# Regularity lemma—applications

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Recall:

**Definition 1.** *Let $G$ be a graph and let $\delta, \varepsilon > 0$ be real numbers. A pair $(A, B)$ of disjoint non-empty sets $A, B \subset V(G)$ is* $(\delta, \varepsilon)$-regular *if*

- *all $A' \subseteq A$ and $B' \subseteq B$ such that $|A'| \geq \delta|A|$ and $|B'| \geq \delta|B|$ satisfy*

$$|d(A', B') - d(A, B)| \leq \varepsilon.$$

*We say that the pair is* $\varepsilon$-regular *if it is* $(\varepsilon, \varepsilon)$-regular.

**Lemma 1.** *Let $G$ be a graph and let $(A, B)$ be a $(\delta, \varepsilon)$-regular pair in $G$ for some $0 < \delta, \varepsilon \leq 1$. Let $B' \subseteq B$ have size at least $\delta|B|$. Then*

- *the number of vertices of $A$ with more than $(d(A, B) + \varepsilon)|B'|$ neighbors in $B'$ is less than $\delta|A|$, and*
- *the number of vertices of $A$ with less than $(d(A, B) - \varepsilon)|B'|$ neighbors in $B'$ is less than $\delta|A|$.*

**Lemma 2.** *Let $G$ be a graph and let $(A, B)$, $(B, C)$ and $(A, C)$ be $(\delta, \varepsilon)$-regular pairs in $G$ for some $0 < \delta, \varepsilon \leq 1/2$, with $|A| = |B| = |C| = n$ and $d(A, B), d(B, C), d(A, C) \geq \delta + \varepsilon$. The number of triangles $v_1v_2v_3$ of $G$ with $v_1 \in A$, $v_2 \in B$, and $v_3 \in C$ is at least*

$$(1 - 2\delta)(d(B, C) - \varepsilon)(d(A, B) - \varepsilon)(d(A, C) - \varepsilon)n^3.$$

**Definition 2.** *For a graph $G$, a partition $V_0$, $V_1$, ..., $V_m$ of $V(G)$ is* $\varepsilon$-regular *if*

- $|V_0| \leq \varepsilon|V(G)|$,
- $|V_1| = |V_2| = \ldots = |V_m|$, *and*

- *for all but at most $\varepsilon m^2$ values of $1 \leq i < j \leq m$, the pair $(V_i, V_j)$ is $\varepsilon$-regular.*

*The integer $m$ is called the* order *of the partition.*

**Theorem 3** (Regularity lemma)**.** *For any positive integer $m_0$ and real number $\varepsilon > 0$, there exists an integer $M \geq m_0$ such that the following holds. Every graph $G$ with at least $m_0$ vertices has an $\varepsilon$-regular partition of order at least $m_0$ and at most $M$.*

# 1 Removal lemma for triangles

Intuitively, either a graph contains a large (cubic) number of triangles, or we should be able to kill all triangles by removing a small (subquadratic) set of edges.

**Theorem 4** (Triangle removal lemma)**.** *For every $0 < \alpha \leq 1$, there exists $\beta > 0$ and $n_0$ such that if $G$ is a graph with $n \geq n_0$ vertices, then either*

- *$G$ contains at least $\beta n^3$ triangles, or*
- *there exists a set $X \subseteq E(G)$ such that $|X| \leq \alpha n^2$ and $G - X$ contains no trianges.*

*Proof.* Let $\varepsilon = \alpha/5$ and $m_0 = \lceil 1/\varepsilon \rceil$. Let $M$ be the upper bound from the Regularity lemma for these values. Let $n_0 = m_0$ and $\beta = (1 - 2\varepsilon)\varepsilon^3(1 - \varepsilon)^3/M^3$.

Regularity lemma implies that $G$ has an $\varepsilon$-regular partition $V_0$, $V_1$, $\ldots$, $V_m$ such that $m_0 \leq m \leq M$. Let $k = |V_1| = \ldots = |V_m|$ and note that $(1 - \varepsilon)n/m \leq k \leq n/m$.

- Let $X_1$ consist of the edges of $G$ incident with $V_0$. We have $|X_1| \leq |V_0|n \leq \varepsilon n^2$.
- Let $X_2 = E(G[V_1]) \cup E(G[V_2]) \cup \ldots \cup E(G[V_m])$. We have $|X_2| \leq mk^2 \leq m(n/m)^2 = n^2/m \leq n^2/m_0 \leq \varepsilon n^2$.
- Let $X_3$ consist of the edges between sets $V_i$ and $V_j$ with $1 \leq i < j \leq m$ such that $d(V_i, V_j) \leq 2\varepsilon$. Note that $|X_3| \leq m^2(2\varepsilon k^2) \leq 2\varepsilon m^2(n/m)^2 = 2\varepsilon n^2$.
- Let $X_4$ consist of the edges between sets $V_i$ and $V_j$ with $1 \leq i < j \leq m$ such that $(V_i, V_j)$ is not an $\varepsilon$-regular pair. Since the partition is $\varepsilon$-regular, we have $|X_4| \leq (\varepsilon m^2)k^2 \leq \varepsilon m^2(n/m)^2 = \varepsilon n^2$.

Let $X = X_1 \cup X_2 \cup X_3 \cup X_4$. Note that $|X| \le 5\varepsilon n^2 = \alpha n^2$, hence if $G - X$ is triangle-free, then the second outcome of the Removal lemma holds.

Therefore, suppose that $G - X$ contains a triangle $v_1v_2v_3$. By the choice of $X$, there exist distinct $i_1, i_2, i_3 \in \{1, \ldots, m\}$ such that $v_1 \in V_{i_1}$, $v_2 \in V_{i_2}$, $v_3 \in V_{i_3}$, and each of $(V_{i_1}, V_{i,2})$, $(V_{i_2}, V_{i_3})$ and $(V_{i_1}, V_{i_3})$ is an $\varepsilon$-regular pair of density at least $2\varepsilon$. By Lemma 2, $G$ contains at least $(1 - 2\varepsilon)\varepsilon^3 k^3 \ge (1 - 2\varepsilon)\varepsilon^3[(1 - \varepsilon)n/M]^3 = \beta n^3$ triangles, as required by the first outcome of the Removal lemma. □

Similar claims can be proved for all other graphs.

## 2 Erdös-Stone theorem

Turán gave an upper bound on the number of edges of a graph without $K_k$: every $n$-vertex graph with more than $\left(1 - \frac{1}{k-1}\right)\frac{n^2}{2}$ edges contains $K_k$ (and the bound is tight, since a complete $(k-1)$-partite graph does not contain $K_k$). Erdös and Stone generalized this to all forbidden subgraphs.

**Theorem 5** (Erdös-Stone theorem)**.** *Let $H$ be a graph with chromatic number $k \ge 2$. For every $\alpha > 0$, there exists $n_0$ such that if a graph $G$ with $n \ge n_0$ vertices has at least $\left(1 - \frac{1}{k-1} + \alpha\right)\frac{n^2}{2}$ edges, then $G$ contains $H$ as a subgraph.*

Again, since a complete $(k-1)$-partite graph cannot contain a graph with chromatic number $k$ as a subgraph, this result is almost tight both in the exponent and in the multiplicative constant, with the exception of excluded bipartite graphs $H$—for those, the right bound on the number of edges is subquadratic, and the problem of determining the exponent and multiplicative constant of such bound is open for most graphs $H$).

In the proof of Erdös-Stone theorem, we will need the following generalization of Lemma 2.

**Lemma 6.** *Let $k > 0$ be an integer, let $H$ be a $k$-colorable graph and let $d > 0$ be a real number. There exists a positive real number $\varepsilon < d$ and an integer $n_1$ as follows. Let $G$ be a graph and let $V_1, \ldots, V_k$ be pairwise disjoint subsets of its vertices, such that $|V_1| = \ldots = |V_k| \ge n_1$. If $(V_i, V_j)$ is an $\varepsilon$-regular pair of density at least $d$ for $1 \le i < j \le k$, then $H$ is a subgraph of $G$.*

*Proof.* Choose $\varepsilon > 0$ such that $(d - \varepsilon)^t = 2\varepsilon$, and let $n_1 = \lceil t/\varepsilon \rceil$.

Let $n = |V_1| = \ldots = |V_k|$. Let $\psi : V(H) \to \{1, \ldots, k\}$ be a $k$-coloring. Let $x_1, \ldots, x_t$ be the vertices of $H$. We will choose pairwise distinct vertices $v_1, \ldots, v_t \in V(G)$ such that

1. For $i = 1, \ldots, t$, the vertex $v_i$ belongs to $V_{\psi(x_i)}$.

2. For $1 \le i < j \le t$, if $\psi(x_i) \ne \psi(x_j)$, then $v_i v_j$ is an edge of $G$.

The second condition implies that $\{v_1, \ldots, v_t\}$ induces a supergraph of $H$.

We will choose the vertices $v_1, \ldots, v_t$ one by one. Suppose that we already fixed $v_1, \ldots, v_p$. For $1 \le i \le k$, let $C_{p,i}$ denote the set of vertices of $V_i$ that are adjacent to all vertices in $\{v_1, \ldots, v_p\} \setminus V_i$. We will maintain the following invariant:

$$|C_{p,i}| \ge (d - \varepsilon)^p n \text{ for } 1 \le i \le k \tag{1}$$

Let $c = \psi(x_{p+1})$. We now describe how to choose $v_{p+1}$ from the set $C_{p,c}$, which will ensure that $v_{p+1}$ is adjacent to all vertices $v_i$ with $1 \le i \le p$ such that $\psi(x_i) \ne \psi(x_{p+1})$.

Note that by (1), we have $|C_{p,i}| \ge (d-\varepsilon)^t n > \varepsilon n$ for $1 \le i \le k$. Hence, by Lemma 1, the number of vertices of $V_c$ with less than $(d-\varepsilon)|C_{p,i}|$ neighbors in $C_{p,i}$ is less than $\varepsilon n$, for any $i$ distinct from $c$. Since $n \ge n_1$, we have $|C_{p,c}| \ge (d-\varepsilon)^t n \ge k\varepsilon n + t$, and thus there exists a vertex $v_{p+1} \in C_{p,c} \setminus \{v_1, \ldots, v_p\}$ such that $v_{p+1}$ has at least $(d - \varepsilon)|C_{p,i}|$ neighbors in $C_{p,i}$ for any $i$ distinct from $c$. This ensures that (1) holds for $p + 1$. $\square$

Let us now proceed with the proof of Erdös-Stone theorem.

*Proof of Theorem 5.* Let $d = \alpha/8$. Let $\varepsilon > 0$ and $n_1$ be the constants of Lemma 6 applied for $H$. Let $m_0 = \lceil 1/\varepsilon \rceil$. Let $M$ be the constant of the Regularity lemma for $m_0$ and $\varepsilon$. Let $n_0 = \lceil \max(m_0, n_1 M/(1-\varepsilon)) \rceil$.

Let $V_0, V_1, \ldots, V_m$ be an $\varepsilon$-regular partition of $G$. Let $s = |V_1| = \ldots = |V_m|$ and note that $(1-\varepsilon)n/m \le s \le n/m$.

- Let $X_1$ consist of the edges of $G$ incident with $V_0$. We have $|X_1| \le |V_0| n \le \varepsilon n^2$.
- Let $X_2 = E(G[V_1]) \cup E(G[V_2]) \cup \ldots \cup E(G[V_m])$. We have $|X_2| \le m s^2 \le m(n/m)^2 = n^2/m \le n^2/m_0 \le \varepsilon n^2$.
- Let $X_3$ consist of the edges between sets $V_i$ and $V_j$ with $1 \le i < j \le m$ such that $d(V_i, V_j) \le d$. Note that $|X_3| \le m^2(dk^2) \le dm^2(n/m)^2 = dn^2$.
- Let $X_4$ consist of the edges between sets $V_i$ and $V_j$ with $1 \le i < j \le m$ such that $(V_i, V_j)$ is not an $\varepsilon$-regular pair. Since the partition is $\varepsilon$-regular, we have $|X_4| \le (\varepsilon m^2)s^2 \le \varepsilon m^2(n/m)^2 = \varepsilon n^2$.

Let $X = X_1 \cup X_2 \cup X_3 \cup X_4$. Note that $|X| \le (3\varepsilon + d)n^2 < 4dn^2 = \alpha\frac{n^2}{2}$, and thus $G - X$ has at least $\left(1 - \frac{1}{k-1}\right)\frac{n^2}{2}$ edges. Let $G'$ be the graph with vertex set $\{1, \ldots, m\}$ such that $ij \in E(G)$ if and only if $(V_i, V_j)$ has non-zero density in $G - X$. Observe that

$$|E(G - X)| \leq s^2|E(G')| \leq \frac{n^2}{m^2}|E(G')|,$$

and thus $|E(G')| > \left(1 - \frac{1}{k-1}\right)\frac{m^2}{2}$. By Turán's theorem, $K_k$ is a subgraph of $G'$. By renumbering the vertices if necessary, we can assume that $\{1, \ldots, k\}$ is a clique in $G'$. By the choice of $X$ and $G'$, $(V_i, V_j)$ is an $\varepsilon$-regular pair in $G$ with density at least $d$ for $1 \leq i < j \leq k$. Since $s \geq (1 - \varepsilon)n_0/M \geq n_1$, Lemma 6 implies that $H$ is a subgraph of $G$. $\square$