Perfect graphs

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For every graph G, we have the simple inequalities

$$\omega(G) \leq \chi(G) \leq \Delta + 1.$$

In Brooks Theorem, we have discussed the upper bound (and noted that up to some exceptional graphs, it can be improved by 1). Let us now have a look at the relationship between the chromatic number and the clique number. Note that $\chi(G)$ can in general be much larger than $\omega(G)$.

Exercise 1. *Show there exist triangle-free graphs (clique number 2) of arbitrarily large chromatic number.*

A graph G is *perfect* if for every induced subgraph H of G, we have $\chi(H) = \omega(H)$. Let us remark that we would not get anything interesting if we just assumed that $\chi(G) = \omega(G)$; indeed, this equality is for example satisfied by the disjoint union of any graph F with the clique on $|V(F)|$ vertices.

Examples of classes of perfect graphs (all the classes listed here are closed on induced subgraphs, e.g., an induced subgraph of a bipartite graph is also bipartite; hence, to show the perfectness, we only need to argue that $\chi(G) = \omega(G)$ for every graph from the class):

- Bipartite graphs.
- Chordal graphs, as we have seen in the last lecture.
- Complements of bipartite graphs: They have independence number two, and to use as few colors as possible, as many color classes as possible must have size two. Hence, for a bipartite graph G, we have $\chi(\overline{G}) = |V(G)| - \beta(G)$, where $\beta(G)$ is the size of the largest matching in G. On the other hand, we also know that $\omega(\overline{G}) = \alpha(G) = |V(G)| - \beta(G)$, see Corollary 5 in the lecture notes from the first lesson.

- Linegraphs of bipartite graphs: For any bipartite graph G, we have $\chi(L(G)) = \chi'(G) = \Delta(G)$. Moreover, for a bipartite graph G, any clique in $L(G)$ consists of edges incident with the same vertex, and thus $\omega(L(G)) = \Delta(G)$.

Exercise 2. *A* comparability graph *is a graph G which has a transitive orientation $\vec{G}$ (i.e., an orientation such that for every $(u,v),(v,w) \in E(\vec{G})$, we also have $(u,w) \in E(\vec{G})$). Equivalently, there exists a partial ordering $\prec$ on $V(G)$ such that $uv \in E(G)$ iff u and v are comparable in $\prec$. Observe that cliques and independent sets in G correspond to the chains and antichains in $\prec$, and show that comparability graphs are perfect.*

Exercise 3. *Show that a graph does not contain an induced 4-vertex path if and only if it can be obtained (starting from single-vertex graph) by disjoint unions and complementations (this class of graphs is called* cographs*). Show that these graphs are perfect.*

1 Algorithms

There exists a polynomial-time algorithm to determine the chromatic number (and the clique number) of a perfect graph. For a real number $r \ge 2$, a *vector r-coloring* of a graph G is a function φ that, for some Euclidean space S of finite dimension, assigns a vector in S of norm 1, and such that for every $uv \in E(G)$, $\langle \varphi(u), \varphi(v) \rangle \le -\frac{1}{r-1}$. The *vector chromatic number* $\chi_v(G)$ of G is the infimum of the real numbers $r \ge 2$ such that G has a vector r-coloring.

Lemma 4. *For every graph G with at least one edge, we have $\chi_v(G) \le \chi(G)$.*

Proof. Let $c = \chi(G)$. Let $v_1, \ldots, v_c$ be unit vectors in $\mathbb{R}^{c-1}$ forming the vertices of a regular simplex. Let $s = \langle v_1, v_2 \rangle$; by symmetry, we have $\langle v_i, v_j \rangle = s$ for any $i \ne j$. We have $\sum_{i=1}^{c} v_i = 0$, and thus

$$0 = \Big| \sum_{i=1}^{c} v_i \Big|^2 = \sum_{i=1}^{c} |v_i|^2 + \sum_{i \ne j} \langle v_i, v_j \rangle = c + c(c-1)s.$$

It follows that $s = -1/(c-1)$, and thus assigning to each vertex of color i the vertex v_i, we obtain a vector c-coloring of G. □

Lemma 5. *For every graph G with at least one edge, we have $\chi_v(G) \ge \omega(G)$.*

Proof. Note that a vector r-coloring of a graph G is also a vector r-coloring of each subgraph H of G, and thus $\chi_v(H) \le \chi_v(G)$. Therefore, it suffices to show that $\chi_v(K_c) \geq c$ for every $c \geq 2$. Let φ be a vertex r-coloring of K_c, with $V(K_c) = \{1, \ldots, c\}$; hence, $\langle \varphi(i), \varphi(j) \rangle \leq -1/(r-1)$ for every $i \neq j$, and $|\varphi(i)| = 1$ for every i. Then

$$0 \leq \Big| \sum_{i=1}^{c} \varphi(i) \Big|^2 = \sum_{i=1}^{c} |\varphi(i)|^2 + \sum_{i \neq j} \langle \varphi(i), \varphi(j) \rangle \leq c - c(c-1)/(r-1),$$

implying $r \geq c$. □

Therefore, for a perfect graph G, we have $\chi(G) = \chi_v(G) = \omega(G)$. Moreover, the vector chromatic number is equal to $1 - 1/t$, where t is computed by the following semidefinite program:

$$\begin{aligned} &\text{minimize } t \text{ such that} \\ &\quad \langle v_z, v_z \rangle = 1 && \text{for every } z \in V(G) \\ &\quad \langle v_y, v_z \rangle \leq t && \text{for every } yz \in E(G) \end{aligned}$$

The solution to a semidefinite program can be approximated arbitrarily well in a polynomial time (and we only need to determine the solution to a limited precision, since in our case, $\chi_v(G) \leq V(G)$ is an integer, and thus we only need to distinguish between a finite set of possible values).

2 Characterization

There are two natural families of graphs that are not perfect: odd cycles of length at least 5 and their complements. Indeed, for any $k \geq 2$, $\chi(C_{2k+1}) = 3$ while $\omega(C_{2k+1}) = 2$, and $\chi(\overline{C_{2k+1}}) = 2k + 1 - \beta(C_{2k+1}) = k + 1$ while $\omega(\overline{C_{2k+1}}) = \alpha(C_{2k+1}) = k$. It was conjectured by Berge in the 60's that there are the only minimal non-perfect graphs, and thus a graph is perfect if and only if it contains none of them as an induced subgraph. This was proven to be true in 2002. An induced subgraph H of a graph G is a *hole* in G if H is a (≤ 4)-cycle, and an *antihole* if H is the complement of a (≤ 4)-cycle.

Theorem 6 (Chudnovsky, Robertson, Seymour, Thomas)**.** *A graph is perfect if and only if it does not contain any odd hole or antihole.*

This result is also known as the Strong Perfect Graph Theorem, and its proof is quite involved. However, note a simple consequence.

Corollary 7. *A graph is perfect if and only if its complement is perfect.*

This corollary is known as the Weak Perfect Graph Theorem, and it has been proven much earlier. It is a consequence of the following characterization of perfect graphs.

Lemma 8. *A graph G is perfect if and only if every induced subgraph H of G satisfies $\alpha(H)\omega(H) \geq |V(H)|$.*

Since $\alpha(\overline{H}) = \omega(H)$ and $\omega(\overline{H}) = \alpha(H)$, the condition on the right hand side is satisfied by G if and only if it is satisfied by $\overline{G}$, and thus Corollary 7 holds. Note that one of the implications from Lemma 8 is easy: Note that every graph F satisfies $\chi(F) \geq |V(F)|/\alpha(F)$, since we need at least $|V(F)|/\alpha(F)$ independent sets to cover all vertices of F. Consequently, if G is perfect, then every induced subgraph H of G satisfies $\omega(H) = \chi(H) \geq |V(H)|/\alpha(H)$, as required. To prove the opposite implication, we need the following lemma, proved using a linear-algebraic argument.

Lemma 9. *Let k and n be positive integers. Let $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ be subsets of $\{1, \ldots, n\}$ such that $|A_i \cap B_j| = 1$ for all $i, j \in \{1, \ldots, k\}$ such that $i \neq j$. If $A_i \cap B_i = \emptyset$ for every $i \in \{1, \ldots, k\}$, then $k \leq n$.*

Proof. The claim is trivial if $k \leq 1$, and thus assume that $k \geq 2$. Let S be the $k \times n$ matrix such that $S_{i,j} = 1$ if $j \in A_i$ and $S_{i,j} = 0$ otherwise. Let T be the $n \times k$ matrix such that $T_{i,j} = 1$ if $i \in B_j$ and $T_{i,j} = 0$ otherwise. Then $M = ST$ is the $k \times k$ matrix such that $M_{ij} = |A_i \cap B_j|$ for each $i, j \in \{1, \ldots, k\}$. Hence, M has 0's on the diagonal and 1's everywhere else. Observe that M has rank k. Indeed, $j = \frac{1}{k-1}\sum_{i=1}^{k} M_{i,\star}$ is the vector with all entries equal to 1, and subtracting j from every row of M (which does not change the rank of M) results in the matrix $-I$ of rank k. On the other hand, $\mathrm{rk}(M) = \mathrm{rk}(ST) \leq \mathrm{rk}(S) \leq n$, since S has n columns. □

Proof of Lemma 8. We prove the claim by induction on $|V(G)|$. The claim is trivial if $E(G) = \emptyset$, and thus we can assume this is not the case, and in particular $|V(G)| \geq 2$. Let $\alpha = \alpha(G)$ and $\omega = \omega(G)$. By the induction hypothesis, we can assume that every proper induced subgraph of G is perfect, and thus it suffices to show that $\chi(G) = \omega$.

We claim that G contains an independent set A that intersects every largest clique in G. If that is the case, then $\omega(G - A) = \omega - 1$, and since $G - A$ is perfect, $G - A$ has a proper coloring by $\omega - 1$ colors. Giving the vertices of A a new color, we obtain a coloring of G by ω colors, finishing the proof.

Hence, suppose for a contradiction that for every independent set A in G, there exists a clique $K(A)$ in G of size ω and disjoint from A. Let $A_0 = \{v_1, v_2, \ldots, v_\alpha\}$ be an arbitrary largest independent set in G. For

$i \in \{1, \ldots, \alpha\}$, we have $\chi(G - v_i) \leq \omega$ by the induction hypothesis, and thus there exist independent sets $A_{i,1}$, …, $A_{i,\omega}$ covering $G - v_i$. Note that $K(A_0)$ is a clique in $G - v_i$ of size ω, and thus it must intersect each of these independent sets in exactly one vertex. For $j \in \{1, \ldots, \omega\}$, $K(A_{i,j}) - v_i$ is a clique in $G - v_i$ of size (at least) $\omega - 1$ disjoint from $A_{i,j}$, and thus $K(A_{i,j})$ intersects each seta $A_{i,j'}$ for $j' \neq i$ in exactly one vertex.

Moreover, consider any $i' \neq i$ and $j' \in \{1, \ldots, \omega\}$. Since $K(A_{i',j'})$ is a clique of size ω disjoint from $A_{i',j'}$ and intersects each of the independent sets $A_{i',j''}$ for $j'' \neq j'$ in (at most) one vertex, we must have $v_i' \in K(A_{i',j'})$. The clique $K(A_{i',j'})$ can intersect A_0 only in one vertex, and thus $v_i \notin K(A_{i',j'})$ and $K(A_{i',j'})$ is a clique of size ω in $G - v_i$. Hence, $K(A_{i',j'})$ must intersect each of the independent sets $A_{i,1}$, …, $A_{i,\omega}$ in exactly one vertex.

Therefore, we can apply Lemma 9 to the sets A_0, $A_{1,1}$, …, $A_{\alpha,\omega}$ and $K(A_0)$, $K(A_{1,1})$, …, $K(A_{\alpha,\omega})$. This gives $\alpha\omega + 1 \leq |V(G)|$, contradicting the assumption that $\alpha(G)\omega(G) \geq |V(G)|$. □

Exercise 10. *Let $\prec$ be an arbitrary partial ordering of a finite set and let a be the size of the largest antichain of $\prec$. Apply Lemma 8 to the comparability graph of $\prec$ and conclude that the elements of $\prec$ can be covered by a chains.*