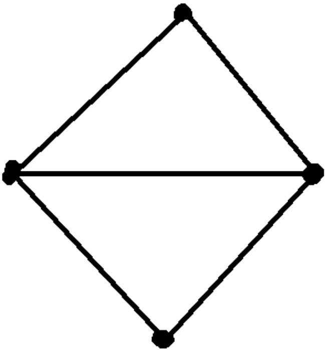

Q: How many perfect matchings does this graph have?

Determining whether a graph has a perfect matching:

- in bipartite graphs: via maximum flow algorithms in $O(n^{1/2}m)$
- in general graphs:
  - Edmonds (blossom) algorithm in $O(n^2 m)$
  - Micali-Vazirani algorithm in $O(n^{1/2}m)$

Determining the number of matchings:

- #P-hard
  - no polynomial-time algorithm unless $\mathrm{P} = \mathrm{NP}$.
  - even for bipartite graphs
- in planar graphs: in $O(n^{2.373})$

Q: Define the sign of a permutation π of $\{1, \ldots, n\}$.

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$$\operatorname{sgn}(\pi) = (-1)^{n + \text{number of cycles of } \pi}$$

Example: The permutation π given by

x	1	2	3	4	5	6	7
$\pi(x)$	3	2	4	1	5	7	6

has cycles $(134), (2), (5), (67)$ and sign -1.

Determinant of an $n \times n$ matrix C:

$$\det(C) = \sum_{\pi} \operatorname{sgn}(\pi) \prod_{i=1}^{n} C_{i,\pi(i)}.$$

Permanent of an $n \times n$ matrix C:

$$\operatorname{per}(C) = \sum_{\pi} \prod_{i=1}^{n} C_{i,\pi(i)}.$$

Q: What is the determinant and the permanent of the following matrix?

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

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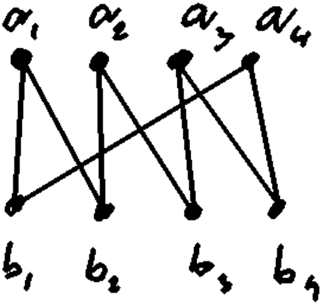
$\det = 0$

$\operatorname{per} = 2$

For a bipartite graph G with parts $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$, the bipartite adjacency matrix C has

$$C_{i,j} = \begin{cases} 1 & \text{if } a_i b_j \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

Example: The bipartite adjacency matrix of



is $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$.

Observation

Number of matchings of $G = \mathrm{per}(C)$.

Theorem (Bregman-Minc inequality)

Let C be an $n \times n$ $\{0,1\}$-matrix, where the i-th row contains r_i ones. Then

$$\operatorname{per}(C) \le \prod_{i=1}^{n} \sqrt[r_i]{r_i!}.$$

Corollary

If G is a d-regular bipartite graph with parts of size n, then G has at most

$$\left(\sqrt[d]{d!}\right)^n \le \left(\sqrt[d]{de} \cdot d/e\right)^n$$

perfect matchings.

Q: Suppose n is divisible by d. Find a d-regular bipartite graph with parts of size n that has $(d!)^{n/d}$ perfect matchings.

A matrix is bistochastic if it is non-negative and all rows and columns sum to 1.

Theorem (Van der Waerden inequality)

If C is an $n \times n$ bistochastic matrix, then

$$\operatorname{per}(C) \geq n!/n^n.$$

Q: Which bistochastic $n \times n$ matrix satisfies $\operatorname{per}(C) = n!/n^n$?

Corollary

If G is a d-regular bipartite graph with parts of size n and C is the bipartite adjacency matrix of G, then C/d is bistochastic, and G has

$$\operatorname{per}(C) = d^n \operatorname{per}(C/d) \geq d^n n!/n^n \geq (d/e)^n$$

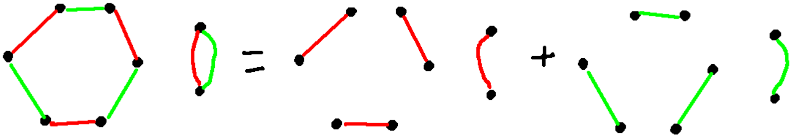
perfect matchings.

- If G is a 3-regular bipartite graph with parts of size n, then the number m of perfect matchings of G satisfies

$$1.1^n \leq m \leq 2.23^n.$$

- There exists $c > 1$ such that every 3-regular 2-edge-connected graph with n vertices has at least c^n perfect matchings.

- Even 2-factor: graph F whose components are even cycles
  - 2-cycles are allowed
  - $c(F)$: number of components of F.
  - $c_2(F)$: number of 2-cycles of F.
- For perfect matchings M_1 and M_2: their union $M_1 + M_2$ is an even 2-factor.
- $M(F) = \{(M_1, M_2) : F = M_1 + M_2\}$.



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Q: Express $|M(F)|$ in terms of $c(F)$ and $c_2(F)$.

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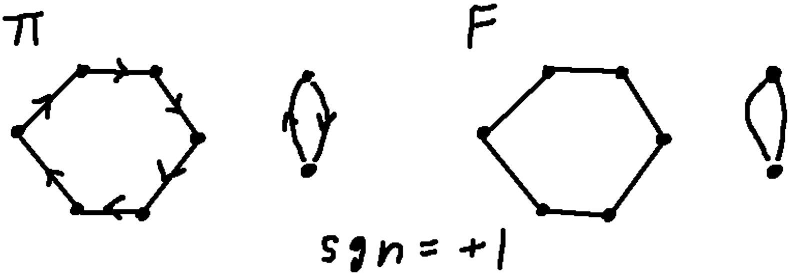
$$|M(F)| = 2^{c(F)-c_2(F)}$$

- A permutation π is even-cycled if all its cycles have even length.
- For an even 2-factor F, $\Pi(F) =$ permutations with cycles F.
- $\mathrm{sgn}(F) = \mathrm{sgn}(\pi)$ for $\pi \in \Pi(F) = (-1)^{c(F)}$



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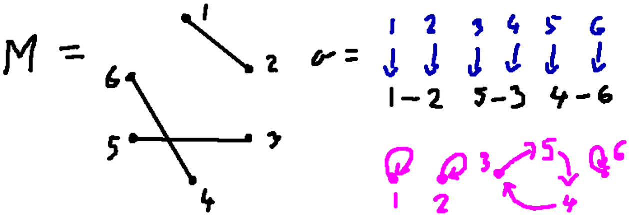
Let C be the adjacency matrix of a graph G with vertex set $\{1, \ldots, n\}$.

$$\begin{aligned}
\sum_{\pi \text{ even-cycled}} \prod_{i=1}^{n} C_{i,\pi(i)} &= \sum_{F \text{ even 2-factor in } G} 2^{c(F)-c_2(F)} \\
&= \sum_{M_1, M_2 \text{ perfect matchings in } G} 1 \\
&= \big(\text{number of perfect matchings in } G\big)^2.
\end{aligned}$$

$\lambda(x, y) = 1$ if $x < y$ and -1 if $x > y$

For a matching M with vertices $\{1, \ldots, n\}$:

- Let σ be a permutation such that $\sigma(1)\sigma(2), \sigma(3)\sigma(4), \ldots \in E(M)$.
- $$\operatorname{sgn}(M) = \operatorname{sgn}(\sigma) \prod_{i=1}^{n/2} \lambda(\sigma(2i-1), \sigma(2i)).$$
- Note: $\operatorname{sgn}(M)$ is the same for all choices of Σ.



$$\operatorname{sgn}(M) = \operatorname{sgn}(\sigma) \cdot \lambda(1, 2) \cdot \lambda(5, 3) \cdot \lambda(4, 6)$$
$$= 1 \cdot 1 \cdot (-1) \cdot 1 = -1.$$

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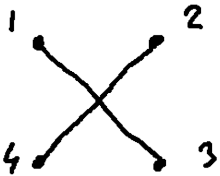
Q: What is the sign of this matching?



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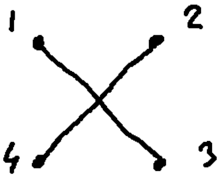
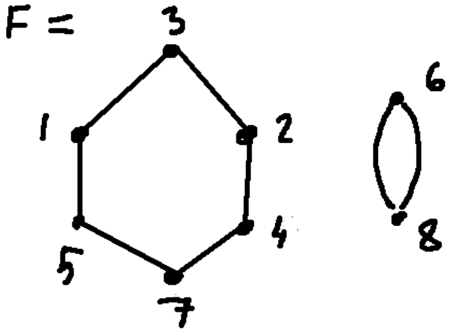


$\sigma = 1324 \sim (1)(23)(4)$, $\mathrm{sgn} = \mathrm{sgn}(\sigma) \cdot \lambda(1,3) \cdot \lambda(2,4) = -1$

- $C = v_1 \ldots v_t$ even cycle:
  $\lambda(C) = \lambda(v_1, v_2) \cdot \lambda(v_2, v_3) \cdots \lambda(v_t, v_1)$.
- F even 2-factor: $\lambda(F) = \prod_{C \text{ cycle of } F} \lambda(C)$.



Q: Determine $\lambda(F)$.

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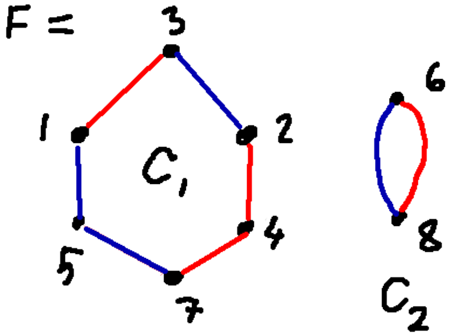


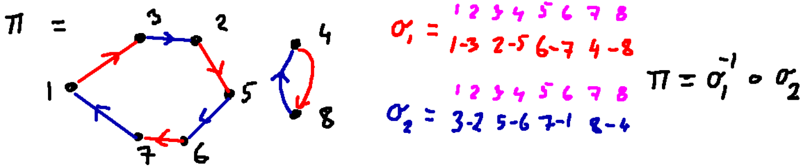
$\lambda(F) = \lambda(C_1) \cdot \lambda(C_2) = (-1) \cdot (-1) = 1$

Q: Determine $\lambda(F)$.

Lemma

$$\mathrm{sgn}(M_1)\mathrm{sgn}(M_2) = \mathrm{sgn}(M_1 + M_2)\lambda(M_1 + M_2)$$

For π =permutation with cycles $M_1 + M_2$:



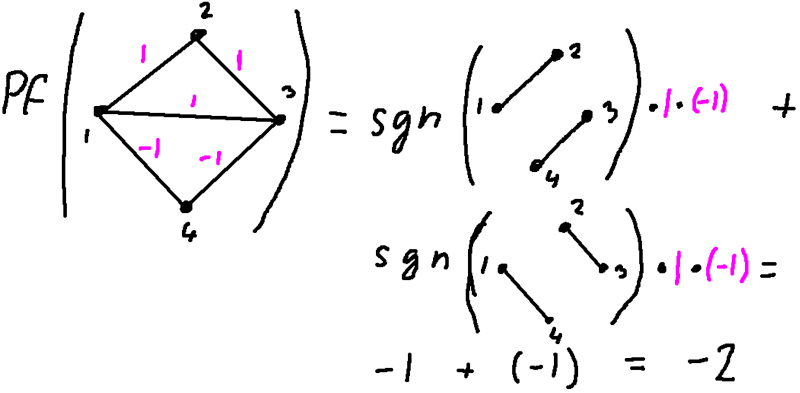
$$\mathrm{sgn}(M_1 + M_2) = \mathrm{sgn}(\pi) = \mathrm{sgn}(\sigma_1^{-1}\sigma_2) = \mathrm{sgn}(M_1)\mathrm{sgn}(M_2)\prod_{i=1}^{n}\lambda(i, \pi(i))$$

$$= \mathrm{sgn}(M_1)\mathrm{sgn}(M_2)\lambda(M_1 + M_2)$$

For $b : E(G) \to \mathbb{R}$, the Pfaffian of (G, b) is

$$\mathrm{Pf}(G,b) = \sum_{M \text{ perfect matching of } G} \mathrm{sgn}(M) \prod_{e \in E(M)} b(e).$$

Example:



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Pfaffian function: $b : E(G) \to \{-1, 1\}$ such that

$$\mathrm{sgn}(M) \cdot \prod_{e \in E(M)} b(e)$$

is the same for every perfect matching M of G.

Observation

If b is a Pfaffian function, then

$$|\mathrm{Pf}(G, b)| = \textit{number of perfect matchings in } G.$$

Lemma

For any graph G and a function $b : E(G) \to \mathbb{Z}$*,* $|\mathrm{Pf}(G, b)|$ *can be computed in polynomial time.*

Theorem (Kasteleyn)

For every planar graph G, we can find a Pfaffian function b in polynomial time.

Corollary

Polynomial-time algorithm to find the number of perfect matchings in a planar graph G.