Hamiltonicity

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A *Hamiltonian cycle* in a graph G is a cycle in G containing all vertices of G. If G has such a cycle, we say G is *Hamiltonian.* In contrast, recall that *Euler tour* in a graph G is a tour passing through each edge exactly once. An Euler tour in a graph is thus related to a Hamiltonian cycle in the linegraph of G (they are not the same, though; e.g., $K_{1,3}$ does not have Euler tour, but $L(K_{1,3}) = K_3$ has a Hamiltonian cycle). However, while it is easy to decide whether a graph contains an Euler tour, deciding whether it has a Hamiltonian cycle is NP-hard. Hence, the best we can do is to obtain some necessary and/or sufficient conditions for the existence of a Hamiltonian cycle (or to give algorithms for special graph classes).

Let us start with a useful sufficient condition. Suppose that G has a Hamiltonian cycle C. Then for any $S \subseteq V(G)$, $C - S$ has at most $|S|$ components, and thus $G-S$ also has at most $|S|$ components. This motivates the following definition: We say that a graph G is *t-tough* if for every $S \subseteq V(G)$, $G - S$ has at most $\max(1, |S|/t)$ connected components.

Observation 1. *Every Hamiltonian graph is* 1*-tough.*

For every $r < 9/4$, there are r-tough graphs that are not Hamiltonian. It has been conjectured that there exists some constant c such that all c-tough graphs are Hamiltonian, but this is still an open question. However, it is known that 10-tough chordal graphs are Hamiltonian.

1 Degrees

What is the minimum degree that ensures that a graph is Hamiltonian? For an odd integer n, the graph $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ is not Hamiltonian, since deleting the $\lfloor n/2\rfloor$ vertices of the smaller part results in $\lceil n/2\rceil$ components.

Theorem 2 (Dirac)**.** *If $|V(G)| \ge 3$ and $\delta(G) \ge |V(G)|/2$, then G is Hamiltonian.*

In fact, the following weaker condition is also sufficient.

Theorem 3 (Ore)**.** *If* $|V(G)| \geq 3$ *and every pair* x, y *of non-adjacent vertices of* G *satisfies* $\deg x + \deg y \geq |V(G)|$, *then* G *is Hamiltonian.*

Both of these results can be proven using *Chvtal's closure operation*.

Lemma 4. *Let* x *and* y *be non-adjacent vertices of a graph* G. *If* $\deg x + \deg y \geq |V(G)|$, *then* G *is Hamiltonian iff* $G + xy$ *is Hamiltonian.*

Proof. Suppose $G + xy$ contains a Hamiltonian cycle C. If $xy \notin E(C)$, then C is also a Hamiltonian cycle in G. Otherwise, consider the path $C - xy = v_1v_2 \ldots v_n$, where $v_1 = x$ and $v_n = y$. Let $S \subseteq \{1, \ldots, n-1\}$ be the set of indices i such that $v_1v_{i+1} \in E(G)$. Note that $|\{1, \ldots, n-1\} \setminus S| = |V(G)| - 1 - \deg x < \deg y$, and thus there exists $i \in S$ such that $v_iv_n \in E(G)$. Then $v_1v_2 \ldots v_iv_nv_{n-1} \ldots v_{i+1}v_1$ is a Hamiltonian cycle in G. □

The graph G' obtained from G by repeating the operation from the statement of Lemma 4 as long as possible is called the *Chvtal's closure* of G (it can be easily seen that G' is uniquely determined, even though you can add the edges to G in any order). By Lemma 4, G is Hamiltonian if and only if G' is Hamiltonian. If G satisfies the assumptions of Theorem 3 (or Theorem 2), then G' is the complete graph, which clearly is Hamiltonian, and thus both theorems follow.

Let us now give "the most general" theorem about degrees and Hamiltonicity. Let $\vec{a} = (a_1, a_2, \ldots, a_n)$ be a sequence of integers such that $0 \leq a_1 \leq \ldots \leq a_n \leq n-1$. We say that a graph G with n vertices *dominates* $\vec{a}$ if there exists an ordering $v_1, \ldots, v_n$ of vertices of G such that $\deg v_i \geq a_i$ for $i = 1, \ldots, n$. We say that $\vec{a}$ is *Hamiltonian* if every graph that dominates $\vec{a}$ is Hamiltonian.

Theorem 5. *For* $n \geq 3$, *a sequence* $\vec{a} = (a_1, a_2, \ldots, a_n)$ *is Hamiltonian if and only if every* i *such that* $a_i \leq i < n/2$ *satisfies* $a_{n-i} \geq n - i$.

Proof. Suppose for a contradiction $\vec{a}$ satisfies the assumptions and G is a non-Hamiltonian graph dominating $\vec{a}$ with the largest number of edges. Let x_1 and x_n be non-adjacent vertices of G such that $\deg x_1 + \deg x_n$ is maximum and $\deg x_1 \leq \deg x_n$. Note that $G + x_1x_n$ dominates $\vec{a}$, and by the maximality of G, the graph $G + x_1x_n$ is Hamiltonian. Since G is not Hamiltonian, the Hamiltonian cycle in $G + x_1x_n$ necessarily contains the edge x_1x_n, and thus G contains a path $P = x_1x_2 \ldots x_n$. Let $A = \{x_i : x_1x_{i+1} \in E(G)\}$. If x_n had a neighbor x_i in A, then $P - x_ix_{i+1} + \{x_1x_{i+1}, x_nx_i\}$ would be a Hamiltonian cycle in G.

Hence, this is not the case, and thus x_n is non-adjacent to all vertices in A. In particular, $\deg x_1 + \deg x_n \le \deg x_1 + (n-1-|A|) = n-1$, and $h = \deg x_1 < n/2$. By the maximality of $\deg x_1 + \deg x_n$, we have $\deg v \le h$ for all $v \in A$, and thus G contains at least h vertices of degree at most h. Consequently, $a_1, \ldots, a_h \le h$, and by the assumptions, $a_{n-h} \ge n-h$. Hence, G contains at least $h+1$ vertices of degree at least $n-h$, and at least one of them, say y, is non-adjacent to the vertex x_1 of degree h. But then $\deg x_1 + \deg y \ge n > \deg x_1 + \deg x_n$, contradicting the choice of x_1 and x_n.

Conversely, suppose that $\vec{a}$ does not satisfy the assumptions, and thus there exists $i < n/2$ such that $a_i \le i$ and $a_{n-i} \le n-i-1$. Then consider the graph G consisting of a clique on vertices $v_{i+1}, \ldots, v_n$ and a complete bipartite graph with parts $\{v_1, \ldots, v_i\}$ and $\{v_{n-i+1}, \ldots, v_n\}$. The vertices $v_1, \ldots, v_i$ have degree $i \ge a_i \ge \ldots \ge a_1$, the vertices $v_{i+1}, \ldots, v_{n-i}$ have degree $n-i-1 \ge a_{n-i} \ge \ldots \ge a_{i+1}$, and the vertices $v_{n-i+1}, \ldots, v_n$ have degree $n-1 \ge a_n \ge \ldots \ge a_{n-i+1}$, and thus G dominates $\vec{a}$. Moreover, G is not Hamiltonian, as deleting i vertices $v_{n-i+1}, \ldots, v_n$ results in a graph with $i+1$ components. $\square$

2 Planar graphs

One might hope that all plane triangulations are Hamiltonian. However, this is not the case: Consider any plane triangulation H with $n > 4$ vertices, and let G be the plane triangulation obtained from H by adding a vertex of degree three into each face of H. Then $G - V(H)$ has $|F(H)| = 2n-4 > n$ components, and thus G is not 1-tough. Observation 1 implies G is not Hamiltonian. On the other hand, the fact that this graph G contains a lot of separating triangles (giving cuts of size three) turns out to be the only problem.

Theorem 6 (Tutte)**.** *Every* 4*-connected planar graph is Hamiltonian.*

The proof is somewhat involved and we will not give it here. It is based on the following lemma for 2-connected planar graphs. Given a set $S \subseteq V(G)$ and a component C of $G - S$, the *attachments* of C are the vertices of S that have a neighbor in C.

Lemma 7. *Let G be a* 2*-connected plane graph, let u and v be distinct vertices incident with the outer face of G, and let e be an edge incident with the outer face of G. Then there exists a path P in G from u to v such that $e \in E(P)$ and every component of $G - V(P)$ has at most three attachments.*

To prove Theorem 6, suppose G is a 4-connected plane graph and let $e' = vw$ be an edge incident with the outer face of G. Then $G - e'$ is 3-connected and the outer face of $G - e'$ has length at least four. Let uv and $e = wz$ be edges incident with the outer face of $G - e'$ such that $u \neq z$. By Lemma 7 applied in $G - e'$, there exists a path P in G from u to v containing the edge wz such that each component of $G - V(P)$ has at most three attachments. However, since G is 4-connected and $|V(P)| \geq 4$, no such component can exist, and thus $V(P) = V(G)$. Consequently, $P + uv$ is a Hamiltonian cycle in G.

Let us also note the following relationship between Hamiltonicity and edge colorings.

Observation 8. *If G is 3-regular and Hamiltonian, then $\chi'(G) = 3$.*

Proof. Since G is 3-regular, $|V(G)|$ is even. Color the edges of a Hamiltonian cycle of G alternatingly by two colors, then use the third color on the perfect matching formed by the edges not belonging to the Hamiltonian cycle. □

The Four Color Theorem is known to be equivalent to the fact that every planar 3-regular 3-connected graph is 3-edge-colorable. Motivated by this, it was conjectured that planar 3-regular 3-connected graphs are Hamiltonian; however, this is false (Tutte found a counterexample with 46 vertices).

3 Number of Hamiltonian cycles

In general, we cannot say much about the number of Hamiltoninan cycles in a graph. However, there is a nice argument that shows that certain graphs cannot have one or two Hamiltoninan cycles (i.e., either they have none or at least three).

Lemma 9. *Suppose every vertex of a graph G has odd degree. Then for every edge $e \in E(G)$, the number of Hamiltonian cycles of G containing the edge e is even.*

Proof. Let $e = xy$. A connected subgraph H of G with $V(H) = V(G)$ is a *lollipop* if $e \in E(H)$ and either

- H is a cycle, or
- $\deg_H x = 1$, some vertex $v \in V(H)$ satisfies $\deg_H v = 3$, and all other vertices have degree two in H.

In the latter case, H consists of a path starting with the edge xy and ending in a cycle. If H is a cycle, the *tail* of H is the edge of H incident with x and different from e. Otherwise, the *tails* of H are the edges of the cycle of H incident with the vertex of degree three.

Let L be the graph whose vertices are the lollipops of G, with distinct lollipops H_1 and H_2 adjacent iff there exists a tail e_1 of H_1 and a tail e_2 of H_2 such that $H_1 - e_1 = H_2 - e_2$. Note that if $H_1 = wzxy\ldots$ is a Hamiltonian cycle, then necessarily $e_1 = zx$, and each neighbor of H_1 are obtained by adding to $H_1 - e_1$ an edge incident with z and different from zx and zw; hence, $\deg_L H_1 = \deg_G z - 2$ is odd.

Suppose now that H_1 is not a cycle, let z be the vertex of H_1 of degree three, and let w_1z and w_2z be the tails of H_1. Each neighbor of H_1 in L is obtained by, for some $i \in \{1, 2\}$, deleting the edge w_iz and adding an edge incident with w_i and not belonging to H_1. Hence, $\deg_L H_1 = (\deg_G w_1 - 2) + (\deg_G w_2 - 2)$ is even.

The graph L contains an even number of vertices of odd degree, and these vertices are exactly the Hamiltonian cycles of G containing the edge e. $\square$

Corollary 10. *Suppose every vertex of a graph G has odd degree. If G is Hamiltonian, then G has at least three distinct Hamiltonian cycles.*

Proof. Let C_1 be a Hamiltonian cycle of G. By Lemma 9 applied to any edge of C_1, there exist a Hamiltonian cycle C_2 of G distinct from C_1. Since $C_1 \neq C_2$, there exists an edge $e \in E(C_1) \setminus E(C_2)$. By Lemma 9 applied to e_1, there exists a Hamiltonian cycle $C_3 \neq C_1$ containing e. Since $e \notin E(C_2)$, we also have $C_3 \neq C_2$. $\square$

This Corollary is the best possible; for example, K_4 has exactly three Hamiltonian cycles.