Q: Maximum number of edges of a triangle-free graph with 5 vertices?

Theorem (Mantel)

An n-vertex triangle-free graph G has at most

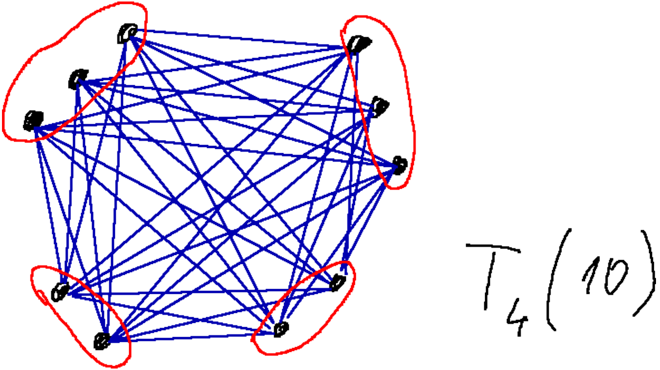
$$\lfloor \tfrac{n}{2} \rfloor \cdot \lceil \tfrac{n}{2} \rceil$$

edges. Equality iff

$$G = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}.$$

Definition

The *r*-partite Turán graph $T_r(n)$: the complete *r*-partite *n*-vertex graph with parts of sizes $\lfloor \frac{n}{r} \rfloor$ and $\lceil \frac{n}{2} \rceil$



$$t_r(n) = |E(T_r(n)|$$

Observation

$$t_r(n) \leq (1 - 1/r)\frac{n^2}{2}$$

Theorem (Turán)

An n-vertex graph G with $\omega(G) \le r$ *has at most* $t_r(n)$ *edges. Equality iff* $G = T_r(n)$.

Corollary

An n-vertex graph H of average degree at most d contains an independent set of size at least $n/(d+1)$.

Proof.

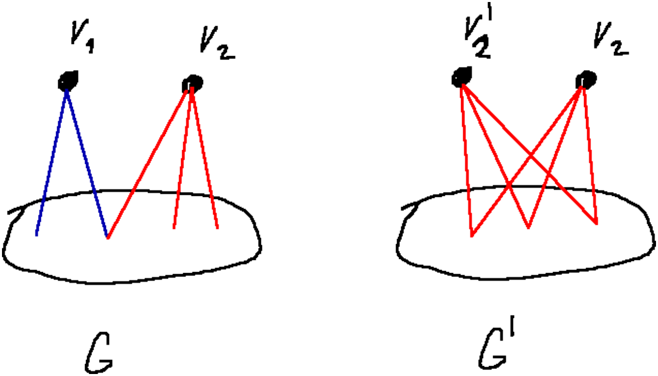
$$|E(\overline{H})| \geq \frac{(n-d-1)n}{2} = \left(1 - \tfrac{d+1}{n}\right)\frac{n^2}{2} > \left(1 - \frac{1}{\lfloor \frac{n}{d+1} \rfloor - 1}\right)\frac{n^2}{2}$$

Turán: $\omega(\overline{H}) > \lfloor \frac{n}{d+1} \rfloor - 1$ □

n-vertex graph G such that

- $\omega(G) \leq r$
- $|E(G)|$ is maximum

If $v_1 v_2 \notin E(G)$, then $\deg v_1 = \deg v_2$

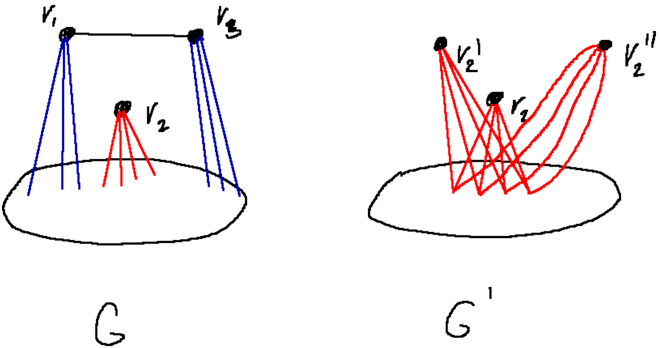


$$|E(G')| = |E(G)| - \deg v_1 + \deg v_2 > |E(G)|$$

n-vertex graph G such that

- $\omega(G) \leq r$
- $|E(G)|$ is maximum

If $v_1v_2, v_2v_3 \notin E(G)$, then $v_1v_3 \notin E(G)$

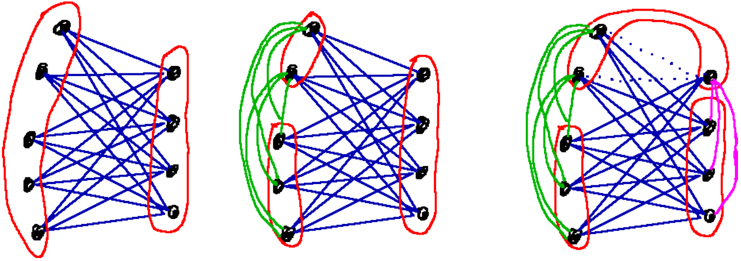


$$|E(G')| = |E(G)| - (\deg v_1 + \deg v_3 - 1) + 2\deg v_2 > |E(G)|$$

n-vertex graph *G* such that

- $\omega(G) \leq r$
- $|E(G)|$ is maximum

G is a complete *r*-partite graph, sizes of parts differ by ≤ 1.



n-vertex graph G such that

- $\omega(G) \leq r$
- $|E(G)|$ is maximum

$$G = T_r(n)$$

Observation

If $\chi(F) > r$, *then* $F \not\subseteq T_r(n)$.

Theorem (Erdős-Stone)

Suppose $\chi(F) = r + 1$. *For every* $\varepsilon > 0$, *there exists* n_0 *such that:*

Every graph with $n \geq n_0$ *vertices and at least* $(1 - 1/r + \varepsilon)\frac{n^2}{2}$ *edges contains* F *as a subgraph.*

Better bounds if F is bipartite:

Theorem

Every n-vertex graph without 4-cycles has $O(n^{3/2})$ edges.

Lemma

For F is bipartite with one of its parts of size a:

Every n-vertex graph without F as a subgraph has $O(n^{2-1/a})$ edges.

Q: How many 2-element subsets of $\{1, \ldots, 5\}$ can you choose so that every two intersect?

Theorem (Erdős-Ko-Rado)

For $n \geq 2r$: The largest system of pairwise intersecting r-element subsets of $\{1, \ldots, n\}$ has size

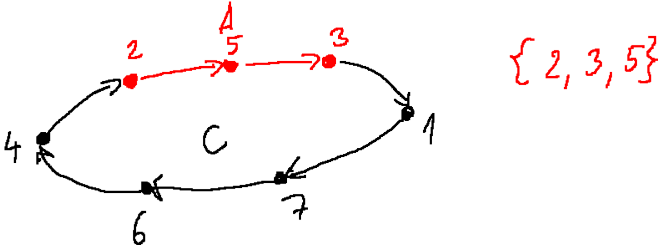
$$\binom{n-1}{r-1}.$$

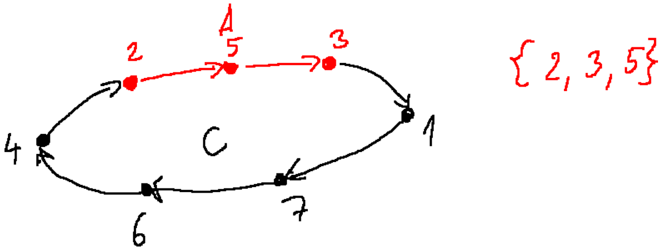
- $\binom{n-1}{r-1}$ subsets containing n.
- If $n < 2r$: We can take all $\binom{n}{r}$ subsets.

$A_1, A_2, \ldots, A_m$ pairwise intersecting r-element subsets of $\{1, \ldots, n\}$.

$c = \#$ of pairs (C, A), where

- C directed cycle with $V(C) = \{1, \ldots, n\}$.
- A subpath of C with $V(A) \in \{A_1, \ldots, A_m\}$.

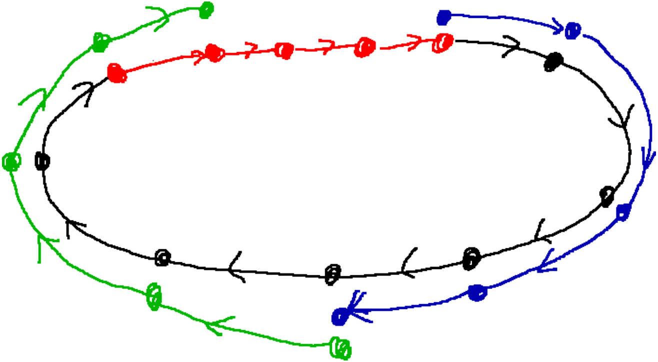




$$c = mr! \cdot (n - r)!$$

Observation

C has at most r pairwise intersecting r-vertex subpaths.



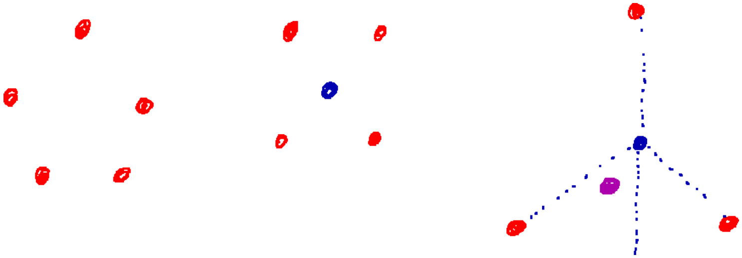
$$c \leq (n-1)! \cdot r$$

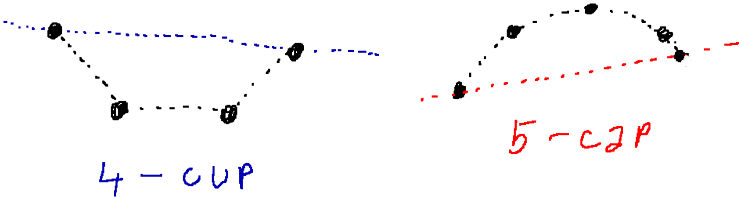
$$mr! \cdot (n-r)! = c \leq (n-1)! \cdot r$$

$$m \leq \frac{(n-1)!r}{r!(n-r)!} = \binom{n-1}{r-1}$$

Observation

Among any 5 points in general position, one can choose 4 in convex position.





$f(a,b) =$ minimum number such that any $f(a,b)$ points in general position contain an a-cup or a b-cap.

$f(2,b) = f(a,2) = 2$.

Lemma

$$f(a,b) \leq f(a-1,b) + f(a,b-1) - 1$$

X: A set of $f(a-1,b)+f(a,b-1)-1$ points

$A =$ rightmost points of $(a-1)$-cups in X.

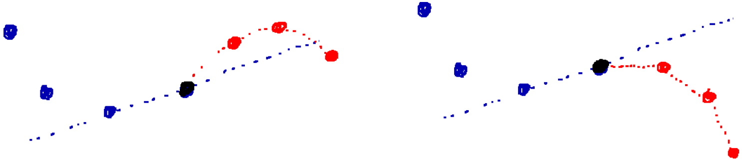
- $X \setminus A$ contains a b-cap: Win.
- $X \setminus A$ contains neither an $(a-1)$-cup nor a b-cap.

$$|X \setminus A| \leq f(a-1,b)-1.$$

X: A set of $f(a-1,b)+f(a,b-1)-1$ points

$A =$ rightmost points of $(a-1)$-cups in X.
$B =$ leftmost points of $(b-1)$-caps in X.

$$|X \setminus A| \le f(a-1,b)-1$$
$$|X \setminus B| \le f(a,b-1)-1$$
$$|X \setminus A| + |X \setminus B| \le f(a-1,b)+f(a,b-1)-2 < |X|$$



$$f(2,b) = f(a,2) = 2$$
$$f(a,b) \leq f(a-1,b) + f(a,b-1) - 1$$
$$f(a,b) \leq \binom{a+b-4}{a-2} + 1$$

Corollary

Any set of $\binom{a+b-4}{a-2} + 1$ points in general position contains an a-cup or a b-cap.

Corollary (Erdős-Szekeres)

Any set of $\binom{2n-4}{n-2} + 1$ points in general position contains n points on convex position.