $S \subseteq V(G)$ induces a clique in G and (T, β) is a tree decomposition of G, then there exists $n \in V(T)$ such that $S \subseteq \beta(n)$.*

Theorem 2. *A graph is chordal if and only if it has a tree decomposition where every bag induces a clique.*

1 Tree-width

The *width* of the tree decomposition is the size of the largest bag minus one. The *tree-width* $\mathrm{tw}(G)$ of a graph G is the minimum width of a tree decomposition of G.

Corollary 3. *A graph G has tree-width at most k, if and only if $G \subseteq G'$ for some chordal graph G' with $\omega(G') = k + 1$.*

Corollary 4. *Every graph of tree-width at most k contains a vertex of degree at most k.*

Examples:

- G has tree-width at most 1 if and only if G is a forest.
- Any cycle has tree-width 2.
- The complete graph K_n has tree-width $n-1$.

A graph H is a *minor* of a graph G ($H \leq_m G$) if H is obtained from G by removing vertices and edges and by contracting edges.

Lemma 5. *If $H \leq_m G$, then* $\mathrm{tw}(H) \leq \mathrm{tw}(G)$.

Proof. Let (T, β) be a tree decomposition of G of width $\mathrm{tw}(G)$.

- If H is obtained from G by removing an edge, then (T, β) is a tree decomposition of H.
- If H is obtained from G by removing a vertex v, then (T, β') is a tree decomposition of H, where $\beta'(n) = \beta(n) \setminus \{v\}$ for every $n \in V(T)$.
- If H is obtained from G by contracting the edge xy to a new vertex w, then (T, β') is a tree decomposition of H, where $\beta'(n) = (\beta(n) \setminus \{x, y\}) \cup \{w\}$ for every $n \in V(T)$ such that $\{x, y\} \cap \beta(n) \neq \emptyset$, and $\beta'(n) = \beta(n)$ for every $n \in V(T)$ such that $\{x, y\} \cap \beta(n) = \emptyset$.

□

A *model* of H in G is a function $\mu : V(H) \to$ connected subgraphs of G such that

- $\mu(u) \cap \mu(v) = \emptyset$ for every $u \neq v$, and
- if $uv \in E(H)$, then G has an edge with one end in $\mu(u)$ and the other end in $\mu(v)$. We denote one such edge by $\mu(uv)$.

Observation 6. *H is a minor of G if and only if H has a model in G.*

Corollary 7. *If H is a minor of G and H has maximum degree at most 3, then H is a topological minor of G.*

Proof. Let μ be a model of H in G, chosen so that $H' = \bigcup_{x \in V(H) \cup E(H)} \mu(x)$ is minimal. Then for every $v \in V(H)$, $\mu(v) \cup \bigcup_{uv \in E(H)} \mu(uv)$ is a subdivision of K_1, K_2 or $K_{1,3}$, and thus H' is a subdivision of H. □

We can now give characterizations of graphs with tree-width 1 and 2 in terms of forbidden (topological) minors.

Theorem 8.

$$\textit{tree-width at most } 1 = \mathrm{Forb}_{\le_m}(K_3) = \mathrm{Forb}_{\le_t}(K_3)$$
$$\textit{tree-width at most } 2 = \mathrm{Forb}_{\le_m}(K_4) = \mathrm{Forb}_{\le_t}(K_4)$$
$$\textit{tree-width at most } k \subsetneq \mathrm{Forb}_{\le_m}(K_{k+2}) \subsetneq \mathrm{Forb}_{\le_t}(K_{k+2}) \textit{ for } k \ge 3.$$

Proof. By Lemma 1, we have $\mathrm{tw}(K_n) = n-1$, and thus a graph of tree-width at most k cannot contain K_{k+2} as a minor. For $k \ge 3$, $\mathrm{Forb}_{\le_m}(K_{k+2})$ includes all planar graphs, which have unbounded tree-width (see below).

Hence, it remains to prove that

- Every graph without K_3-minor (or K_3-topological minor) has tree-width at most 1.
- Every graph without K_4-minor (or K_4-topological minor) has tree-width at most 2.

Graphs without K_3-minor are precisely forests, which have tree-width at most 1. For K_4-minor-free graphs, we proceed by induction on the number of vertices. That is, we are given a graph G without without K_4-minor, and assume that every K_4-minor-free graph with less than $|V(G)|$ vertices has tree-width at most 2.

Note that G is not 3-connected: otherwise, let C be an induced cycle in G. Since G is 3-connected, we have $G \ne C$, and thus $G - V(C)$ is non-empty. Since G is 3-connected, every component K of $G - V(C)$ has at least three distinct neighbors in C. Consequently, contracting K and all but three edges of C results in K_4, contrary to the assumption that $K_4 \not\le_m G$.

If $|V(G)| \le 3$, then G has tree-width at most two. Hence, assume that $|V(G)| \ge 4$, and thus G contains a minimum cut S (of size at most 2). Let G_1 and G_2 be subgraphs of G such that $V(G_1) \ne S \ne V(G_2)$, $V(G_1) \cap V(G_2) = S$ and $G_1 \cup G_2 = G$. If $|S| \le 1$, then let $G_1' = G_1$ and $G_2' = G_2$. If $S = \{u, v\}$, then let $G_1' = G_1 + uv$ and $G_2' = G_2 + uv$. Observe that G_1' and G_2' are minors of G, and thus they do not contain K_4 as a minor. For $i \in \{1, 2\}$, induction hypothesis implies that G_i' has a tree decomposition (T_i, β_i) of width at most 2. By Lemma 1, there exists $n_i \in V(T_i)$ such that $S \subseteq \beta_i(n_i)$. Let $T = (T_1 \cup T_2) + n_1 n_2$, and let $\beta(n) = \beta_1(n)$ for $n \in V(T_1)$ and $\beta(n) = \beta_2(n)$ for $n \in V(T_2)$. Then (T, β) is a tree decomposition of G of width at most two. □

1.1 Minors and coloring

Clearly, if G contains a clique of size k, its chromatic number is at least k. The converse is false, however Hadwiger conjectured a "weak" converse in the terms of minors.

Conjecture 1. *For every $k \geq 1$, every graph that does not contain K_k as a minor has chromatic number at most $k-1$.*

For $k=5$, this strengthens the 4-color theorem (since a planar graph does not contain K_5 as a minor, it is 4-colorable). Hadwiger's conjecture is known to be true for $k=5$ and $k=6$, in both cases by showing its equivalence with the 4-color theorem. For $k \geq 7$, the conjecture is open. For $k \leq 4$, we can prove even stronger statement.

Lemma 9. *For $1 \leq k \leq 4$, every graph that does not contain K_k as a topological minor has chromatic number at most $k-1$.*

Proof. A graph without K_1 has no vertices, and thus it can be colored by 0 colors. A graph without K_2 has no edges, and thus it can be colored by 1 color.

For $3 \leq k \leq 4$, Theorem 8 implies that every graph G without K_k as a topological minor has tree-width at most $k-2$, and by Corollary 4, G contains a vertex v of degree at most $k-2$. Hence, we can color G by removing v, coloring $G-v$ by at most $k-1$ colors, and assigning v a color different from the colors of its neighbors. $\square$

Let us note that the analogue of Lemma 9 for arbitrary k was conjectured by Hajós. Hajós conjecture is known to be false for $k \geq 7$, and it is open for $k \in \{5,6\}$.

1.2 Tree-width and cuts

Let (T,β) be a tree decomposition of a graph G. Every edge of the tree decomposition defines a cut in G, as follows. For an edge $n_1n_2 \in E(T)$, let T_{n_1,n_2} be the component of $T - n_1n_2$ that contains n_2, and let $G_{n_1,n_2} = G\left[\bigcup_{n \in V(T_{n_1,n_2})} \beta(n)\right]$.

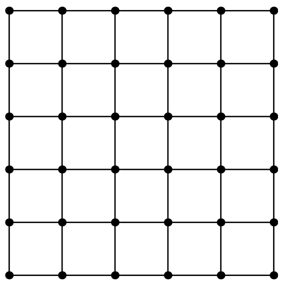
Observation 10. *Let G be a graph, let (T,β) be its tree decomposition, and let n_1n_2 be an edge of T. Then $G = G_{n_1,n_2} \cup G_{n_2,n_1}$ and $V(G_{n_1,n_2}) \cap V(G_{n_2,n_1}) = \beta(n_1) \cup \beta(n_2)$. In particular, G contains no edge with one end in $V(G_{n_1,n_2}) \setminus V(G_{n_2,n_1})$ and the other end in $V(G_{n_2,n_1}) \setminus V(G_{n_1,n_2})$.*

Hence, graphs of bounded tree-width contain many small cuts.

Lemma 11. *If G is a graph of tree-width at most k, then there exists $S \subseteq V(G)$ of size at most $k+1$ such that every component of $G-S$ has at most $|V(G)|/2$ vertices.*

Proof. Let (T, β) be a tree decomposition of G of width at most k. Each of the bags of the decomposition has size at most $k+1$. We define an orientation of T as follows. For every edge $nn' \in E(T)$ such that $|V(G_{n,n'}) \setminus \beta(n)| > |V(G)|/2$, orient the edge nn' towards n'. Since T is a tree, we have $|E(T)| < |V(T)|$, and thus T contains a vertex n such that no edge of T is oriented away from n. Hence, we can set $S = \beta(n)$. □

The $n \times n$ *grid* is the graph with vertex set $\{(i, j) : 1 \leq i, j \leq n\}$ and two vertices (i_1, j_1) and (i_2, j_2) adjacent if $|i_1 - i_2| + |j_1 - j_2| = 1$.



Corollary 12. *The $n \times n$ grid G_n has tree-width at least $\lfloor n/2 \rfloor$.*

Proof. Suppose that G_n has tree-width at most $\lfloor n/2 \rfloor - 1$. By Lemma 11, there exists $S \subseteq V(G_n)$ of size at most $\lfloor n/2 \rfloor$ such that every component of $G_n - S$ has at most $|V(G_n)|/2$ vertices. However, at least $\lceil n/2 \rceil$ of rows and columns of the grid are not intersected by S, which gives a component with more than $|V(G_n)|/2$ vertices. □

A slightly more involved argument shows that the $n \times n$ grid has tree-width exactly n. The following result of Robertson and Seymour gives an approximate characterization of graphs with bounded tree-width.

Theorem 13. *For every n, there exists k such that every graph of tree-width at least k contains the $n \times n$ grid as a minor.*

2 Exercises

1. $(\star)$ Prove that if G has tree-width at most k, then G has a tree decomposition (T, β) of width at most k such that $|V(T)| \leq n$.

2. $(\star)$ A graph G is *outerplanar* if it can be drawn in plane so that every vertex of G is incident with the outer face. Prove that every outerplanar graph has tree-width at most 2.

3. ($\star\star$) Prove that outerplanar $= \mathrm{Forb}_{\leq_t}(K_4, K_{2,3})$.

4. ($\star\star\star$) A 2-*terminal graph* is a triple (G, u, v), where u and v are distinct vertices of G. For two 2-terminal graphs (G_1, u_1, v_1) and (G_2, u_2, v_2), their *series composition* is (G', u_1, v_2), where G' is obtained from the disjoint union of G_1 and G_2 by identifying v_1 with u_2; and their *parallel composition* is (G'', u, v), where G'' is obtained from the disjoint union of G_1 and G_2 by identifying u_1 with u_2 to a single vertex u, and v_1 with v_2 to a single vertex v. Let $K_2^{\cdot\cdot}$ denote the 2-terminal graph (G, u, v), where $V(G) = \{u, v\}$ and $E(G) = \{uv\}$. A 2-terminal graph (G, u, v) is *series-parallel* if can be obtained by a sequence of series and parallel compositions from copies of $K_2^{\cdot\cdot}$. Prove that (G, u, v) is series-parallel, if and only if $G + uv$ is 2-connected and has tree-width at most 2.

5. ($\star\star$) Prove that if G is a graph of tree-width at most k, then it has induced subgraphs G_1 and G_2 such that $G = G_1 \cup G_2$, $|V(G_1) \cap V(G_2)| \leq k + 1$, $|V(G_1) \setminus V(G_2)| \leq \frac{2}{3}|V(G)|$ and $|V(G_2) \setminus V(G_1)| \leq \frac{2}{3}|V(G)|$.