# Matchings in general graphs

Zdeněk Dvořák

October 6, 2020

**Definition 1.** *A* matching $M$ *in a graph* $G$ *is a* 1*-regular subgraph of* $G$*.*

Matching is often equivalently defined as a set of edges such that each vertex of $G$ is incident with at most one of them. We will use the above definition, so we can speak about the vertices and edges of a matching.

The *size* of the matching $M$ is the number of its edges $|E(M)|$. Let $\beta(G)$ denote the size of the largest matching in $G$. A set $X \subseteq V(G)$ is *covered* by the matching $M$ if $X \subseteq V(M)$. The matching $M$ is *perfect* if $V(G)$ is covered by $M$, or equivalently, if the size of $M$ is $|V(G)|/2$.

Before we focus on the main topic of this lesson, the matchings in general graphs, let us review the results on matchings in bipartite graphs that you should be familiar with from other courses.

## 1 Matchings in bipartite graphs

Let $G$ be a bipartite graph. Every edge of a matching in $G$ is incident with exactly one vertex of each part of the bipartition of $G$. Consequently, if $G$ has a perfect matching, then both parts of $G$ must have the same size. The most fundamental result about matchings in bipartite graphs is Hall's theorem.

**Theorem 2** (Hall's theorem)**.** *Let $G$ be a bipartite graph and let $A$ be one of the parts of the bipartition of $G$. The following claims are equivalent:*

- *The graph $G$ contains a matching that covers $A$.*
- *For every $X \subseteq A$, we have $|N(X)| \geq |X|$.*

In particular, if both parts of $G$ have the same size, Theorem 2 tells us precisely when $G$ has a perfect matching. Let us recall some further results related to Hall's theorem.

Observe that if $S \subseteq V(G)$ and $V(G) \setminus S$ is an independent set in $G$, then $G$ cannot have a matching of size larger than $|S|$, since each edge of the matching must have at least one end in $S$. The converse holds as well.

**Lemma 3.** *For every bipartite graph $G$, there exists a set $S \subseteq V(G)$ such that $|S| = \beta(G)$ and $V(G) \setminus S$ is an independent set.*

*Proof.* Let $A$ and $B$ be the parts of the bipartition of $G$. Let $Z \subseteq A$ be a set such that $|Z| - |N_G(Z)|$ is maximum among all subsets of $A$ (note that $Z$ can be the empty set). Let $S = (A \setminus Z) \cup N_G(Z)$. Since $A$ and $B$ are independent sets and no edge of $G$ has one end in $Z$ and the other end in $B \setminus N_G(Z)$, the set $V(G) \setminus S$ is independent. As we have argued above, $G$ cannot have a matching of size larger than $|S|$, and thus $\beta(G) \le |S|$.

Let $G'$ be the graph obtained from $G$ by adding a set $B'$ of $|Z| - |N_G(Z)|$ vertices adjacent to all vertices of $A$. Since $|Z| - |N_G(Z)|$ is maximum among all subsets of $A$, every $X \subseteq A$ satisfies

$$|N_{G'}(X)| = |B'| + |N_G(X)| = (|Z| - |N_G(Z)|) - (|X| - |N_G(X)|) + |X| \ge |X|.$$

Hence, by Theorem 2, $G'$ has a matching covering $A$. Deleting the edges incident with $B'$, we obtain a matching in $G$ of size at least $|A| - |B'| = |A| - |Z| + |N_G(Z)| = |S|$, and thus $\beta(G) \ge |S|$.

Hence, $\beta(G) = |S|$. □

**Exercise 4.** *Show that conversely, Lemma 3 implies Hall's theorem.*

Recall that $\alpha(G)$ is the size of the largest independent set in $G$.

**Corollary 5.** *If $G$ is bipartite, then $\alpha(G) + \beta(G) = |V(G)|$.*

Remark: Since the size of a largest matching in $G$ can be determined in polynomial time, we can for bipartite graphs determine the size of the largest independent set in polynomial time. This is in contrast to the situation in general graphs, where the problem is NP-hard.

**Exercise 6.** *Prove that in a general (not necessarily bipartite) graph we have*

$$|V(G)| - 2\beta(G) \le \alpha(G) \le |V(G)| - \beta(G),$$

*and show there exist arbitrarily large graphs $G$ such that $\alpha(G) = |V(G)| - 2\beta(G)$.*

Finally, let us mention a nice sufficient condition for the existence of perfect matchings.

**Lemma 7.** *For every integer $d \ge 1$, every $d$-regular bipartite graph has a perfect matching.*

*Proof.* Suppose $G$ is a $d$-regular bipartite graph. By Lemma 3, there exists $S \subseteq V(G)$ such that $|S| = \beta(G)$ and $T = V(G) \setminus S$ is an independent set. Since $T$ is an independent set, all edges incident with $T$ have the other end in $S$. There are $d|T|$ such edges, and each vertex of $S$ is incident with at most $d$ of them, and thus $|S| \geq d|T|/d = |T|$. Since $|S| + |T| = |V(G)|$, this implies $\beta(G) = |S| \geq |V(G)|/2$, and thus $G$ has a perfect matching. $\square$

**Exercise 8.** *Prove Lemma 7 using Hall's theorem instead of Lemma 3.*

# 2 Edmonds-Gallai decomposition

We now aim to derive a condition for the existence of perfect matchings in general graphs strengthening Theorem 2 and Lemma 3.

**Exercise 9.** *Show that Lemma 3 does not hold for general graphs; i.e., there exist (non-bipartite) graphs $G$ such that $\beta(G) < |S|$ for every $S \subseteq V(G)$ such that $V(G) \setminus S$ is an independent set.*

*Also, why you cannot straightforwardly generalize Hall's theorem to non-bipartite graphs?*

We will actually "overshoot" a bit in our effort and give a much stronger result than we usually need. Let us start with a few definitions.

A graph $G$ is *hypomatchable* if $G$ does not have a perfect matching, but for every $v \in V(G)$, the graph $G - v$ has a perfect matching. In particular, this means $G$ has an odd number of vertices. Moreover, note that if $G$ is hypomatchable, then every supergraph of $G$ with the same vertex set is hypomatchable.

**Exercise 10.** *List all minimal hypomatchable graphs (i.e., hypomatchable graphs with no hypomatchable proper spanning subgraphs) with at most five vertices.*

For a graph $G$ and $S \subseteq V(G)$, let $G_S$ denote the bipartite graph with one part formed by $S$ and the other part consisting of the connected components of $G - S$, such that $v \in S$ and a connected component $C$ of $G - S$ are adjacent if and only if $G$ contains an edge from $v$ to $V(C)$.

**Definition 11.** *An* EG-set *in $G$ is a set $S \subseteq V(G)$ such that*

- *every component of $G - S$ is hypomatchable, and*
- *the bipartite graph $G_S$ has a matching that covers $S$.*

As a special case, if $G$ is hypomatchable, then $\emptyset$ is an EG-set. Otherwise, any EG-set must be non-empty.

**Observation 12.** *If $S$ is an EG-set in $G$ and $G-S$ has $c$ components, then $\beta(G)=(|V(G)|+|S|-c)/2$.*

*Proof.* Let $M_S$ be a matching in $G_S$ that covers $S$. For each edge $vC\in E(M_S)$, let us choose a neighbor $m_v$ of $v$ in $V(C)$, and let $M_0$ be the matching in $G$ with the edge set $\{vm_v : v\in S\}$. For each component $C$ of $G-S$, if $C$ has a neighbor $v$ in $M_S$, then let $M_C$ be a perfect matching in $C-m_v$, otherwise let $M_C$ be an arbitrary matching in $C$ covering all but one vertex; in both cases such a matching exists, since $C$ is hypomatchable. Then $M_0\cup\bigcup_C M_C$ is a matching in $G$ covering all but $c-|S|$ vertices of $G$, and thus this matching has size $(|V(G)|+|S|-c)/2$. Therefore, $\beta(G)\ge(|V(G)|+|S|-c)/2$.

Conversely, consider any matching $M$ in $G$, and let $M'$ be the submatching of $M$ consisting of the edges with one end in $S$ and the other end in $G-S$. Clearly $|E(M')|\le|S|$, and thus at least $c-|S|$ components of $G-S$ are not incident with any edge of $M'$. Since all components of $G-S$ have an odd number of vertices, each component of $G-S$ not incident with $M'$ contains a vertex not covered by $M$. Hence, at least $c-|S|$ vertices are not covered by $M$, and thus $|E(M)|\le(|V(G)|+|S|-c)/2$. Since this holds for any matching $M$, we have $\beta(G)\le(|V(G)|+|S|-c)/2$. □

**Exercise 13.** *Observe that every largest matching in $G$ can be obtained by the procedure described in the first paragraph of the proof of Observation 12.*

*Suppose you are given a blackbox that for an input graph $G$ outputs (in polynomial time) an EG-set in $G$. Describe an algorithm to enumerate all largest matchings in the input graph. The algorithm should be efficient in the sense that the time complexity should be the number of largest matchings in $G$ times a polynomial in $|V(G)|+|E(G)|$.*

Crucially, every graph actually contains an EG-set. For a graph $H$, let $o(H)$ denote the number of components of $H$ of odd size.

**Theorem 14.** *Every graph $G$ contains an EG-set.*

*Proof.* We will prove this claim by induction on $|V(G)|$, and thus we can assume that every graph with less than $|V(G)|$ vertices contains an EG-set.

Let $S\subseteq V(G)$ be chosen so that

- $o(G-S)-|S|$ is maximum possible, and
- among such sets, $|S|$ is maximum.

First, consider any component $C$ of $G - S$ and let us argue that $C$ is hypomatchable. If $C$ had an even number of vertices, then choose any $v \in V(C)$ and let $S' = S \cup \{v\}$. We have $o(G-S') \geq o(G-S)+1$, since every connected component of $G-S$ other than $C$ is also a component of $G-S'$, while the even-size component $C$ is replaced by $C - v$ which has at least one component of odd size. Therefore, $o(G-S')-|S'| \geq (o(G-S)+1)-(|S|+1) = o(G-S)-|S|$ and $|S'| > |S|$, contradicting the choice of $S$.

Hence, $C$ has an odd number of vertices. If $C$ were not hypomatchable, then there would exist $v \in V(C)$ such that $C - v$ does not have a perfect matching, i.e., $\beta(C - v) < (|V(C)| - 1)/2$. Since $|V(C)| - 1$ is even and $\beta(C - v)$ is an integer, it follows that $\beta(C - v) \leq (|V(C)| - 3)/2$. By the induction hypothesis, $C - v$ contains an EG-set $S_C$. By Observation 12, $C - v - S_C$ has at least $|S_C| + 2$ (hypomatchable, and thus odd) components. Letting $S' = S \cup \{v\} \cup S_C$, we have $o(G-S') \geq (o(G-S)-1)+(|S_C|+2)$, and thus $o(G-S')-|S'| \geq (o(G-S)+|S_C|+1)-(|S|+|S_C|+1) = o(G-S)-|S|$. Since $|S'| > |S|$, this contradicts the choice of $S$. Therefore, every component of $G - S$ must be hypomatchable.

Let us now argue the bipartite graph $G_S$ has a matching that covers $S$. Otherwise, by Theorem 2, there would exist a set $X \subseteq S$ such that $|N_{G_S}(X)| < |X|$. Let $S' = S \setminus X$. Each component of $G - S$ not contained in $N_{G_S}(X)$ is also a component of $G - S'$, and thus $o(G - S') \geq o(G - S) - |N_{G_S}(X)| > o(G - S) - |X|$. Consequently, $o(G - S') - |S'| > (o(G - S) - |X|) - (|S| - |X|) = o(G - S) - |S|$, contradicting the choice of $S$.

Therefore, $S$ is an EG-set. □

# 3 Tutte's theorem

For applications, it is often more convenient to use the following less detailed result.

**Theorem 15** (Tutte's theorem)**.** *A graph $G$ has a perfect matching if and only if every set $S \subseteq V(G)$ satisfies $|S| \geq o(G - S)$.*

*Proof.* Suppose $G$ has a perfect matching $M$ and consider any $S \subseteq V(G)$. Each odd-size component of $G - S$ must be incident with an edge of $M$ with the other end in $S$, and thus $o(G - S) \leq |S|$.

Suppose now that $|S| \geq o(G - S)$ holds for every $S \subseteq V(G)$. By Theorem 14, $G$ contains an EG-set $S$. By Observation 12, we have $\beta(G) = (|V(G)| + |S| - o(G - S))/2 \geq |V(G)|/2$, and thus $G$ has a perfect matching. □

**Exercise 16.** *Prove that for every graph $G$,*

$$\beta(G) = \min_{S \subseteq V(G)} (|V(G)| + |S| - o(G-S))/2.$$

Let us now give an application to matchings in regular graphs.

**Exercise 17.** *Find a 3-regular connected graph without a perfect matching.*

**Theorem 18** (Petersen's theorem)**.** *Every 3-regular 2-edge-connected graph $G$ has a perfect matching.*

*Proof.* Consider any non-empty set $S \subseteq V(G)$. Let $C$ be an odd-size component of $G - S$. Consider the graph $G'$ obtained from $G$ by contracting all vertices outside of $S$ to a single vertex $z$ and suppressing loops. The vertices of $V(C)$ have degree three in $G'$ and there is an odd number of them. Since every graph has an even number of vertices of odd degree, $z$ must have odd degree. Therefore, the number of edges of $G$ between $C$ and $S$ is odd. There cannot be only one such edge, since otherwise deleting it would disconnect the graph, contradicting the assumption that $G$ is 2-edge-connected.

Therefore, at least three edges leave every odd-size component of $G - S$, and thus there are at least $3o(G-S)$ such edges. Each vertex of $S$ is incident with at most three such edges, and thus $|S| \geq 3o(G-S)/3 = o(G-S)$.

Note that since $G$ is 3-regular, it has an even number of vertices, and thus for $S = \emptyset$, we have $o(S) = 0$. Therefore, $|S| \geq o(G-S)$ holds for every $S \subseteq V(G)$, and Theorem 15 implies $G$ has a perfect matching. $\square$

**Exercise 19.** *Prove that for every odd $d \geq 1$, if $G$ is $d$-regular and $(d-1)$-edge-connected, then $G$ has a perfect matching. This is false if $d$ is even (consider an odd cycle); formulate and prove a correct statement for $d$ even.*