Chordal graphs

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A graph G is *chordal* if it does not contain any induced cycle of length at least four; i.e., any (≥ 4)-cycle in G has a *chord* (an edge between non-consecutive vertices of the cycle).

Exercise 1. *Let S be a finite set of closed intervals of real numbers, and let G be the graph with vertex set S such that the intervals $I_1, I_2 \in S$ are adjacent iff they intersect (we say that G is the* intersection graph *of S; the intersection graphs of intervals are called* interval graphs*). Show that G is chordal.*

We now give a number of alternate characterizations of chordal graphs. For a connected graph G, a set $Z \subseteq V(G)$ is a *minimal cut* if $G - Z$ is not connected but for every $X \subsetneq Z$, $G - X$ is connected.

Lemma 2. *Let G be a connected graph. If G is chordal, then every minimal cut in G induces a clique.*

Proof. Let Z be a minimal cut in G. If $G[Z]$ were not a clique, then there would exist non-adjacent vertices $u, v \in Z$. Let A and B be components of $G - Z$. We claim that u and v have neighbors in both A and B. Indeed, if say u had no neighbor in A, then $Z \setminus \{u\}$ would still separate A from B, contradicting the minimality of Z. Hence, there exists a path P_A in G from u to v with all internal vertices in A, and a path P_B in G from u to v with all internal vertices in B. By choosing shortest such paths, we can assume P_A and P_B are induced paths in G. Moreover, since $uv \notin E(G)$, both of these paths have length at least two. Consequently, $P_A \cup P_B$ is an induced (≥ 4)-cycle in G, which is a contradiction. $\square$

A vertex $v \in V(G)$ is *simplicial* if the neighborhood of v induces a clique in G.

Lemma 3. *A graph G is chordal if and only if every induced subgraph of G has a simplicial vertex.*

Proof. Note that a (≥ 4)-cycle does not have a simplicial vertex; hence, if every induced subgraph of G has a simplicial vertex, then G is chordal.

Conversely, we will show by induction on the number of vertices that if G is chordal, then either G is a clique, or G contains at least two non-adjacent simplicial vertices. Without loss of generality, we can assume that G is connected, as otherwise the claim follows by applying the induction hypothesis to the components of G (if all components are cliques, then every vertex is simplicial, and we can choose two vertices from different components). The claim is also clear if G is a clique, and thus there exists a set $Z \subseteq V(G)$ such that $G - Z$ is not connected (e.g., this is true for the complement of two non-adjacent vertices); let us choose Z to be the a smallest such set. Then Z is a minimal cut, and thus by Lemma 2, $G[Z]$ is a clique. Let $G = G_1 \cup G_2$, where G_1 and G_2 are proper induced subgraphs of G intersecting in Z. By the induction hypothesis, G_1 contains a simplicial vertex v_1 not belonging to Z (if G_1 is a clique, we can choose $v_1 \in V(G_1) \setminus Z$ arbitrarily; otherwise, G_1 contains two non-adjacent simplicial vertices, and at most one of them belongs to the clique Z). Similarly, G_2 contains a simplicial vertex v_2 not belonging to Z. Then v_1 and v_2 are non-adjacent simplicial vertices of G.

Hence, if G is chordal, then it contains a simplicial vertex. Moreover, every induced subgraph of G is also chordal, and thus the same argument shows it has a simplicial vertex. □

An *elimination ordering* in a graph G is an ordering $v_1, \ldots, v_n$ of the vertices of G with the property that for each m, the set $\{v_i : i < m, v_i v_m \in E(G)\}$ induces a clique.

Corollary 4. *A graph G is chordal if and only if G has an elimination ordering.*

Proof. Suppose G has an elimination ordering, and consider any (≥ 4)-cycle K in G. Let v be the last vertex of K in the elimination ordering. Then the neihbors of v in K belong to the clique induced by the neighbors of v preceding it in the ordering, and thus they are adjacent. Consequently, K has a chord. Therefore, G is chordal.

Suppose now G is a chordal graph. Let $G_n = G$, and for $i = n, \ldots, 1$, let v_i be a simplicial vertex in G_i (which exists by Lemma 3) and let $G_{i-1} = G_i - v_i$. Then $v_1, \ldots, v_n$ is an elimination ordering in G. □

This gives us a way to test in polynomial time whether a graph is formal: Repeatedly find a simplicial vertex (this can be easily done in a polynomial time) and construct an ordering as described in the proof of Corollary 4. If we finish with an elimination ordering, then the graph is chordal. If the construction fails (the currently considered subgraph does not have a simplicial vertex), then the graph is not chordal by Lemma 3.

Moreover, using the elimination ordering, we can also compute three graph invariants of a chordal graph that are hard to compute for a general graph: Its chromatic number, clique number, and independence number.

Lemma 5. *Let $v_1, \ldots, v_n$ be an elimination ordering of a chordal graph G, and for $i \in \{1, \ldots, n\}$, let $G_i = G - \{v_{i+1}, \ldots, v_n\}$. Then $\chi(G) = \omega(G) = \max\{\deg_{G_i} v_i : i \in \{1, \ldots, n\}\} + 1$.*

Proof. Let $D = \max\{\deg_{G_i} v_i : i \in \{1, \ldots, n\}\} + 1$. Let us color vertices $v_1, \ldots, v_n$ in order by colors in $\{1, 2, \ldots\}$, giving v_i the smallest color not appearing on the neighbors of v_i that precede it in the ordering. Clearly, the color of v_i ends up being smaller or equal to $\deg_{G_i}(v_i) + 1 \leq D$, and thus we obtain a proper D-coloring of G. Hence, $\chi(G) \leq D$.

Moreover, by the definition of the elimination ordering, the neighborhood of v_i in G_i induces a clique, and thus G contains a clique of size $\deg_{G_i}(v_i)+1$. The largest of these cliques has size D, implying $\omega(G) \geq D$.

Finally, note that we need at least $\omega(G)$ colors to properly color G. Hence,

$$D \leq \omega(D) \leq \chi(G) \leq D.$$

□

The graphs with this property (the chromatic number of each induced subgraph is equal to its clique number) are called *perfect*; we will speak more about them in the next lecture.

Lemma 6. *If a graph G is chordal, then there exist cliques $K_1, \ldots, K_{\alpha(G)}$ such that $V(G) = K_1 \cup \ldots \cup K_{\alpha(G)}$.*

Proof. We prove the claim by induction on the number of vertices of G. If G is a clique, then $\alpha(G) = 1$ and we can take $K_1 = V(G)$. Hence, suppose that G is not a clique. Since G is chordal, Lemma 3 implies it contains a simplicial vertex v. Let K be the clique induced by v and the neighbors of v, and let $G' = G - K$. By the induction hypothesis, there exist cliques K_1, $\ldots$, $K_{\alpha(G')}$ covering the vertex set of G'. Then the cliques $K_1, \ldots, K_{\alpha(G')}$, K cover $V(G)$, and thus it suffices to show that $\alpha(G) = \alpha(G') + 1$.

Note that it is not possible to cover vertices of G by fewer than $\alpha(G)$ cliques, since every clique contains at most one vertex of the largest independent set of G; hence, $\alpha(G) \leq \alpha(G') + 1$. On the other hand, let A' be an independent set in G' of size $\alpha(G')$; then $A \cup \{v\}$ is an independent set in G of size $\alpha(G') + 1$, and thus $\alpha(G) \geq \alpha(G') + 1$. Therefore, we have $\alpha(G) = \alpha(G') + 1$, as required. □

Exercise 7. *What is the time complexity of the algorithms to determine the chromatic, clique, and independence number of chordal graphs following from Lemmas 5 and 6?*