Theorem (Tutte)

If $G \neq K_4$ is 3-connected, then there exists $e \in E(G)$ such that G/e is 3-connected.

Corollary

Every 3-connected graph can be obtained from K_4 by decontracting edges.

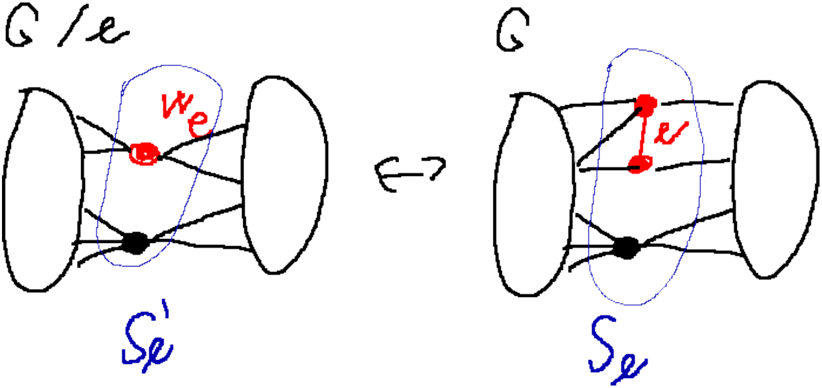
Compare:

Lemma

Every 2-connected graph can be obtained from a cycle by adding ears.

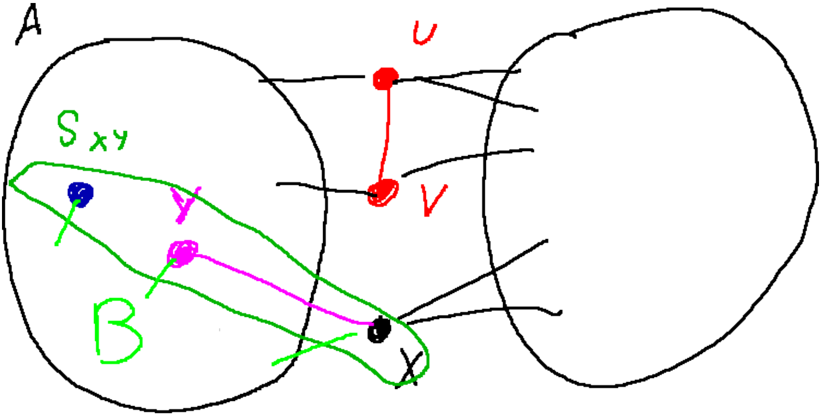
By contradiction: Suppose $(\forall e \in E(G)) G/e$ is not 3-connected.

- w_e: Vertex created by contracting $e = uv$.
- S'_e: A (≤ 2)-cut in G/e.
  - $w_e \in S'_e$.
- $S_e = (S'_e \setminus \{w_e\}) \cup \{u, v\}$ is a 3-cut in G.



Choose $e = uv \in E(G)$ and a component A of $G - S_e$ so that $|V(A)|$ is minimum.

- $S_e = \{u, v, x\}$.
- G 3-connected $\Rightarrow$ x has a neighbor y in A.
- B: The component of $G - S_{xy}$ disjoint from $\{u, v\}$.
  - $B \subset G - \{x, y, u, v\}$.
- G 3-connected $\Rightarrow$ y has a neighbor in B.
  - All neighbors of y are in $V(A) \cup \{u, v, x\}$: $B \cap A \neq \emptyset$.
- B connected, $B \subset G - S_e$, $y \notin V(B) \Rightarrow B \subsetneq A$.



Theorem (Wagner, the hard implication)

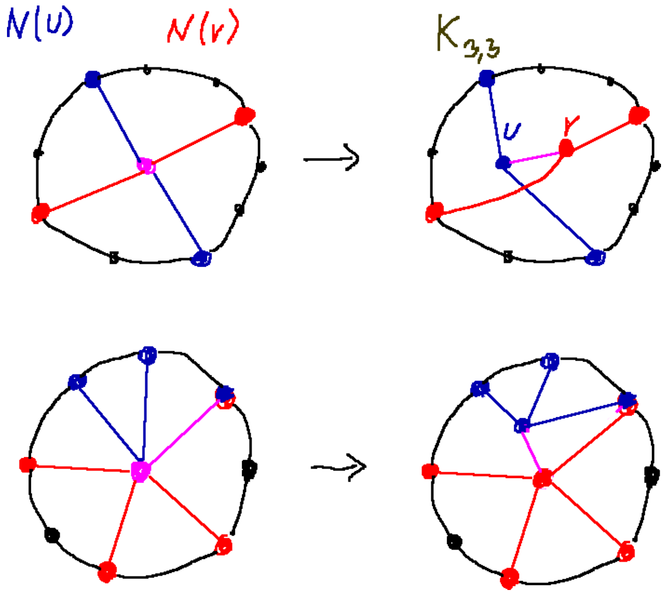
If $K_5, K_{3,3} \not\preceq_m G$, then G is planar.

Lemma

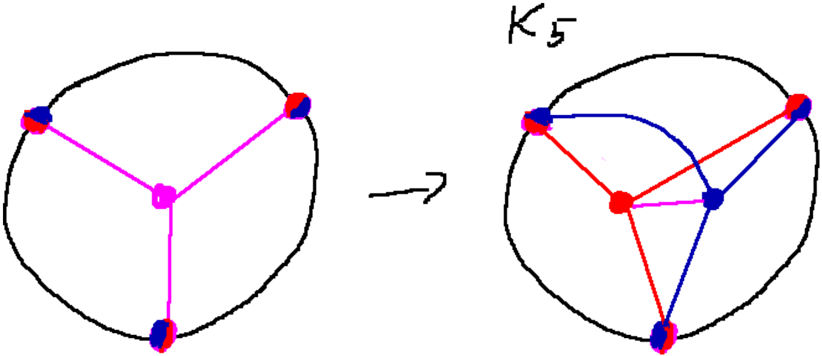
If G is 3-connected and $K_5, K_{3,3} \not\preceq_m G$, then G is planar.

- Choose uv such that G/uv is 3-connected.
- By the induction hypothesis, G/uv is planar.
- $G - \{u, v\}$ planar, 2-connected $\Rightarrow$ faces bounded by cycles.
- Transform the drawing of G/uv to a drawing of G.

Case 1: $N(u) \setminus \{v\} \not\subseteq N(v) \setminus \{u\}$, or vice versa.



Case 2: $N(u) \setminus \{v\} = N(v) \setminus \{u\}$.

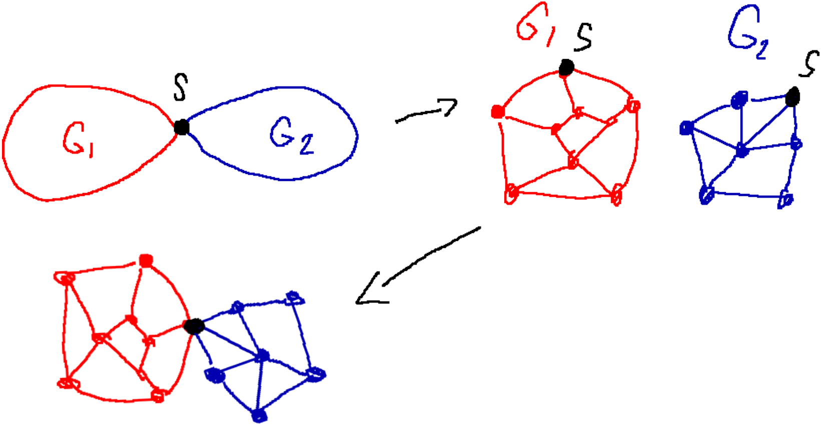


Theorem (Wagner, the hard implication)

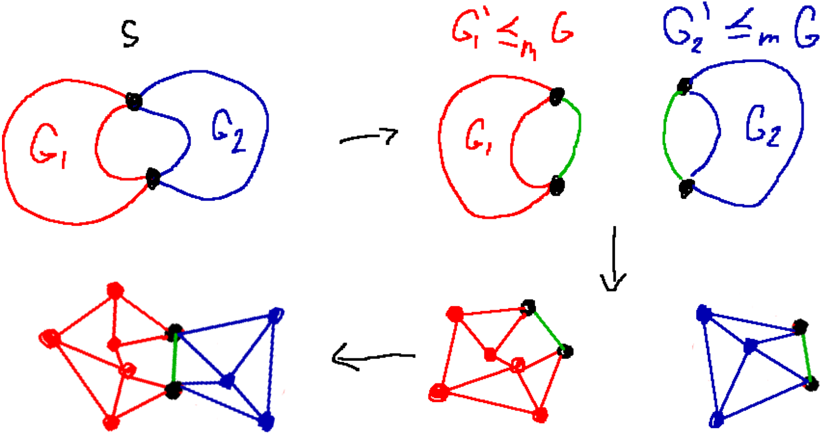
If $K_5, K_{3,3} \not\preceq_m G$, then G is planar.

- $S =$ smallest cut in G, WLOG $|S| \leq 2$.

Case 1: $|S| \leq 1$.



Case 2: $|S| = 2$.



Q: Define the chromatic number of a graph.

Q: Define the chromatic number of a graph.

Definition

A function $\varphi : V(G) \to \{1, \ldots, k\}$ is a proper k-coloring if for every $uv \in E(G)$, we have $\varphi(u) \neq \varphi(v)$.

Definition

The chromatic number $\chi(G)$ of G is the smallest k such that G has a proper k-coloring.

Q: What is the largest possible chromatic number of a graph G such that $K_3 \not\preceq_m G$?

Lemma

If G has $n \geq 4$ vertices and $|E(G)| \geq 2n - 2$, then $K_4 \preceq_m G$.

By induction on $n + |E(G)|$:

- $n = 4$, $|E(G)| \geq 6 \Rightarrow G = K_4$.
- $|E(G)| > 2n - 2 \Rightarrow$ for any $e \in E(G)$, $K_4 \preceq_m G - e$ by the induction hypothesis.
- $n \geq 5$, $|E(G)| = 2n - 2$, average degree

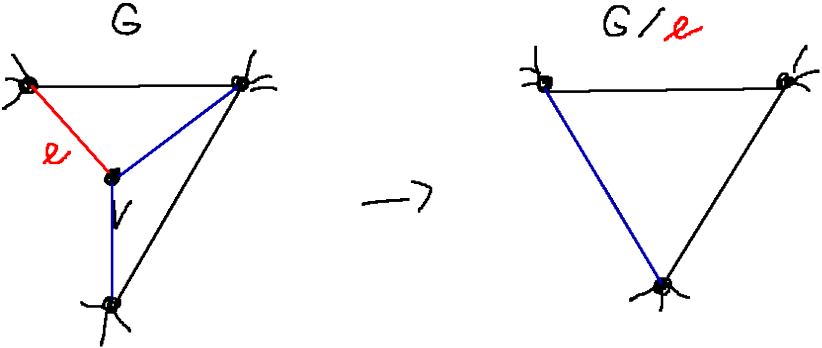
$$\frac{2|E(G)|}{|V(G)|} = 4 - \frac{4}{n} < 4.$$

Case 1: $\delta(G) \leq 2$.

If $\deg(v) \leq 2$, then

- $|E(G - v)| \geq |E(G)| - 2 = (2n - 2) - 2 = 2(n - 1) - 2$
- $K_4 \preceq_m G - v$ by the induction hypothesis.

Case 2: $\delta(G) = 3$.



- $|E(G/e)| \geq |E(G)| - 2 = (2n - 2) - 2 = 2(n - 1) - 2$
- $K_4 \preceq_m G/e$ by the induction hypothesis.

Q: For every $n \geq 4$, find a graph with n vertices and $2n - 3$ edges not containing K_4 as a minor.

Lemma

If G has $n \geq 4$ vertices and $K_4 \not\preceq_m G$, then $|E(G)| \leq 2n - 3$.

Corollary

If $K_4 \not\preceq_m G$, then G has

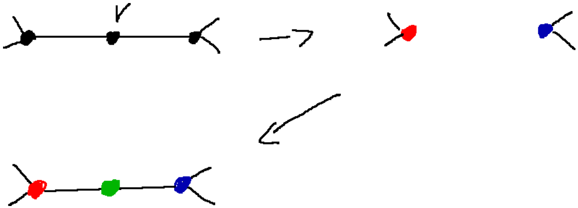
- *average degree at most $4 - 6/n$ and*
- *minimum degree at most 3.*

Remark: Actually $\delta(G) \leq 2$.

Corollary

If $K_4 \not\leq_m G$, then $\chi(G) \leq 3$.

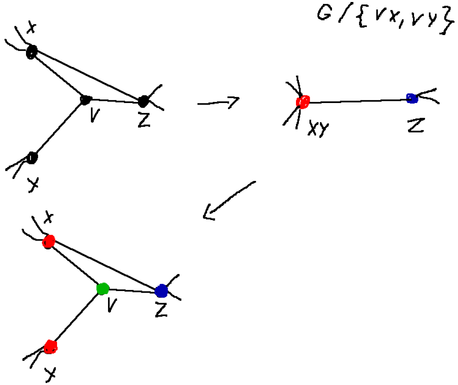
- v a vertex of degree at most 3
- $\deg(v) \leq 2$: 3-color $G - v$ and extend to v.



Corollary

If $K_4 \not\preceq_m G$, then $\chi(G) \leq 3$.

- v a vertex of degree at most 3
- $\deg(v) = 3$: x, y non-adjacent neighbors of v
- $G/\{vx, vy\}$ 3-colorable by induction hypothesis.



Q: What is the maximum possible chromatic number of a graph G such that $K_5, K_{3,3} \not\preceq_m G$?

Theorem (Wagner)

$$\max\{\chi(G) : K_5 \not\preceq_m G\} = \max\{\chi(G) : G \text{ planar}\}$$

Corollary

For $k \leq 5$, if $K_k \not\preceq_m G$, then $\chi(G) \leq k - 1$.

Conjecture (Hadwiger)

For every k, if $K_k \not\preceq_m G$, then $\chi(G) \leq k - 1$.

- True also for $k = 6$ (Robertson, Seymour, Thomas'93).
- $K_k \not\preceq_m G \Rightarrow \chi(G) = O(k \cdot (\log \log k)^6)$. (Postle'20)