Extremal theory of hypergraphs; Density of hypergraphs avoiding the Fano plane

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November 20, 2020

A <u>k-uniform hypegraph</u> G is a pair (V, E), where E is a set of k-element subsets of V, called <u>hyperedges</u>. Notation: $|G|$ number of vertices of G, $\|G\|$ number of hyperedges of G. Similarly to the graph case (see Lesson 1), we have

Definition 1. *Let $F_1, \ldots, F_m$ be k-uniform hypergraphs. Maximum number of edges of a k-uniform hypergraph with n vertices that does not contain any subhypergraph isomorphic to $F_1, \ldots, F_m$:*

$$\mathrm{ex}(n; F_1, \ldots, F_m).$$

Density version:

$$\overline{\mathrm{ex}}(n; F_1, \ldots, F_m) = \frac{\mathrm{ex}(n; F_1, \ldots, F_m)}{\binom{n}{k}}.$$

Asymptotic density:

$$\overline{\mathrm{ex}}(\infty; F_1, \ldots, F_m) = \inf\{\overline{\mathrm{ex}}(n; F_1, \ldots, F_m) : n \in \mathbb{N}\}.$$

Lemma 2. *If $n_1 \geq n_2$, then $\overline{\mathrm{ex}}(n_1; F_1, \ldots, F_m) \leq \overline{\mathrm{ex}}(n_2; F_1, \ldots, F_m)$.*

Corollary 3.

$$\overline{\mathrm{ex}}(\infty; F_1, \ldots, F_m) = \lim_{n\to\infty} \overline{\mathrm{ex}}(n; F_1, \ldots, F_m),$$

and for every n_0 we have

$$\overline{\mathrm{ex}}(\infty; F_1, \ldots, F_m) \leq \overline{\mathrm{ex}}(n_0; F_1, \ldots, F_m).$$

Asymptotically, for $n \to \infty$, we have

$$\mathrm{ex}(n; F_1, \ldots, F_m) = (\overline{\mathrm{ex}}(\infty; F_1, \ldots, F_m) + o(1))\frac{n^2}{k}.$$

In general, the extremal theory for hypergraphs is much more challenging than for graphs. For example, we do not even know $\overline{\mathrm{ex}}(\infty; K_4^{(3)})$ for the complete 3-uniform hypergraph $K_4^{(3)}$ with four vertices. We will need the following simple (non-tight bound) following straightforwardly from Corollary 3.

Example 4. *Every 4-vertex 3-uniform hypergraph not containing $K_4^{(3)}$ has at most three edges, i.e. $\overline{\mathrm{ex}}(4; K_4^{(3)}) = 3/4$. Consequently, we also have $\overline{\mathrm{ex}}(4; K_4^{(3)}) \leq 3/4$, and thus $\mathrm{ex}(5; K_4^{(3)}) \leq \frac{3}{4}\binom{5}{3} = 7.5$. Since $\mathrm{ex}(5; K_4^{(3)})$ is an integer, we have $\mathrm{ex}(5; K_4^{(3)}) \leq 7$, and thus $\overline{\mathrm{ex}}(5; K_4^{(3)}) \leq 7/10$. Therefore, $\mathrm{ex}(n; K_4^{(3)}) \leq \frac{7}{10}\binom{n}{3}$ for every $n \geq 5$ and $\overline{\mathrm{ex}}(\infty; K_4^{(3)}) \leq 7/10$.*

Our goal now is to investigate the extremal function for the Fano hypergraph Fano, whose vertex set is formed by vectors 001, 010, ..., 111 and $\{x, y, z\}$ is a hyperedge iff $x+y+z = 0$ holds in $\mathbb{Z}_2^3$ (equivalently, Fano is the hypegraph whose vertices are the points and edges are the lines of the Fano plane, the finite projective plane of order two). A short case analysis shows that for any assignment of colors 1 and 2 to the vertices of of Fano, there exists a monochromatic hyperedge. Hence, Fano is not a subgraph of the hypergraph $B_n^{(3)}$ whose vertex set consists of two parts A and B such that $|A| = \lfloor n/2 \rfloor$ and $|B| = \lceil n/2 \rceil$ and the edges are all triples intersecting both A and B.

Theorem 5. *For sufficiently large n, we have $\mathrm{ex}(n; \mathrm{Fano}) = \|B_n^{(3)}\|$, and $B_n^{(3)}$ is the only n-vertex 3-uniform hypergraph with this many hyperedges not containing* Fano *as a subhypergraph.*

To prove Theorem 5, we need a couple of auxiliary results.

Lemma 6. *Let G be a multigraph with $n \geq 3$ vertices. If any three vertices induce a submultigraph with at most 10 edges, then $\|G\| \leq 3\binom{n}{2} + n - 2$.*

Proof. We prove the claim by induction on n. The claim clearly holds for $n = 3$, and thus we can assume $n \geq 4$. If all edges have multiplicity at most 3, then $\|G\| \leq 3\binom{n}{2}$. Otherwise, consider vertices u and v joined by an edge of multiplicity $m \geq 4$. The number of edges between $\{u, v\}$ and any other vertex is at most $10 - m$ by the assumptions, and thus the number of edges between $\{u, v\}$ and the rest of the graph is at most $(10 - m)(n - 2)$. Hence, $\deg u + \deg v \leq (10 - m)(n - 2) + 2m$, and by symmetry we can assume $\deg v \leq (10 - m)(n - 2)/2 + m$. By the induction hypothesis, we have $\|G - v\| \le 3\binom{n-1}{2} + n - 3$, and thus

$$\begin{aligned}\|G\| &\le 3\binom{n-1}{2} + n - 3 + (10-m)(n-2)/2 + m \\ &\le 3\binom{n-1}{2} + n + 3(n-2) + 1 = 3\binom{n}{2} + n - 2.\end{aligned}$$

□

A similar argument gives the following.

Lemma 7. *Let G be a multigraph with $n \ge 4$ vertices. If any four vertices induce a submultigraph with at most 20 edges, then $\|G\| \le 3\binom{n}{2} + (4n-10)/3$.*

Proof. We prove the claim by induction on n. The claim clearly holds for $n = 4$, and thus we can assume $n \ge 5$. If any three vertices induce a submultigraph with at most 10 edges, then the claim follows from Lemma 6. Hence, suppose that the submultigraph induced by $\{u, v, w\}$ has $m \ge 11$ edges. he number of edges between $\{u, v, w\}$ and any other vertex is at most $20 - m$ by the assumptions, and thus $\deg u + \deg v + \deg w \le (20-m)(n-3) + 2m$. By symmetry, we can assume $\deg v \le ((20-m)(n-3)+2m)/3$. By the induction hypothesis, we have $\|G - v\| \le 3\binom{n-1}{2} + 4n/3 - 14/3$, and thus

$$\begin{aligned}\|G\| &\le 3\binom{n-1}{2} + 4n/3 - 14/3 + ((20-m)(n-3)+2m)/3 \\ &\le 3\binom{n-1}{2} + 13(n-23)/3 = 3\binom{n}{2} + (4n-10)/3.\end{aligned}$$

□

For a 3-uniform hypergraph G and a set $S \subseteq V(G)$, the <u>link multigraph</u> of S in G is the multigraph with vertex set $V(G) \setminus S$, where each pair xy of vertices is an edge whose multiplicity is the number of vertices $z \in S$ such that $\{x, y, z\}$ is a hyperedge of G. For $z \in S$, let $L(z)$ denote the set of edges of this link graph such that the corresponding hyperedge contains S.

Lemma 8. *Let G be a 3-uniform hypergraph and let S be a set of four vertices such that $G[S]$ is complete. Let L be the link multigraph of S in G. If some four vertices of L induce a submultigraph with at least 21 edges, then* Fano $\subseteq G$.

Proof. Let $S = \{v_1, v_2, v_3, v_4\}$. Suppose $\|L[\{x_1, x_2, x_3, x_4\}]\| \geq 21$. For $i \in \{1, 2, 3\}$, let M_i be the perfect matching on $\{x_1, \ldots, x_4\}$ containing the edge x_1x_{i+1}. We can assume $|L(v_1)| \geq \ldots \geq |L(v_4)|$. Since $21 \leq \|L[\{x_1, x_2, x_3, x_4\}]\| = ||L(v_1)|+\cdots+|L(v_4)|$, we have $|L(v_2)|, |L(v_3)| \geq 5$, and thus we can by symmetry assume $M_2 \subseteq L(v_2)$ and $M_3 \subseteq L(v_3)$. Moreover, we have $|L(v_1)| \geq 6$, and thus $M_1 \subseteq L(v_1)$. Then the hyperedges $\{v_i, x, y\}$ for $i \in \{1, 2, 3\}$ and $xy \in M_i$ together with the hyperedge $\{v_1, v_2, v_3\}$ form Fano. $\square$

We are now ready to prove the asymptotic form of Theorem 5.

Theorem 9.

$$\overline{\mathrm{ex}}(\infty; \mathrm{Fano}) = 3/4.$$

Proof. Recall that Fano $\not\subseteq B_n^{(3)}$, and observe that $\lim_{n\to\infty} \frac{\|B_n^{(3)}\|}{\binom{n}{3}} = 3/4$. Hence, $\overline{\mathrm{ex}}(\infty; \mathrm{Fano}) \geq 3/4$.

We now aim to prove that $\overline{\mathrm{ex}}(\infty; \mathrm{Fano}) \leq 3/4$. To this end, we need to prove that for every $\varepsilon > 0$ and sufficiently large n, if G is a 3-uniform n-vertex hypergraph with at least $(3/4 + \varepsilon)n^3/6$ hyperedges, then Fano $\subseteq G$. First, as long as G contains a vertex incident with less than $(3/4 + \varepsilon/2)n^2/2$ hyperedges, keep deleting such vertices. We end up with an n_0-vertex hypergraph G_0 of minimum degree at least $(3/4 + \varepsilon/2)n_0^2/2$, and as usual, it is easy to argue that $n_0 = \Omega(\varepsilon^{1/3}n)$.

Since $\|G_0\| \geq (3/4+\varepsilon/2)n_0^3/6$, Example 4 implies that $K_4^{(3)} \subseteq G_0$. Let S be the vertex set of this $K_4^{(3)}$ and L the corresponding link multigraph. Since G_0 has minimum degree at least $(3/4+\varepsilon/2)n_0^2/2$, each vertex of S contributes at least $(3/4+\varepsilon/2)n_0^2/2 - 3n_0 \geq (3/4+\varepsilon/4)\binom{n_0}{2}$, and thus $\|L\| \geq (3+\varepsilon)\binom{n_0}{2}$. By Lemma 7, this implies there exists a set of four vertices of L inducing more than 20 edges. By Lemma 8, this implies Fano $\subseteq G$. $\square$

Next, we need the corresponding stability result, showing that near-extremal graphs are close to $B_n^{(3)}$.

Theorem 10. *For every $\varepsilon > 0$, there exists $\gamma > 0$ and n_0 such that for every 3-uniform hypergraph G with $n \geq n_0$ vertices and at least $(3/4 - \gamma)n^3/6$ hyperedges, if* Fano $\not\subseteq G$*, then there exists a partition of $V(G)$ to parts A and B such that $\|G[A]\| + \|G[B]\| \leq \varepsilon n^3$.*

The proof of Theorem 10 follows an idea similar to the one of the proof of Theorem 9, but is somewhat more lengthy; if you are interested, see [1, Theorem 1.2]. We now need one more simple claim. Let $\mu(G)$ denote the maximum size of a matching in a graph G.

Lemma 11. *For every n-vertex graph G, we have $\|G\| \leq 2\mu(G)n$.*

Proof. Let M be a largest matching in G. Then every edge is incident with at least one vertex of $V(M)$ and $|V(M)| = 2\mu(G)$. $\square$

Next, let us prove the weaker variant of Theorem 5 for graphs of bounded minimum degree.

Lemma 12. *There exists n_0 such that the following holds. Let G be a 3-uniform hypergraph with $n \geq n_0$ vertices and $\mathrm{ex}(n; \mathrm{Fano})$ hyperedges. If $\mathrm{Fano} \not\subseteq G$ and $\delta(G) \geq \delta(B_n^{(3)})$, then $\|G\| \leq \|B_n^{(3)}\|$, with equality iff $G = B_n^{(3)}$.*

Proof. Choose sufficiently small $\varepsilon > 0$ and sufficiently large n_0. Let A and B form a partition of $V(G)$ minimizing $\|G[A]\| + \|G[B]\|$; by Theorem 10, we have $\|G[A]\| + \|G[B]\| \leq \varepsilon n^3$. Note that for $x \in A$, we have $\left|L(x) \cap \binom{A}{2}\right| \leq \left|L(x) \cap \binom{B}{2}\right|$, as otherwise moving x to B would decrease the number of hyperedges within the parts. Symmetrically, $\left|L(y) \cap \binom{B}{2}\right| \leq \left|L(y) \cap \binom{A}{2}\right|$ for every $y \in B$.

Let q denote the number of triples intersecting both A and B that are not hyperedges of G. Let $\Delta = ||A| - n/2| = ||B| - n/2|$. Since $\delta(G) \geq \delta(B_n^{(3)}) \geq (3/8 - 3\varepsilon)n^2$, we have

$$\begin{aligned}(1/8 - \varepsilon)n^3 \leq \|G\| &= \binom{|A|}{2}|B| + \binom{|B|}{2}|A| - q + \|G[A]\| + \|G[B]\| \\ &\leq |A||B|n/2 - q + \varepsilon n^3 = (1/8 + \varepsilon)n^3 - \Delta^2 n/2 - q.\end{aligned}$$

Hence, we have $q \leq 2\varepsilon n^3$ and $\Delta \leq 2\varepsilon^{1/2} n$.

Suppose next that there exists $x \in A$ such that $\left|L(x) \cap \binom{A}{2}\right| > 4\varepsilon^{1/3} n^2$, and thus also $\left|L(x) \cap \binom{B}{2}\right| > 4\varepsilon^{1/3} n^2$. By Lemma 11, there exist matchings $M_A \subseteq L(x) \cap \binom{A}{2}$ and $M_B \subseteq L(x) \cap \binom{B}{2}$ of size at least $2\varepsilon^{1/3} n$. If there existed an edge $a_1a_2 \in M_A$ and distinct edges $b_1b_2, c_1c_2 \in M_B$ such that $\{a_i, b_j, c_j\} \in E(G)$ for all $i, j, k \in \{1, 2\}$, then we would have $\mathrm{Fano} \subseteq G$. Hence, this is not the case, and thus for each edge of M_A and a pair of distinct edges of M_B, there exists a non-hyperedge of G intersecting all of them. Consequently, $q \geq |M_A|\binom{|M_B|}{2} \geq 8\varepsilon n^3/3$, which is a contradiction.

Therefore, we have $\left|L(x) \cap \binom{A}{2}\right| \leq 4\varepsilon^{1/3} n^2$ for every $x \in A$, and symmetrically, $\left|L(y) \cap \binom{B}{2}\right| \leq 4\varepsilon^{1/3} n^2$ for every $y \in B$. Suppose now that there exists a hyperedge $\{x_1, x_2, x_3\} \in E(G[A])$. For $i \in \{1, 2, 3\}$, we have

$$\begin{aligned}\left|L(x_i) \cap \binom{B}{2}\right| &\geq \deg x - \left|L(x) \cap \binom{A}{2}\right| - |A||B| \\ &\geq (3/8 - \varepsilon)n^2 - 4\varepsilon^{1/3}n^2 - n^2/4 + \Delta^2 \\ &\geq (1/8 - 9\varepsilon^{1/3})n^2.\end{aligned}$$

Since $\binom{|B|}{2} \leq (n/2 + \Delta)^2/2 \leq (1/8 + 2\varepsilon^{1/2})n^2$, $L(x_i)$ has at most $11\varepsilon^{1/3}n^2$ non-edges in $\binom{B}{2}$. Consequently, all but at most $33\varepsilon^{1/3}n^2$ pairs of vertices in $\binom{B}{2}$ are joined by a triple edge in $(L(x_1) \cup L(x_2) \cup L(x_3)) \cap \binom{B}{2}$. By Turán's theorem, there exist four vertices $y_1, \ldots, y_4 \in B$ such that any pair of them is joined by a triple edge in $L(x_1) \cup L(x_2) \cup L(x_3)$. However, this implies Fano $\subseteq G$, which is a contradiction.

Therefore, $E(G[A]) = \emptyset$, and by symmetry, $E(G[B]) = \emptyset$. It follows that $\|G\| \leq \|B_n^{(3)}\|$, with equality only if $G = B_n^{(3)}$. □

We are now ready to finish the argument.

Proof of Theorem 5. Let $n_1 = 8n_0^3$, where n_0 is the constant from Lemma 12. For a 3-uniform hypergraph H, let $m(H) = \|H\| - \|B_{|H|}^{(3)}\|$. Let G be a 3-uniform hypergraph with $n \geq n_1$ vertices and $\mathrm{ex}(n; \mathrm{Fano})$ hyperedges such that Fano $\not\subseteq G$. Since Fano $\not\subseteq B_n^{(3)}$, we have $m(G) \geq 0$.

Let $G_0 = G$. For $i \geq 0$, as long as G_i contains a vertex v of degree less than $\delta(B_{|G_i|}^{(3)})$, we let $G_{i+1} = G_i - v$. Note that $B_{|G_i|-1}^{(3)}$ is obtained from $B_{|G_i|}^{(3)}$ by deleting a vertex of minimum degree, and thus $m(G_{i+1}) \geq m(G_i) + 1$. Let G_k be the last member of this sequence; we have $m(G_k) \geq k$. On the other hand, $m(G_k) \leq \|G_k\| \leq (n-k)^3$, and thus $k + k^{1/3} \leq n$ and $k \leq n - n^{1/3}/2$. Consequently, $|G_k| = n - k \geq n^{1/3}/2 \geq n_0$.

By the choice of G_k, we have $\delta(G_k) \geq \delta(B_{|G_k|}^{(3)})$, and thus Lemma 12 implies $m(G_k) = 0$ (and thus $k = 0$ and $G_k = G$) and $G_k = B_n^{(3)}$. □

References

[1] Peter Keevash, Benny Sudakov: *The Turán number of the Fano plane*, Combinatorica **25** (2005), 561–574.