Definition

The (ordinary) generating function of a sequence $a_0, a_1, a_2, \ldots$ is

$$A(x) = \sum_{n \geq 0} a_n x^n.$$

- The generating function of $1, 1, 1, \ldots$ is

$$\sum_{n \geq 0} x^n = \frac{1}{1-x}.$$

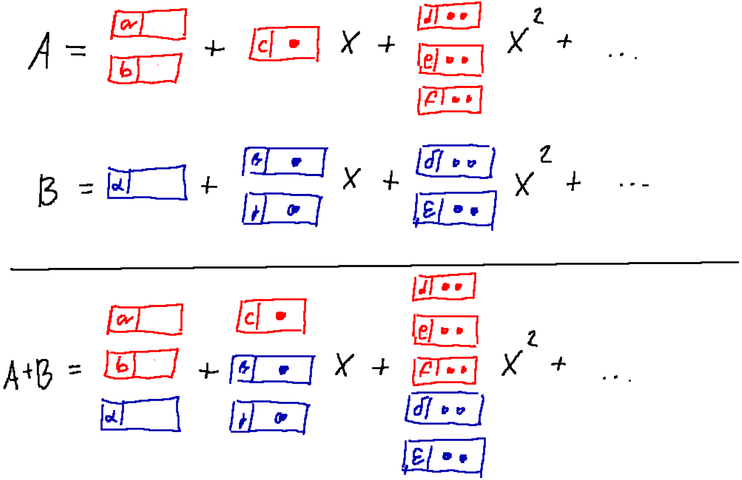
- The generating function of the sequence with elements

$$a_n = \text{number of strings of letters } \mathtt{a} \text{ and } \mathtt{b} \text{ of length } n$$

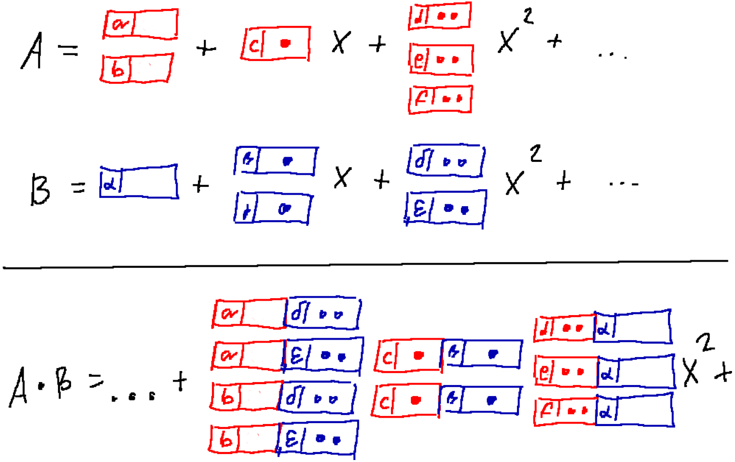
  is

$$\sum_{n \geq 0} 2^n x^n = \frac{1}{1-2x}.$$

$$\left(\sum_{n\geq 0} a_n x^n\right) + \left(\sum_{n\geq 0} b_n x^n\right) = \sum_{n\geq 0} (a_n + b_n) x^n$$



$$\left(\sum_{n\geq 0} a_n x^n\right) \cdot \left(\sum_{n\geq 0} b_n x^n\right) = \sum_{n\geq 0} \left(\sum_{i=0}^{n} a_i b_{n-i}\right) x^n$$



$a_n =$ number of strings of letters `a`, `b`, `c` of length n and not containing substring `aa`.

- $a_0 = 1$, $a_1 = 3$, $a_2 = 8$, $a_3 = ?$, $\ldots$
- $A(x) = \sum_{n \geq 0} a_n x^n$.

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A good string is

- empty or `a`: Generating function $1 + x$. Or,
- `b` or `c` followed by a good string: Generating function $2x \cdot A$. Or,
- `ab` or `ac` followed by a good string: Generating function $2x^2 \cdot A$.

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$$A = 1 + x + (2x + 2x^2)A$$
$$A = \frac{1 + x}{1 - 2x - 2x^2}$$

$2x^2 + 2x - 1 = 0$:

$$x_1 = \frac{\sqrt{3} - 1}{2}, \qquad x_2 = -\frac{\sqrt{3} + 1}{2}$$

$$1/x_1 = \sqrt{3} + 1, \qquad 1/x_2 = 1 - \sqrt{3}$$

$$\begin{aligned}
A &= \frac{1 + x}{1 - 2x - 2x^2} = -\frac{1 + x}{2(x_1 - x)(x_2 - x)} \\
&= \frac{(2\sqrt{3} + 3)/6}{1 - x/x_1} - \frac{(2\sqrt{3} - 3)/6}{1 - x/x_2} \\
&= \frac{2\sqrt{3} + 3}{6} \sum_{n \geq 0} (1/x_1)^n x^n - \frac{2\sqrt{3} - 3}{6} (1/x_2)^n x^n \\
a_n &= \frac{2\sqrt{3} + 3}{6} (\sqrt{3} + 1)^n - \frac{2\sqrt{3} - 3}{6} (1 - \sqrt{3})^n
\end{aligned}$$

$t_n =$ number of rooted trees with n vertices where each vertex is either a leaf or has 2 or 3 children; the order of chilren matters.

- $t_0 = 0$, $t_1 = 1$, $t_2 = 0$, $t_3 = 1$, $t_4 = 1$, $t_5 = 2$, $t_6 = ?$, ...
- $T(x) = \sum_{n \geq 0} t_n x^n$.

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A good tree is

- a single-vertex tree: Generating function x. Or,
- a root plus 2 good trees: Generating function $x \cdot T \cdot T$. Or,
- a root plus 3 good trees: Generating function xT^3.

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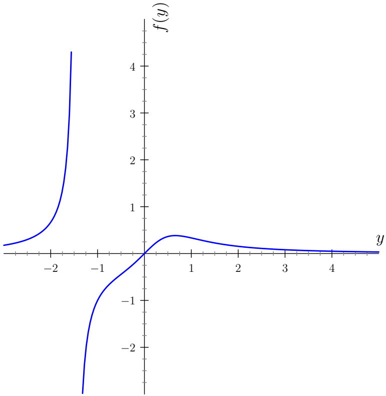
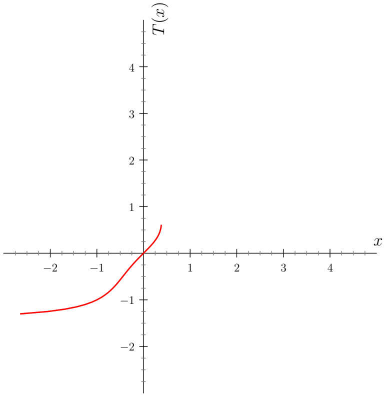
$$T(x) = x(1 + T^2(x) + T^3(x))$$
$$x = \frac{T(x)}{1 + T(x)^2 + T(x)^3}$$

For $f(y) = \frac{y}{1+y^2+y^3}$: we have $f(T(x)) = x$, and $T = f^{-1}$.



f(y)
y



T(x)
x

$$[x^n] \sum_n a_n x^n = a_n$$

Theorem (Lagrange inversion formula)

Suppose $F(y) = \sum_{n \geq 0} f_n y^n$ *with* $f_0 \neq 0$ *and* $A(x) = xF(A(x))$. *Then*

$$[x^n]A(x) = \frac{1}{n}[y^{n-1}]F^n(y)$$

$$\begin{aligned}
T(x) &= x(1 + T^2(x) + T^3(x)) \\
F(y) &= 1 + y^2 + y^3 \\
t_n &= \frac{1}{n}[y^{n-1}](1 + y^2 + y^3)^n \\
&= \frac{1}{n} \sum_{a,b \in \mathbb{Z}_0^+ : 2a+3b=n-1} \binom{n}{n-a-b, a, b}
\end{aligned}$$

Definition

The exponential generating function of a sequence $a_0, a_1, a_2, \ldots$ is

$$A(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}.$$

- The exponential generating function of $1, 1, 1, \ldots$ is

$$\sum_{n \geq 0} \frac{x^n}{n!} = e^x.$$

- The generating function of the sequence with elements

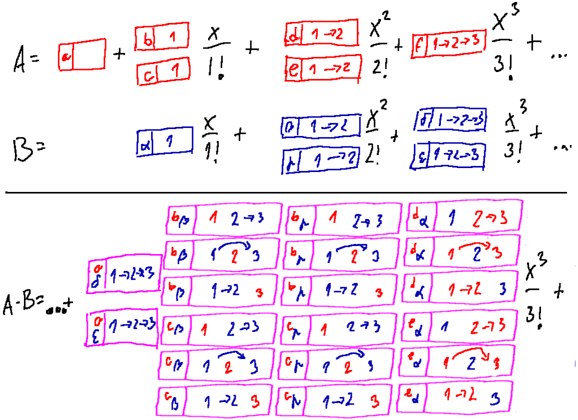
$$a_n = \text{number of strings of letters } \mathtt{a} \text{ and } \mathtt{b} \text{ of length } n$$

  is

$$\sum_{n \geq 0} 2^n \frac{x^n}{n!} = e^{2x}.$$

$$\Big(\sum_{n\geq 0} a_n \frac{x^n}{n!}\Big) \cdot \Big(\sum_{n\geq 0} b_n \frac{x^n}{n!}\Big) = \sum_{n\geq 0} \Big(\sum_{i=0}^{n} \frac{a_i b_{n-i}}{i!(n-i)!}\Big) x^n$$

$$= \sum_{n\geq 0} \Big(\sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}\Big) \frac{x^n}{n!}$$



$p_n =$ number of ordered partitions of $\{1, \ldots, n\}$, i.e., number of tuples $(A_1, \ldots, A_k)$ of non-empty disjoint sets s.t. $A_1 \cup \ldots \cup A_k = \{1, \ldots, n\}$

- $p_0 = 1$, $p_1 = 1$, $p_2 = 3$, $p_3 =?$
- $P(x) = \sum_{n \geq 0} p_n \frac{x^n}{n!}$

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- $p_0 = 1$, $p_1 = 1$, $p_2 = 3$, $p_3 = ?$
- $P(x) = \sum_{n \geq 0} p_n \frac{x^n}{n!}$

$$\begin{aligned} P &= 1 + (e^x - 1) + (e^x - 1) \cdot (e^x - 1) + (e^x - 1)^3 + \ldots \\ &= \frac{1}{2 - e^x} \end{aligned}$$

Definition

The radius of convergence of $A = \sum_{n=0}^{\infty} a_n x^n$ is

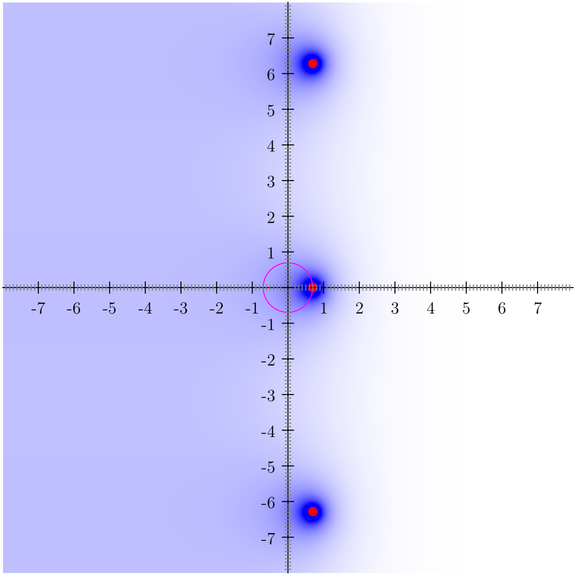
$$R = \sup\{c > 0 : |a_n| \leq (1/c)^n \text{ for all but finitely many } n\}.$$

Lemma

Let $A = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$.

- *A diverges for every $x \in \mathbb{C}$ such that $|x| > R$,*
- *A converges for every $x \in \mathbb{C}$ such that $|x| < R$,*
- *there exists $x \in \mathbb{C}$ such that $|x| = R$ and A diverges at x, and*
- *if $a_n \geq 0$ for all n, then A diverges at R.*

Graph of $|P(x)| = \left|\frac{1}{2-e^x}\right|$ for $x \in \mathbb{C}$:



Observation

For every $\varepsilon > 0$,

$$|a_n| < (1/R + \varepsilon)^n$$

holds for all but finitely many values of n, and thus

$$|a_n| = O\big((1/R + \varepsilon)^n\big).$$

$P(x) = \frac{1}{2-e^x}$:

- Radius of convergence $\log 2$.
- $1/\log 2 < 1.443$

$$\frac{p_n}{n!} = O(1.443^n)$$

Let $q(x) = \frac{\log 2 - x}{2 - e^x}$, so that

$$P(x) = \frac{1}{\log 2 - x} \cdot q(x).$$

Define

$$q(\log 2) = \lim_{x \to \log 2} q(x) = \tfrac{1}{2}$$

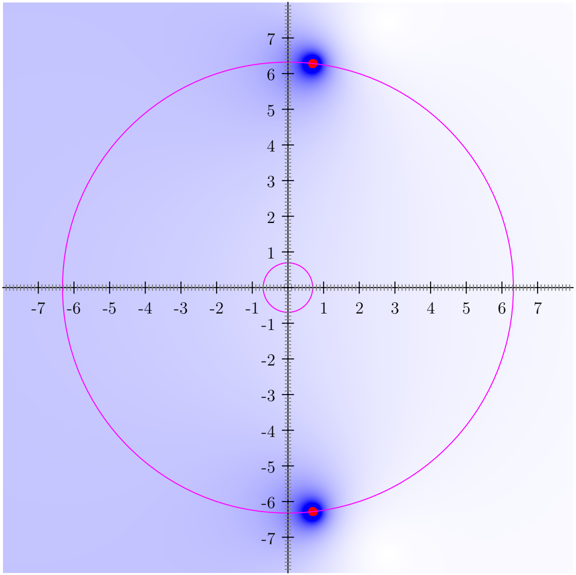
$$h(x) = P(x) - \frac{1/2}{\log 2 - x} = \frac{q(x) - 1/2}{\log 2 - x}$$

We have

$$\begin{aligned} \lim_{x \to \log 2} h(x) &= \lim_{x \to \log 2} \frac{q(x) - 1/2}{\log 2 - x} = \lim_{x \to \log 2} -q'(x) \\ &= \lim_{x \to \log 2} \frac{2 - e^x - (\log 2 - x)e^x}{(2 - e^x)^2} \\ &= \lim_{x \to \log 2} \frac{x - \log 2}{2e^x - 4} = \lim_{x \to \log 2} \frac{1}{2e^x} = \frac{1}{4}. \end{aligned}$$

Define $h(\log 2) = 1/4$.

Graph of $|h(x)|$ for $x \in \mathbb{C}$:



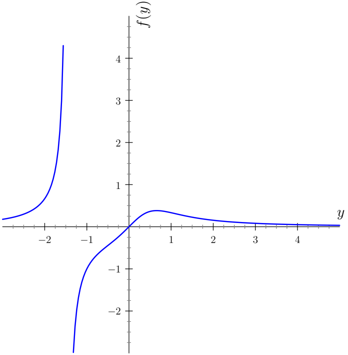
For $h(x)$:

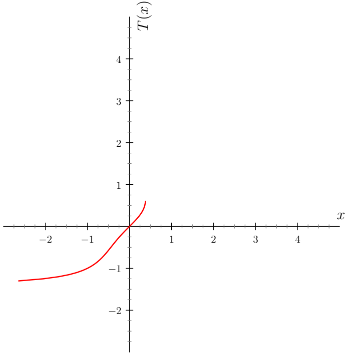
- radius of convergence $|2\pi i + \log 2| = \sqrt{4\pi^2 + \log^2 2} > 1/0.16$,
- $[x^n]h(x) = O(0.16^n)$

$$\begin{aligned}
\frac{p_n}{n!} &= [x^n]P(x) = [x^n]\frac{1/2}{\log 2 - x} + [x^n]h(x) \\
&= [x^n]\frac{1}{2\log 2} \cdot \frac{1}{1 - x/\log 2} + [x^n]h(x) \\
&= \frac{1}{2\log^{n+1} 2} + O(0.16^n) = 0.5 \cdot (1.443\ldots)^{n+1} + O(0.16^n).
\end{aligned}$$

$t_n =$ number of rooted trees with n vertices where each vertex is either a leaf or has 2 or 3 children; the order of chilren matters.

- $T(x) = \sum_{n \geq 0} t_n x^n$.
- $T(x)$ is the inverse to $f(y) = \frac{y}{1+y^2+y^3}$.





- $R = f(y_0)$, where $f'(y_0) = 0$
- $1/R < 2.62 \Rightarrow t_n = O(2.62^n)$