Cographs; chordal graphs and tree decompositions

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Let us now proceed with some more interesting graph classes closed on induced subgraphs.

1 Cographs

The class of *cographs* is defined recursively as follows.

- The graph with one vertex is a cograph.
- The disjoint union of two cographs is a cograph.
- The complement of a cograph is a cograph.

A graph is a cograph if and only if it can be obtained by a finite number of applications of these rules. For example, K_4 is a cograph, because its complement is a union of single-vertex graphs.

Theorem 1. $\mathrm{Forb}_{\sqsubseteq}(P_4) =$ *cographs.*

Proof. Firstly, note that the class of cographs is induced-subgraph-closed. Furthermore, P_4 is not a cograph, since it has more than one vertex and both P_4 and its complement are connected. Hence, if a graph G contains P_4 as an induced subgraph, then G is not a cograph, and thus $\mathrm{Forb}_{\sqsubseteq}(P_4) \subseteq$ cographs.

Therefore, we only need to prove that every graph G such that $P_4 \not\sqsubseteq G$ is a cograph. We prove the claim by induction, and thus we assume that the claim holds for all graphs with less than $|V(G)|$ vertices. Since all graphs with at most 3 vertices are cographs, we can assume that $|V(G)| \geq 4$. If G is not connected, then by induction hypothesis, all the components of G are cographs, and thus G is a cograph. Hence, assume that G is connected.

Consider any vertex $v \in V(G)$, and let $G' = G - v$. Note that $P_4 \not\sqsubseteq G'$, and by the induction hypothesis, G' is a cograph.

Suppose first that G' is not connected, and let $C_1, \ldots, C_n$ be the components of G'. Since G is connected, v has a neighbor $u_i \in V(C_i)$ for $1 \le i \le n$. If v is adjacent to all vertices of G', then $G = \overline{\overline{G'} \cup \{v\}}$ is a cograph. Hence, we can assume that a vertex $w_1 \in V(C_1)$ is not adjacent to v. Since C_1 is connected, we can assume that u_1 is adjacent to w_1. However, then $w_1 u_1 v v_2$ is an induced path in G, which contradicts the assumption that $P_4 \not\sqsubseteq G$.

Finally, consider the case that G' is connected. Since G' is a cograph with more than one vertex, its complement is not connected. Furthermore, $\overline{P_4} = P_4$, and thus $P_4 \not\sqsubseteq \overline{G}$. By the same argument as in the previous paragraph, we prove that $\overline{G}$ is a cograph, and thus G is a cograph. □

A graph G is *chordal* if no cycle of length greater than 3 is induced, that is, chordal = $\mathrm{Forb}_{\sqsubseteq}$(holes). There are several related characterization of chordal graphs.

For a connected graph G, $S \subseteq V(G)$ is a *minimal cut* if $G - S$ is not connected, but $G - S'$ is connected for every $S' \subsetneq S$. Observe that every vertex of a minimal cut S has a neighbor in every component of $G - S$.

Lemma 2. *If a connected graph G is chordal, then every minimal cut $S \subseteq V(G)$ induces a clique.*

Proof. Suppose that $u, v \in S$ are not adjacent. Let C_1 and C_2 be components of $G - S$. Since S is a minimal cut, both u and v have neighbors both in C_1 and in C_2. For $i \in \{1, 2\}$, let P_i be a shortest path between u and v through C_i. Then $P_1 \cup P_2$ is an induced hole in G, which contradicts the assumption that G is chordal. □

Theorem 3. *If G is chordal and not a clique, then there exist $G_1, G_2 \sqsubset G$ such that $G = G_1 \cup G_2$ and $G_1 \cap G_2$ is a clique.*

Proof. If G is not connected, we can take G_1 to be a component of G and $G_2 = G \setminus V(G_2)$. Hence, assume that G is connected. Since G is not a clique, there exist vertices $x, y \in V(G)$ that are not adjacent. Let $S_0 = V(G) \setminus \{x, y\}$. Then $G - S_0$ is not connected. Consequently, there exists $S \subseteq S_0$ such that S is a minimal cut. Let $G - S = C_1 \cup C_2$, where C_1 and C_2 are non-empty and disjoint. Then we can set $G_1 = G - V(C_2)$ and $G_2 = G - V(C_1)$. Note that $G_1 \cap G_2 = G[S]$ is a clique by Lemma 2. □

Observe also that if G_1 and G_2 are chordal and $G_1 \cap G_2$ is a clique, then $G_1 \cup G_2$ is chordal. Hence, every chordal graph can be obtained using a finite number of applications of these rules:

- A complete graph is chordal.

- If G_1 and G_2 are chordal and G is obtained from G_1 and G_2 by gluing on a clique, then G is chordal.

A vertex $v \in V(G)$ is *simplicial* if the neighborhood of v in G induces a clique.

Lemma 4. *If G is chordal and not a clique, then G contains two non-adjacent simplicial vertices.*

Proof. We prove the claim by induction, and thus we assume that it holds for all graphs with less than $|V(G)|$ vertices. By Theorem 3, there exist $G_1, G_2 \sqsubset G$ such that $G = G_1 \cup G_2$ and $G_1 \cap G_2$ is a clique. For $i \in \{1, 2\}$, if G_i is not a clique, then by the induction hypothesis, G_i contains two non-adjacent simplicial vertices, and at most one of them belongs to the clique $G_1 \cap G_2$. Let v_i be a simplicial vertex of G_i not belonging to $G_1 \cap G_2$. If G_i is a clique, then let v_i be any vertex of G_i not belonging to $G_1 \cap G_2$.

Since neither v_1 nor v_2 belong to $G_1 \cap G_2$, observe that v_1 and v_2 are non-adjacent simplicial vertices of G. □

Theorem 5. *A graph G is chordal if and only if every induced subgraph of G contains a simplicial vertex.*

Proof. Every induced subgraph of a chordal graph is chordal, and contains a simplicial vertex by Lemma 4. Hence, it suffices to prove that if every induced subgraph of G contains a simplicial vertex, then G is chordal. We prove the claim by induction on the number of vertices of G, and thus we assume that the claim holds for all graphs with less than $|V(G)|$ vertices.

Let v be a simplicial vertex of G. By the induction hypothesis, $G - v$ is chordal, and thus every induced hole in G contains v. However, for any hole $C = uvwz_1z_2 \ldots \subseteq G$, the vertices u and w are adjacent, and thus C is not induced. It follows that G is chordal. □

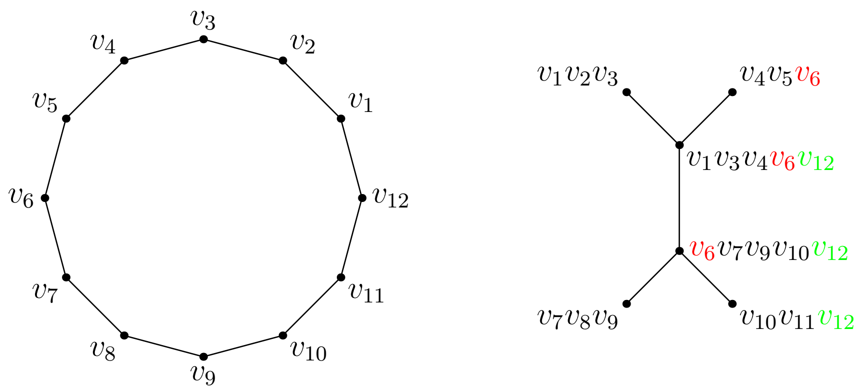
A *tree decomposition* of a graph G is a pair (T, β), where $\beta : V(T) \to 2^{V(G)}$ assigns a *bag* $\beta(n)$ to each vertex of T, such that

- for every $v \in V(G)$, there exists $n \in V(T)$ such that $v \in \beta(n)$ – "every vertex is in some bag",

- for every $uv \in E(G)$, there exists $n \in V(T)$ such that $u, v \in \beta(n)$ – "every edge is in some bag", and

- for every $v \in V(G)$, the set $\{n \in V(T) : v \in \beta(n)\}$ induces a connected subtree of T – "every vertex appears in a connected subtree of the decomposition".

The *width* of the tree decomposition is the size of the largest bag minus one. Example: a tree decomposition of C_{12} of width 4.



A tree decomposition (T, β) is *reduced* if every $n_1n_2 \in E(T)$ satisfies $\beta(n_1) \not\subseteq \beta(n_2)$ and $\beta(n_2) \not\subseteq \beta(n_1)$.

Lemma 6. *For every tree decomposition (T, β) of G, there exists a reduced tree decomposition (T', β') of G such that each bag of (T', β') is equal to some bag of (T, β).*

Proof. Suppose that $n_1n_2 \in E(T)$ satisfies $\beta(n_1) \subseteq \beta(n_2)$. Let T_1 be the tree obtained from T by contracting the edge n_1n_2, and let $n \in V(T_1)$ be the vertex obtained from n_1 and n_2 by this contraction. Let $\beta_1 : V(T_1) \to 2^{V(G)}$ be defined by $\beta_1(n) = \beta(n_2)$ and $\beta_1(n') = \beta(n')$ for every $n' \in V(T_1) \setminus \{n\}$. Then (T_1, β_1) is a tree decomposition of G with fewer vertices. By repeating this operation as long as possible, we eventually obtain a reduced tree decomposition of G. □

Corollary 7. *If G has a tree decomposition of width at most k, then the minimum degree of G is at most k.*

Proof. Let (T, β) be a reduced tree decomposition of G of width at most k, and let n be a leaf of T. Since (T, β) is reduced, there exists $v \in V(G)$ such that v only appears in the bag of n. Hence, all neighbors of v belong to $\beta(n)$, and thus $\deg(v) \leq |\beta(n)| - 1 \leq k$. □

Lemma 8. *If G is a complete graph, then G has only one reduced tree decomposition: tree with one vertex with bag equal to $V(G)$.*

Proof. Let (T, β) be a reduced tree decomposition of G, and suppose that T has an edge n_1n_2. Since (T, β) is reduced, there exists $x \in \beta(n_1) \setminus \beta(n_2)$ and $y \in \beta(n_2) \setminus \beta(n_1)$. Let T_x be the subtree of T induced by the vertices whose bags contain x, and let T_y be the subtree of T induced by the vertices whose bags contain y. Since $n_1 \notin V(T_x)$ and $n_2 \notin V(T_y)$, the edge n_1n_2 belongs neither to T_x nor to T_y. Hence, T_x and T_y are subgraphs of different components of $T - n_1n_2$, and thus they are disjoint. However, $xy \in E(G)$ implies that some bag contains both x and y, which is a contradiction. Therefore, a reduced tree decomposition of G has no edges. □

Corollary 9. *If $S \subseteq V(G)$ induces a clique in G and (T, β) is a tree decomposition of G, then there exists $n \in V(T)$ such that $S \subseteq \beta(n)$.*

Proof. Let $\beta_1 : V(T) \to 2^S$ be defined by $\beta_1(n) = \beta(n) \cap S$ for every $n \in V(T)$. Then (T, β_1) is a tree decomposition of the clique $G[S]$. By Lemma 6, there exists a reduced tree decomposition (T_2, β_2) of $G[S]$ whose every bag is equal to some bag of (T, β_1). By Lemma 8, T_2 has only one vertex n_2, and $\beta_2(n_2) = S$. Let n be a vertex of T such that $\beta_2(n_2) = \beta_1(n)$. We have $S = \beta_2(n_2) = \beta_1(n) \subseteq \beta(n)$. □

Theorem 10. *A graph G is chordal if and only if it has a tree decomposition (T, β) such that every bag induces a clique.*

Proof. Suppose that G is chordal. We proceed by induction, and thus we can assume that the claim holds for all graphs with less than $|V(G)|$ vertices. If G is a clique, then we can set $V(T) = \{n\}$ and $\beta(n) = V(G)$. Hence, assume that G is not a clique, and by Theorem 3, there exist $G_1, G_2 \sqsubset G$ such that $G = G_1 \cup G_2$ and $G_1 \cap G_2$ is a clique. For $i \in \{1, 2\}$, the induction hypothesis implies that there exists a tree decomposition (T_i, β_i) of G_i such that every bag induces a clique. By Corollary 9, there exists $n_i \in V(T_i)$ such that $V(G_1 \cap G_2) \subseteq \beta_i(n_i)$. Let T be the tree obtained from T_1 and T_2 by adding the edge n_1n_2. Let β be defined by $\beta(n) = \beta_1(n)$ if $n \in V(T_1)$ and $\beta(n) = \beta_2(n)$ if $n \in V(T_2)$. Then (T, β) is a tree decomposition of G such that every bag induces a clique.

Conversely, suppose that G has a tree decomposition (T, β) such that every bag induces a clique. Consider a cycle $C \sqsubseteq G$, and let $\beta' : V(T) \to 2^{V(C)}$ be defined by $\beta'(n) = \beta(n) \cap V(C)$ for every $n \in V(T)$. Then, (T, β') is a tree decomposition of C such that every bag induces a clique. By Corollary 7, any tree decomposition of a cycle must contain a bag of size at least three, and thus C contains a clique of size three. Therefore, C is a triangle. It follows that G contains no induced hole, and thus G is chordal. □

2 Exercises

1. ($\star$) Let $\mathcal{C}$ be the class of graphs that can be obtained by a finite number of applications of these rules:

   - The graph with one vertex belongs to $\mathcal{C}$.
   - For any $G_1, G_2 \in \mathcal{C}$, the disjoint union of G_1 and G_2 belongs to $\mathcal{C}$.
   - For any $G_1, G_2 \in \mathcal{C}$, the complete join of G_1 and G_2 (the graph obtained from the disjoint union of G_1 and G_2 by adding all edges with one end in G_1 and the other end in G_2) belongs to $\mathcal{C}$.

   Prove that $\mathcal{C}$ is exactly the class of all cographs.

2. ($\star\star\star$) A graph G is a *split graph* if its vertex set can be partitioned to an independent set and a clique (with arbitrary edges between the two parts), i.e., there exist disjoint $A, B \subseteq V(G)$ such that $A \cup B = V(G)$, $G[A]$ is an independent set and $G[B]$ is a clique. Prove that $\mathrm{Forb}_{\sqsubseteq}(C_4, C_5, 2K_2) =$ split graphs. Hint: first show that all graphs in $\mathrm{Forb}_{\sqsubseteq}(C_4, C_5, 2K_2)$ are chordal.

3. ($\star\star$) Let (T, β) be a tree decomposition of G. Let n_1, n_2 and n_3 be vertices of T such that n_2 lies on the path between n_1 and n_3 in T. Prove that every path in G from $\beta(n_1)$ to $\beta(n_3)$ intersects $\beta(n_2)$.