Graphs on surfaces, the generalized Euler's formula and the classification theorem

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In this lecture, we allow the graphs to have loops and parallel edges. In addition to the plane (or the sphere), we can draw the graphs on the surface of the torus or on more complicated surfaces.

Definition 1. *A* surface *is a compact connected* 2*-dimensional manifold without boundary.*

Intuitive explanation:

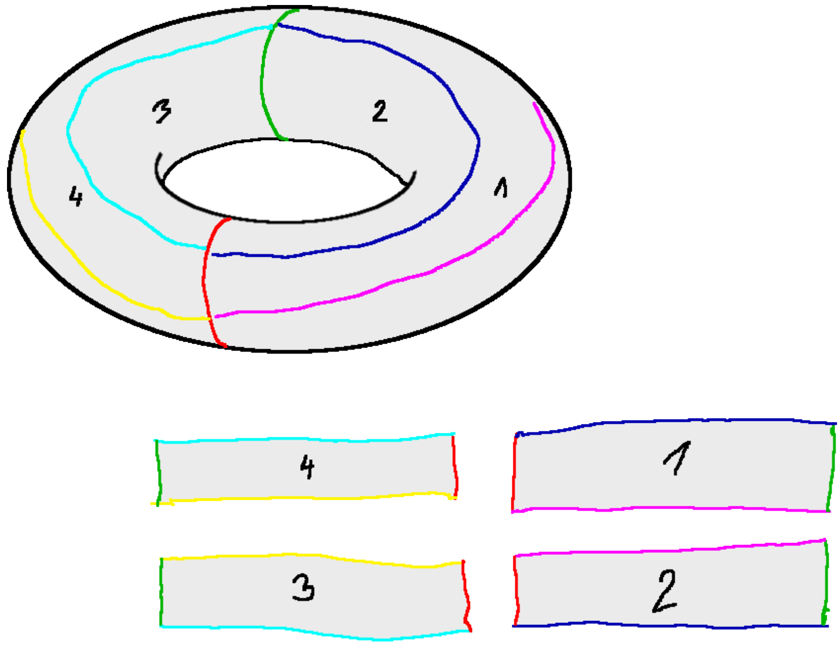
- 2-dimensional manifold without boundary: Each point has a neighborhood homeomorphic to an open disk, i.e., "locally, the surface looks at every point the same as the plane."
- compact: "The surface can be covered by a finite number of such neighborhoods."
- connected: "The surface has just one piece."

Examples:

- The sphere and the torus are surfaces.
- The plane is not a surface, since it is not compact.
- The closed disk is not a surface, since it has a boundary.

From the combinatorial perspective, it does not make sense to distinguish between some of the surfaces; the same graphs can be drawn on the torus and on a deformed torus (e.g., a coffee mug with a handle). For us, two surfaces will be equivalent if they only differ by a homeomorphism; a function $f : \Sigma_1 \to \Sigma_2$ between two surfaces is a *homeomorphism* if f is a bijection, continuous, and the inverse f^{-1} is continuous as well. In particular, this implies that f maps simple continuous curves to simple continuous curves, and thus it maps a drawing of a graph in Σ_1 to a drawing of the same graph in Σ_2. Hence, exactly the same graphs can be drawn in two homeomorphic surfaces.

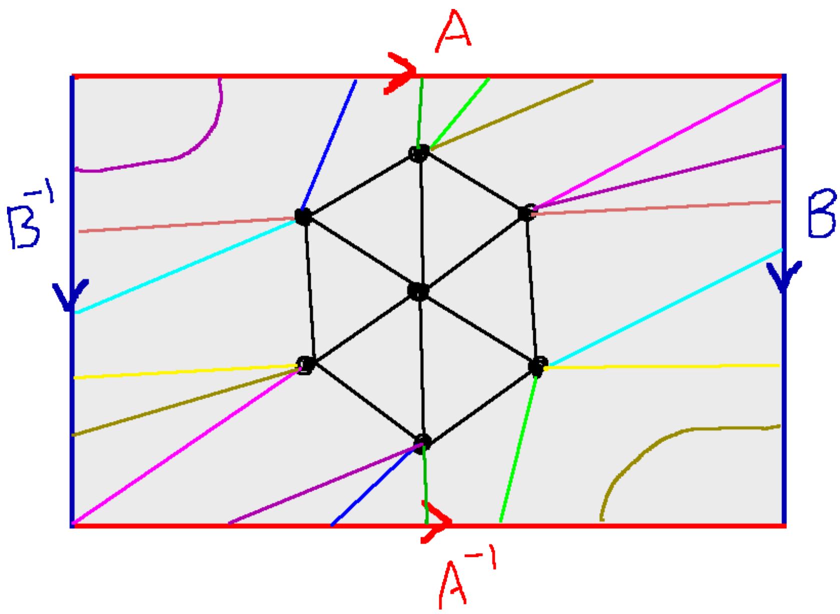
Since a surface is compact, it can be cut into finitely many parts homeomorphic to disks, and conversely glued back together from these parts. It is natural to view these parts as polygons, with each edge of the polygon being a piece of the boundary that is in its entirety glued together with another edge of (the same or different) polygon. For example, the following figure illustrates the way the torus can be cut into such polygons.



Let us keep the edges of the polygons used in the gluing drawn in the surface. This gives a drawing of a graph N in the surface, where all faces of are homeomorphic to the open disks.

Definition 2. *A* net *of a surface is a graph drawn in the surface so that every face is homeomorphic to an open disk.*

How many polygons are there needed to represent a surface? As long as we have more than one polygon, since the resulting surface is connected, two of them can be glued together to a single polygon. Repeating this process, we end up with just a single polygon (with even number of edges) for which we have prescribed a way how the edges should be glued together (equivalently, every surface has a net with just one face; we call such a net *simple*). This gives a *polygonal representation* of the surface, illustrated here for the torus (the edges of matching color are glued together in the direction of the arrows). Note the figure also illustrates how K_7 can be drawn in the torus.



Note that we can succintly describe the polygonal representation by assigning the same letters to the pairs of edges that are glued together, with distinct letters assigned to different pairs, and give the cyclic string of letters in the clockwise order along the polyhedron, marking the edges we traverse against the direction of the arrows by the superscript -1. E.g., the representation of the torus given above is $ABA^{-1}B^{-1}$.

Exercise 3. *Consider a polygonal representation given by a cyclic string $w = AA^{-1}w'$ containing the substring AA^{-1}. Note that gluing the edges marked A first transforms the polygon into one described by w', and thus w and w' are representations of the same surface. Similarly, show that*

- *w_1ABw_2AB can be simplified to w_1Cw_2C.*
- *w_1Aw_2A and $w_1A^{-1}w_2A^{-1}$ represent the same surface.*
- *$w_1Aw_2A^{-1}$ and $w_1A^{-1}w_2A$ represent the same surface.*

For a net G of a surface, let us define

$$g(G) = |E(G_1)| - |V(G_1)| - |F(G_1)| + 2.$$

Lemma 4. *If G_1 and G_2 are nets of the same surface, then $g(G_1) = g(G_2)$.*

Proof. For simplicity, let us assume that the drawings of G_1 and G_2 intersect in a finite number of points (getting rid of this assumption is actually rather nontrivial). Note that subdividing an edge of a net G does not change $g(G)$ (it increases both the number of edges and the number of vertices by 1). Hence, by putting vertices at all the intersections if needed, we can assume that G_1 and G_2 only intersect in vertices. Let $G = G_1 \cup G_2$. We will show that $g(G_1) = g(G) = g(G_2)$. By symmetry, it suffices to prove the first equality.

Note that deleting an edge of G_1 that separates two distinct faces does not change $g(G_1)$ (it decreases both the number of edges and the number of faces by 1, and preserves the property that G_1 is a net). Hence, we can without loss of generality assume that G_1 has only one face. Cutting the surface with G drawn in it along the edges of G_1 gives us a graph G' drawn in the disk. Taking this disk to be a part of the plane, we obtain a drawing of G' in the plane with $|F(G')| = |F(G)|+1 = |F(G)|-|F(G_1)|+2$, $|E(G')| = |E(G)| + |E(G_1)|$ and

$$|V(G')| = |V(G)|-|V(G_1)|+\sum_{v\in V(G_1)} \deg_{G_1}(v) = |V(G)|-|V(G_1)|+2|E(G_1)|.$$

By Euler's formula, we have

$$\begin{aligned}
0 &= |E(G')| - |F(G')| - |V(G')| + 2\\
&= |E(G)| + |E(G_1)| - (|F(G)| - |F(G_1)| + 2) - (|V(G)| - |V(G_1)| + 2|E(G_1)|) + 2\\
&= (|E(G)| - |V(G)| - |F(G)|) - (|E(G_1)| - |V(G_1)| - |F(G_1)|),
\end{aligned}$$

and thus $g(G) = g(G_1)$. □

Definition 5. *The* Euler genus *of a surface with a net G is $g(G)$.*

By Lemma 4, the genus does not depend on which net of the surface we choose. For example, the net for torus depicted in the first picture has 4 vertices, 4 faces, and 8 edges, implying that the torus has Euler genus $8 - 4 - 4 + 2 = 2$.

Corollary 6 (Generalized Euler's formula)**.** *If G is a graph drawn in a surface of Euler genus g, then*

$$|E(G)| \le |V(G)| + |F(G)| + g - 2.$$

The equality holds iff every face of G is homeomorphic to an open disk.

Proof. As long as it is possible to add an edge to G in such a way that the number of faces stays the same, do so, obtaining a supergraph G' of G. Then G' is a net of the surface, and thus

$$|E(G)| \leq |E(G')| = |V(G')| + |F(G')| + g - 2 = |V(G)| + |F(G)| + g - 2$$

by the definition of the genus of a surface. □

Exercise 7. *Observe that every surface other than the sphere has a simple net of minimum degree at least two (deleting a vertex of degree one preserves the fact that the graph is a net), and that every simple net of the sphere is a tree. Consequently, show that the sphere has Euler genus* 0*, and any other surface has positive genus.*

Corollary 8. *If* G *is a simple graph drawn in a surface of Euler genus* g *and* $|E(G)| \geq 2$*, then*

$$|E(G)| \leq 3|V(G)| + 3g - 6.$$

Proof. For a face f of G, let us define $\ell(F)$ as the number of edges incident with f, where an edge incident with f on both sides is counted twice. Since G is simple and $|E(G)| \geq 2$, observe that $\ell(F) \geq 3$, and thus

$$2|E(G)| = \sum_{f \in F(G)} \ell(F) \geq 3|F(G)|.$$

Using the Generalized Euler's formula, we have

$$\begin{aligned} |E(G)| &\leq |V(G)| + |F(G)| + g - 2 \leq |V(G)| + \tfrac{2}{3}|E(G)| + g - 2 \\ |E(G)| &\leq 3|V(G)| + 3g - 6, \end{aligned}$$

as required. □

The following figure shows all non-isomorphic polygonal representations of length at most four (excluding those we can simplify using Exercise 3), and the genus of the corresponding surfaces (the colors indicate which vertices of the polygon get glued together to a single vertex).

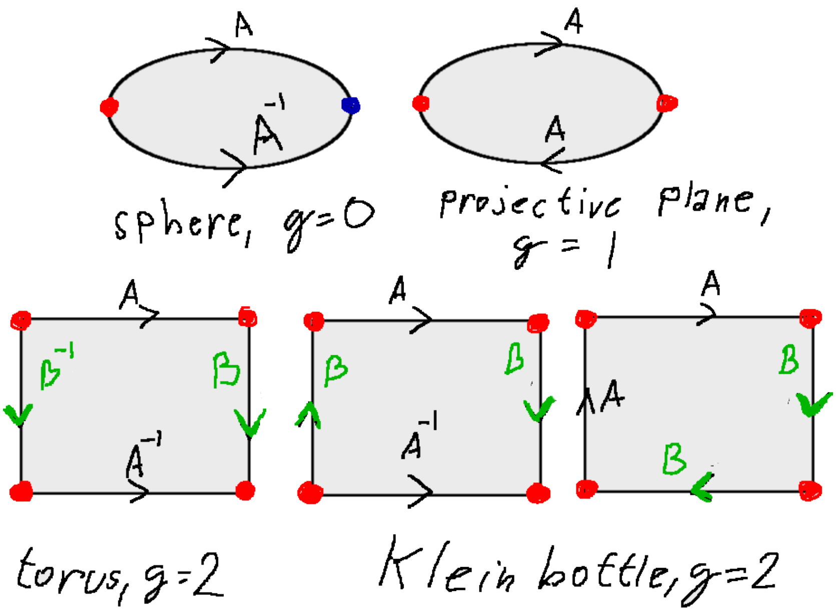
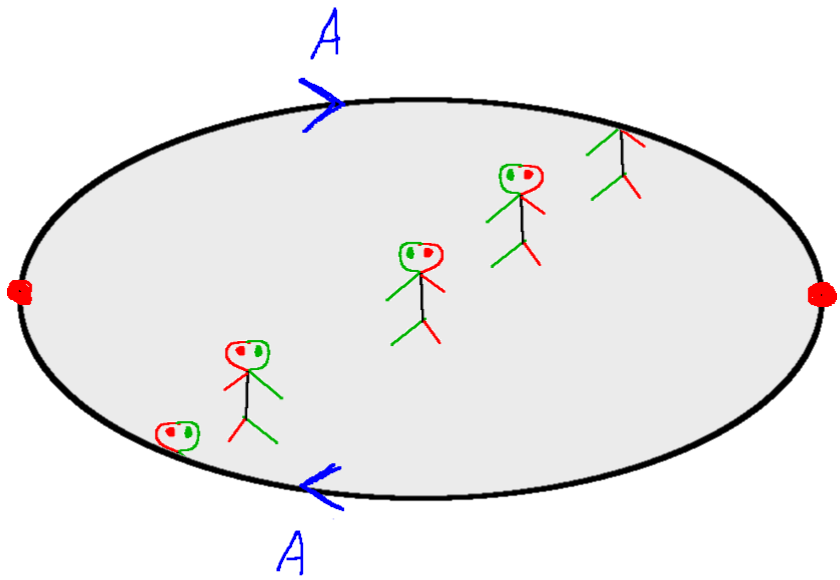


Figure 1: Polygonal representations of basic surfaces

Exercise 9. *In which of these surfaces can we draw K_6 or K_7?*

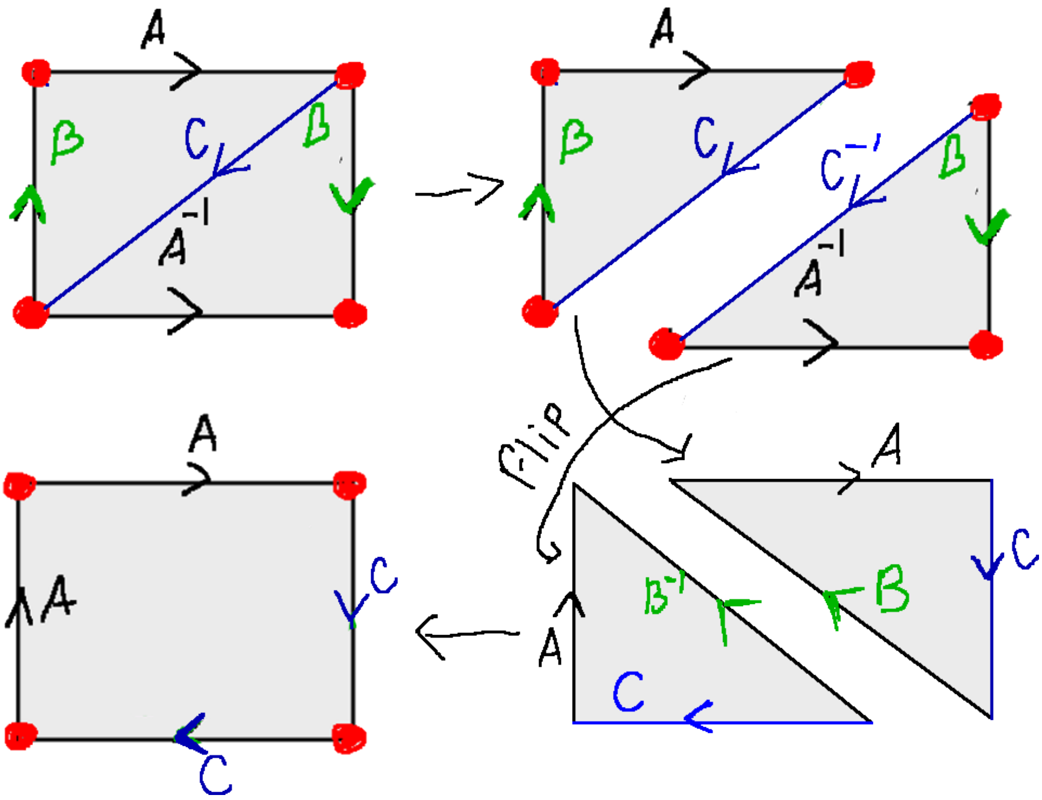
The first three surfaces in the picture above are distinct, as they have pairwise different genus. What about the last three surfaces? For these, we need another invariant. Let us consider the second surface, the *projective plane* (note the gluing prescribed for this surface cannot be accomplished in the 3-dimensional space, but it can be done in the 4-dimensional one). Suppose we walk "around" the projective plane, crossing the edge of its net (the boundary of the polygon) exactly once:



Notice this switched our left and right side. So, in the projective plane, we cannot consistently define "left" and "right". Note also this is not the case on the torus—we can define the orientation (left/right) in the face of the simple net, then observe that the orientation is the same on both sides of each edge of the net.

Definition 10. *A surface is* non-orientable *if in its polygonal representation, two edges that are glued together are directed in the same direction along the boundary of the cycle. The surface is* orientable *otherwise.*

Note this definition does not depend on the choice of the representation of the surface; as we have argued, orientability is equivalent to the possibility to define a consistent orientation at each point of the surface. In Figure 1, the surface in the bottom left is orientable, while the surfaces represented by the remaining two bottom pictures are non-orientable. We claim that they actually represent the same surface (the Klein bottle). Indeed, we can transform one into the other by cutting and gluing as shown here:



Similarly, it turns out that you can bring every representation into one of two "canonical" forms, as shown in the following theorem (which we will not prove).

Theorem 11 (Classification theorem)**.** *Every polygonal representation can be transformed by a series of cuttings and gluings together with simplifications described in Exercise 3 to one of the following two forms:*

- $(ABA^{-1}B^{-1})(CDC^{-1}D^{-1})\dots$
- $(AA)(BB)(CC)\dots$

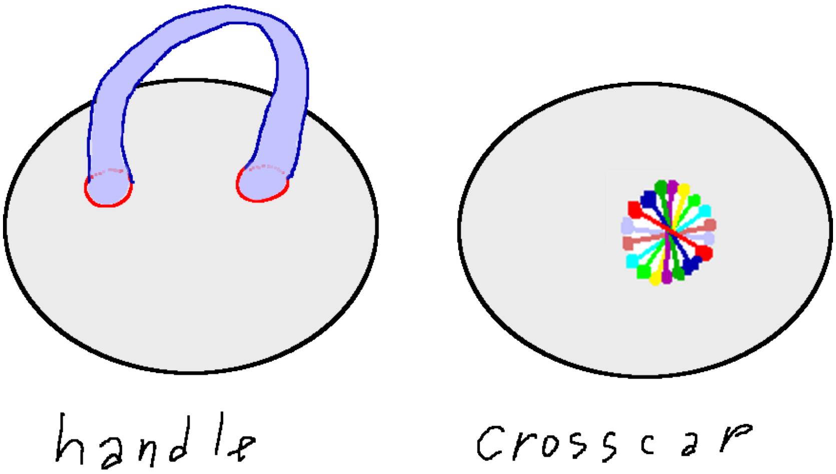
Note that the surfaces of the first form (with the block repeated k times) are orientable and have Euler genus $2k$. The surfaces of the second form are non-orientable and have Euler genus k.

Corollary 12. *Two surfaces are homeomorphic if and only if they have the same Euler genus and orientability.*

Moreover, the Euler genus of an orientable surface is always even; hence, it is sometimes convenient to speak about the *genus* of the surface, which is equal to half the Euler genus for orientable surface and to the Euler genus for the non-orientable ones.

Finally, let us mention another approach to forming surfaces. You can start with the sphere, then perform the following operations (repeatedly, in any order), see the illustration below:

- *Adding a handle*: Drill two holes anywhere in the surface, then attach a “handle” (a cylinder) on them.
- *Adding a crosscap*: Drill a hole anywhere in the surface, then glue together the opposite points of the boundary of this hole (this can only be done in ($\geq$4)-dimensional space).



Exercise 13. *Show that adding a handle increases the Euler genus by two, while adding a crosscap increases it by one (hint: start with a net N in which the holes you drill are the faces of the net, then add an edge to N to obtain a net for the new surface).*

Observation 14. *A surface obtained from the sphere by adding a handles and b crosscaps has Euler genus $2a + b$, and it is orientable iff $b > 0$. Consequently, every surface is homeomorphic to some surface obtained in this way.*