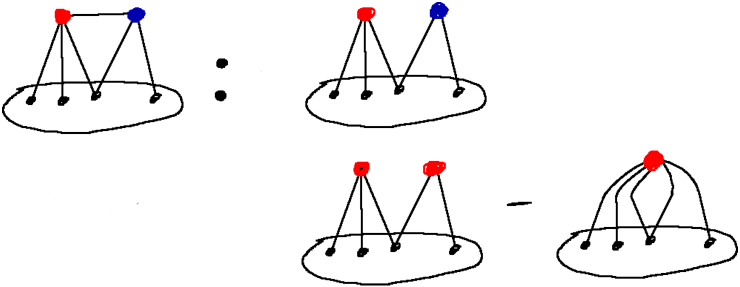
$\pi_G(k)$ = number of k-colorings of G



$$\pi_G(k) = \begin{cases} k^{|V(G)|} & \text{if } E(G) = \emptyset \\ 0 & \text{if } e \text{ is a loop} \\ \pi_{G-e}(k) - \pi_{G/e}(k) & \text{otherwise} \end{cases}$$

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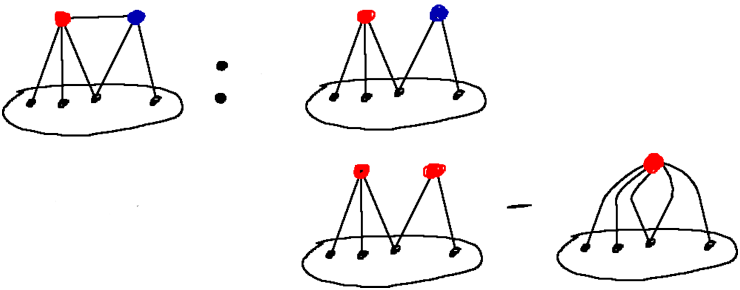


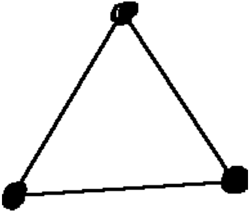
$$\pi_G(k) = \begin{cases} k^{|V(G)|} & \text{if } E(G) = \emptyset \\ (k-1)\pi_{G/e}(k) & \text{if } e \text{ is a bridge} \\ 0 & \text{if } e \text{ is a loop} \\ \pi_{G-e}(k) - \pi_{G/e}(k) & \text{otherwise} \end{cases}$$

Observation

$\pi_G(k)$ is a polynomial in variable k of degree at most $|V(G)|$.

π_G is the chromatic polynomial of G.

Q: Compute the chromatic polynomial of



For a connected graph G and $p \in [0,1]$:

- $G_p =$ the random graph obtained by deleting each edge independently with probability p.
- $R_G(p) =$ probability that G_p is connected

$$R_G(p) = \begin{cases} 1 & \text{if } E(G) = \emptyset \\ (1-p)R_{G/e}(p) & \text{if } e \text{ is a bridge} \\ R_{G-e}(p) & \text{if } e \text{ is a loop} \\ pR_{G-e}(p) + (1-p)R_{G/e}(p) & \text{otherwise} \end{cases}$$

R_G is the reliability polynomial of G.

For a connected graph G,

$$s_G = \text{the number of spanning trees of } G$$

$$s_G = \begin{cases} 1 & \text{if } E(G) = \emptyset \\ s_{G/e} & \text{if } e \text{ is a bridge} \\ s_{G-e} & \text{if } e \text{ is a loop} \\ s_{G-e} + s_{G/e} & \text{otherwise} \end{cases}$$

For a graph G and $A \subseteq E(G)$:

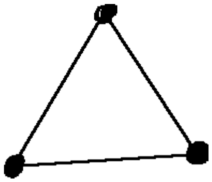
- $\kappa(G)$ = the number of components of G.
- $\kappa_G(A)$ = the number of components of $(V(G), A)$.
- $r_G(A) = \kappa_G(A) - \kappa(G) \geq 0$
- $c_G(A) = \kappa_G(A) - (V(G) - |A|) \geq 0$

Definition

Tutte polynomial of a graph G is

$$T_G(x, y) = \sum_{A \subseteq E(G)} (x-1)^{r_G(A)} (y-1)^{c_G(A)}.$$

Q: Compute Tutte polynomial of



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Tutte polynomial of a graph G is

$$T_G(x, y) = \sum_{A \subseteq E(G)} (x-1)^{r_G(A)}(y-1)^{c_G(A)}.$$

$$T_G(2, 2) = 2^{|E(G)|}$$

For a graph G and $A \subseteq E(G)$:

- $\kappa(G) =$ the number of components of G.
- $\kappa_G(A) =$ the number of components of $(V(G), A)$.
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Definition

Tutte polynomial of a graph G is

$$T_G(x, y) = \sum_{A \subseteq E(G)} (x-1)^{r_G(A)} (y-1)^{c_G(A)}.$$

For G connected:

- $r_G(A) = 0$ iff $(V(G), A)$ is connected.
- $c_G(A) = 0$ iff $(V(G), A)$ is a forest.

For a graph G and $A \subseteq E(G)$:

- $\kappa(G) =$ the number of components of G.
- $\kappa_G(A) =$ the number of components of $(V(G), A)$.
- $r_G(A) = \kappa_G(A) - \kappa(G) \geq 0$
- $c_G(A) = \kappa_G(A) - (V(G) - |A|) \geq 0$

Definition

Tutte polynomial of a graph G is

$$T_G(x, y) = \sum_{A \subseteq E(G)} (x-1)^{r_G(A)} (y-1)^{c_G(A)}.$$

$T_G(1,2) =$ number of connected spanning subgraphs

$T_G(2,1) =$ number of spanning forests

$T_G(1,1) =$ number of spanning trees

Lemma

$$T_G(x, y) = \begin{cases} 1 & \textit{if } E(G) = \emptyset \\ x \cdot T_{G/e}(x, y) & \textit{if e is a bridge} \\ y \cdot T_{G-e}(x, y) & \textit{if e is a loop} \\ T_{G-e}(x, y) + T_{G/e}(x, y) & \textit{otherwise.} \end{cases}$$

$E(G) = \emptyset$:

$$T_G(x, y) = (x - 1)^{r_G(\emptyset)} (y - 1)^{c_G(\emptyset)} = 1$$

e is a bridge:

$$r_G(A) - 1 = r_G(A \cup \{e\}) = r_{G/e}(A)$$
$$c_G(A \cup \{e\}) = c_G(A) = c_{G/e}(A)$$

$$\begin{aligned}
&T_G(x, y) \\
&= \sum_{A \subseteq E(G)} (x-1)^{r_G(A)}(y-1)^{c_G(A)} \\
&= \sum_{A \subseteq E(G) \setminus \{e\}} \left((x-1)^{r_G(A)}(y-1)^{c_G(A)} + (x-1)^{r_G(A \cup \{e\})}(y-1)^{c_G(A \cup \{e\})}\right) \\
&= \sum_{A \subseteq E(G/e)} \left((x-1)^{r_{G/e}(A)+1}(y-1)^{c_{G/e}(A)} + (x-1)^{r_{G/e}(A)}(y-1)^{c_{G/e}(A)}\right) \\
&= x \sum_{A \subseteq E(G/e)} (x-1)^{r_{G/e}(A)}(y-1)^{c_{G/e}(A)} = x \cdot T_{G/e}(x, y).
\end{aligned}$$

e is a loop:

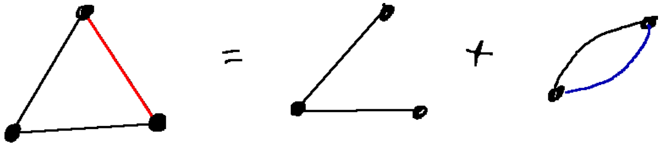
$$r_G(A) = r_G(A \cup \{e\}) = r_{G-e}(A)$$
$$c_G(A \cup \{e\}) - 1 = c_G(A) = c_{G-e}(A)$$

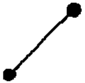
$$\begin{aligned}
&T_G(x, y) \\
&= \sum_{A \subseteq E(G)} (x-1)^{r_G(A)}(y-1)^{c_G(A)} \\
&= \sum_{A \subseteq E(G) \setminus \{e\}} \Big((x-1)^{r_G(A)}(y-1)^{c_G(A)} + (x-1)^{r_G(A \cup \{e\})}(y-1)^{c_G(A \cup \{e\})} \Big) \\
&= \sum_{A \subseteq E(G-e)} \Big((x-1)^{r_{G-e}(A)}(y-1)^{c_{G-e}(A)} + (x-1)^{r_{G-e}(A)}(y-1)^{c_{G-e}(A)+1} \Big) \\
&= y \sum_{A \subseteq E(G-e)} (x-1)^{r_{G-e}(A)}(y-1)^{c_{G-e}(A)} = y \cdot T_{G-e}(x, y).
\end{aligned}$$

e is neither a bridge nor a loop:

$$r_G(A) = r_{G-e}(A) \qquad r_G(A \cup \{e\}) = r_{G/e}(A)$$
$$c_G(A) = c_{G-e}(A) \qquad c_G(A \cup \{e\}) = c_{G/e}(A)$$

$$\begin{aligned}
&T_G(x, y) \\
&= \sum_{A \subseteq E(G)} (x-1)^{r_G(A)}(y-1)^{c_G(A)} \\
&= \sum_{A \subseteq E(G) \setminus \{e\}} \Big((x-1)^{r_G(A)}(y-1)^{c_G(A)} + (x-1)^{r_G(A \cup \{e\})}(y-1)^{c_G(A \cup \{e\})}\Big) \\
&= \sum_{A \subseteq E(G) \setminus \{e\}} \Big((x-1)^{r_{G-e}(A)}(y-1)^{c_{G-e}(A)} + (x-1)^{r_{G/e}(A)}(y-1)^{c_{G/e}(A)}\Big) \\
&= T_{G-e}(x, y) + T_{G/e}(x, y).
\end{aligned}$$



$= x^2 +$  $+$ 

$$= x^2 + x + y$$

$$T_{C_k}(x,y) = T_{P_k}(x,y) + T_{C_{k-1}}(x,y)$$

$$= x^{k-1} + T_{C_{k-1}}(x,y)$$

$$= x^{k-1} + x^{k-2} + \dots + x + y$$

$$U_G(n,b,l,d,c) = n^{\kappa(G)} d^{|E(G)|+\kappa(G)-|V(G)|} c^{|V(G)|-\kappa(G)} T_G(b/c, l/d).$$

Lemma

$$U_G = \begin{cases} n^{|V(G)|} & \text{if } E(G) = \emptyset \\ b \cdot U_{G/e} & \text{if } e \text{ is a bridge} \\ l \cdot U_{G-e} & \text{if } e \text{ is a loop} \\ d \cdot U_{G-e} + c \cdot U_{G/e} & \text{otherwise.} \end{cases}$$

Corollary

- $\pi_G = U_G(k, k-1, 0, 1, -1) = k^{\kappa(G)}(-1)^{|V(G)|-\kappa(G)} T_G(1-k, 0)$
- $R_G = U_G(1, 1-p, 1, p, 1-p) = p^{c_G(E(G))}(1-p)^{|V(G)|-1} T_G(1, p^{-1})$
- $s_G = U_G(1,1,1,1,1) = T_G(1,1)$

e is neither a bridge nor a loop:

$$\kappa(G) = \kappa(G - e) = \kappa(G/e)$$

$$\begin{aligned}
U_G &= n^{\kappa(G)} d^{|E(G)|+\kappa(G)-|V(G)|} c^{|V(G)|-\kappa(G)} T_G(b/c, l/d) \\
&= n^{\kappa(G)} d^{|E(G)|+\kappa(G)-|V(G)|} c^{|V(G)|-\kappa(G)} (T_{G-e} + T_{G/e}) \\
&= n^{\kappa(G-e)} d^{|E(G-e)|+1+\kappa(G-e)-|V(G-e)|} c^{|V(G-e)|-\kappa(G-e)} T_{G-e} \\
&\quad + n^{\kappa(G/e)} d^{|E(G/e)|+\kappa(G-e)-|V(G/e)|} c^{|V(G/e)|+1-\kappa(G/e)} T_{G/e} \\
&= d \cdot U_{G-e}(n, b, l, d, c) + c \cdot U_{G/e}(n, b, l, d, c).
\end{aligned}$$

Lemma

If G_1 and G_2 intersect in at most one vertex, then $T_{G_1 \cup G_2} = T_{G_1} T_{G_2}$.

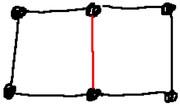
Proof.

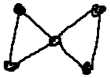
By induction on $|E(G_2)|$. E.g., if G_2 contains a bridge e:

- e is a bridge of $G_1 \cup G_2$.

$$
\begin{aligned}
T_{G_1 \cup G_2} = x \cdot T_{(G_1 \cup G_2)/e} &= x \cdot T_{G_1 \cup (G_2/e)} \\
&= T_{G_1} \cdot x \cdot T_{G_2/e} = T_{G_1} T_{G_2}.
\end{aligned}
$$

□



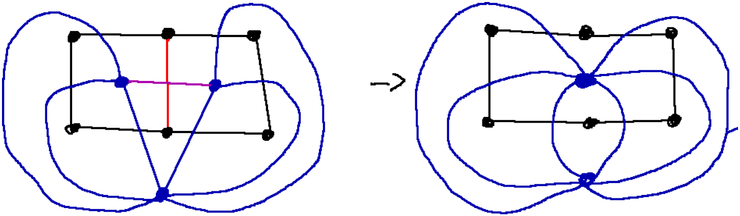
$= C_6 +$  $=$

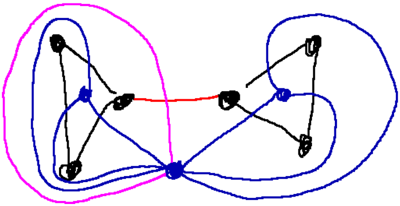
$= C_6 + (C_3)^2 =$

$= x^5 + x^4 + \dots + x + y + (x^2 + x + y)^2$

For a connected plane graph G and its dual $G^\star$:

- e not a bridge: $(G - e)^\star = G^\star / e$
- e not a loop: $(G/e)^\star = G^\star - e$
- e is a loop in G iff e is a bridge in $G^\star$
- e is a bridge in G iff e is a loop in $G^\star$





Lemma

For a connected plane graph G,

$$T_G(x, y) = T_{G^\star}(y, x).$$

Q: Show that a connected plane graph and its dual have the same number of spanning trees.

Computing $T_G(x, y)$

- in P at $(1, 1)$, $(-1, -1)$, $(0, -1)$, $(-1, 0)$, $(x, 1/(x-1)+1)$.
- for planar G in P at $(x, 2/(x-1)+1)$
- #P-hard otherwise