# Applications of stability

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From the last lecture:

**Theorem 1.** *Let $F$ be a graph of chromatic number $r+1$, where $r \geq 1$. For every $\varepsilon > 0$ there exists $\beta > 0$ such that for sufficiently large $n$, if $G$ is an $n$ vertex graph with at least $(1-1/r-\beta)n^2/2$ edges and $F \not\subseteq G$, then there exists a partition of $V(G)$ to parts $A_1, \ldots, A_r$ satisfying*

$$\sum_{i=1}^{r} \|G[A_i]\| \leq \varepsilon n^2.$$

**Corollary 2.** *Let $F$ be a graph of chromatic number $r+1$, where $r \geq 1$, and let $\gamma$ be a positive real number. Let $G$ be an $n$-vertex graph with $\mathrm{ex}(n;F)$ edges such that $F \not\subseteq G$. If $n$ is sufficiently large, then $G$ has minimum degree at least $(1-1/r-\gamma)n$.*

**Observation 3.** *Let $G$ be the complete $r$-partite $n$-vertex graph with parts $A_1, \ldots, A_r$. Then*

$$\|G\| = \Big(1 - 1/r - \sum_{i=1}^{r}(1/r - |A_i|/n)^2\Big)\frac{n^2}{2}.$$

**Corollary 4.** *Let $r$ be a positive integer. Let $G$ be an $n$-vertex graph with at least $(1-1/r-\varepsilon)n^2/2$ edges, and let $A_1, \ldots, A_r$ be a partition of $V(G)$ to parts such that*

$$\sum_{i=1}^{r} \|G[A_i]\| \leq \varepsilon n^2.$$

*Then $|A_i - n/r| \leq \sqrt{3\varepsilon}n$ for every $i$ and $G$ contains at most $\frac{3}{2}\varepsilon n^2$ non-edges with ends in distinct parts.*

*Proof.* Suppose $G$ contains $\mu n^2$ non-edges with ends in different parts. By Observation 3 we have

$$\begin{aligned}(1-1/r-\varepsilon)\frac{n^2}{2} \le \|G\| &\le \varepsilon n^2 - \mu n^2 + \Big(1-1/r-\sum_{i=1}^{r}(1/r-|A_i|/n)^2\Big)\frac{n^2}{2}\\ &= (1-1/r-\varepsilon)\frac{n^2}{2} + \Big(3\varepsilon-2\mu-\sum_{i=1}^{r}(1/r-|A_i|/n)^2\Big)\frac{n^2}{2},\end{aligned}$$

and thus

$$2\mu+\sum_{i=1}^{r}(1/r-|A_i|/n)^2 \le 3\varepsilon.$$

This implies the inequalities from the statement. □

An edge $e \in E(F)$ is critical if $\chi(F-e) < \chi(F)$. For example, all edges of an odd cycle are critical.

**Theorem 5.** *Let $F$ be a graph of chromatic number $r+1$, where $r \ge 1$. If $F$ has a critical edge, then for sufficiently large $n$ we have* $\mathrm{ex}(n;F) = t_r(n)$ *and $T_r(n)$ is the only $n$-vertex graph with* $\mathrm{ex}(n;F)$ *edges not containing $F$ as a subgraph.*

*Proof.* Let $k = |F|$, $\beta = \frac{1}{3kr^2}$ and $\varepsilon = \beta^2/3$. Let $G$ be an $n$-vertex such that $F \not\subseteq G$ and $\|G\| = \mathrm{ex}(n;F)$. Let $A_1, \ldots, A_r$ be a partition of $V(G)$ such that

$$m = \sum_{i=1}^{r} \|G[A_i]\|$$

is minimum. Let $e$ be a critical edge of $F$ and let $w$ be a vertex of $F$ incident with $e$.

By Theorem 1 and Corollaries 2 and 4, for sufficiently large $n$ we have that $m \le \varepsilon n^2$, the minimum degree of $G$ is at least $(1-1/r-\varepsilon)n$, $|A_i - n/r| \le \varepsilon n$ for each $i$, and $G$ contains at most $\varepsilon n^2$ non-edges with ends in different parts.

Suppose first that there exists $i$ such that $\Delta(G[A_i]) \ge \beta n$. Let $v \in A_i$ be a vertex with at least $\beta n$ neighbors in $A_i$. The minimality of $m$ implies that moving $v$ to any other part does not decrease the number of edges within the parts, and thus $v$ has at least $\beta n$ neighbors in each part. Let $N_1, \ldots, N_r$ be sets of neighbors of $v$ in $A_1, \ldots, A_r$ such that $|N_1| = \ldots = |N_r| \ge \beta n$ and let $s = |N_1 \cup \ldots \cup N_r| \ge \beta r n$. The subgraph $G[N_1 \ldots N_r]$ has at least

$$\begin{aligned}\big(1-\frac{1}{r}\big)\frac{s^2}{2} - \varepsilon n^2 &\ge \big(1-\frac{1}{r}\big)\frac{s^2}{2} - \frac{\varepsilon}{\beta^2 r^2}s^2\\ &= \big(1-\frac{1}{r}-\frac{2\varepsilon}{\beta^2 r^2}\big)\frac{s^2}{2} \ge \big(1-\frac{1}{r-1}+\varepsilon\big)\frac{s^2}{2}\end{aligned}$$

edges. The chromatic number of $F - w$ is $r$, and for sufficiently large $n$ (and thus also large $s$), Erdős-Stone theorem implies $F - w \subseteq G[N_1 \dots N_r]$. Using $v$ to represent $w$ gives $F \subseteq G$, which is a contradiction.

Therefore, we can assume $\Delta(G[A_i]) \le \beta n$ for every $i$. Consider a vertex $v \in A_i$. Since $\Delta(G[A_i]) \le \beta n$ and $|A_i| \ge n/r - \varepsilon n$, $v$ has at least $n/r - (\varepsilon + \beta)n$ non-neighbors in $A_i$. Since $\deg(v) \ge (1 - 1/r - \varepsilon)n$, $v$ has at most $(2\varepsilon + \beta)n$ non-neighbors in any other part.

Suppose now that any of the subgraphs $G[A_i]$ (say for $i = 1$) has at least one edge $e'$. Select $B_1 \subset A_1$ of size $k$ such that both ends of $e'$ belong to $B_1$ arbitrarily. For $j = 2, \dots, r$, choose $B_j \subset A_j$ of size $k$ so that every vertex of $B_j$ is adjacent to all vertices of $B_1 \cup \ldots \cup B_{j-1}$; this is possible, since there are at most $k(r-1)(2\varepsilon + \beta)n \le n/r - \varepsilon n - k \le |A_j| - k$ non-edges between $B_1 \cup \ldots \cup B_{j-1}$ and $A_j$. Then $G[B_1 \cup \ldots \cup B_r]$ is a complete $r$-partitite graph with parts of size $k$ plus one edge, and thus it contains $F$ as a subgraph, which is a contradiction.

It follows that $G$ is an $r$-partite graph with parts $A_1, \dots, A_r$. No $r$-partite graph contains $F$ as as subgraph, and since $G$ has the largest number of edges among the graphs with this property, we conclude $G = T_r(n)$. □

Next, let us consider the extremal number for the graph $kK_{r+1}$, that is, $k$ disjoint cliques of size $r + 1$.

**Theorem 6.** *Let $G$ be an $n$-vertex graph such that $kK_{r+1} \not\subseteq G$ and $\|G\| = \mathrm{ex}(n; kK_{r+1})$. For sufficiently large $n$, $G$ is the graph obtained from $T_r(n-k+1)$ by adding $k-1$ universal vertices, and thus*

$$\mathrm{ex}(n; kK_{r+1}) = t_r(n-k+1) + (k-1)(n-k+1) + \binom{k-1}{2}.$$

*Proof.* Let $\beta = \frac{1}{3r^2}$ a $\varepsilon = \beta^2/8$. Let $A_1, \dots, A_r$ be a partition of $V(G)$ such that

$$m = \sum_{i=1}^{r} \|G[A_i]\|$$

is minimum. By Theorem 1 and Corollaries 2 and 4, for sufficiently large $n$ we have $m \le \varepsilon n^2$, $G$ has minimum degree at least $(1 - 1/r - \varepsilon)n$, $|A_i - n/r| \le \varepsilon n$ for each $i$, and $G$ contains at most $\varepsilon n^2$ non-edges with ends in different parts. For every $v \in V(G)$, let $i(v)$ denote the index $i$ such that $v \in A_i$.

Let us start with an observation that we will use several times later. Consider any disjoint sets $U, Z \subseteq V(G)$ such that $|U| \le k$, $|Z| \le k(r+1)$ and every vertex $u \in U$ has at least $\beta n$ neighbors in $A_{i(u)}$. The minimality of $m$ implies that $u$ has at least $\beta n$ neighbors in each part. Therefore, we can choose pairwise disjoint sets $N_{u,t} \subseteq A_t \setminus (U \cup Z)$ for $u \in U$ and $1 \leq t \leq r$ such that $u$ is adjacent to all vertices of $N_{u,t}$ and all these parts have the same size greater or equal to $(\beta n - k(r+2))/k \geq \beta n/2$. Let $s = |N_{u,1} \cup \ldots \cup N_{u,r}| \geq \beta rn/2$. The subgraph $G[N_{u,1} \ldots N_{u,r}]$ has at least

$$\big(1 - \frac{1}{r}\big)\frac{s^2}{2} - \varepsilon n^2 \geq \big(1 - \frac{1}{r}\big)\frac{s^2}{2} - \frac{4\varepsilon}{\beta^2 r^2} s^2$$
$$= \big(1 - \frac{1}{r} - \frac{8\varepsilon}{\beta^2 r^2}\big)\frac{s^2}{2} > \big(1 - \frac{1}{r-1}\big)\frac{s^2}{2}$$

edges. Turán's theorem implies $K_r \subseteq G[N_{u,1} \ldots N_{u,r}]$. Adding $u$ gives a clique of size $r+1$ in $G$, and for disjoint $u \in U$ these cliques are pairwise disjoint. Moreover, these cliques are also disjoint from $Z$.

Let $U \subseteq V(G)$ be the set of all vertices $u \in V(G)$ such that $u$ has at least $\beta n$ neighbors in $A_{i(u)}$. Since $kK_{r+1} \not\subseteq G$, the previous paragraph implies $|U| \leq k-1$. Every vertex $v \in V(G) \setminus U$ has at most $\beta n$ neighbors in $A_{i(v)}$. Since $|A_{i(v)}| \geq n/r - \varepsilon n$, $v$ has at least $n/r - (\varepsilon + \beta)n$ non-neighbors in $A_{i(v)}$. Since $\deg(v) \geq (1 - 1/r - \varepsilon)n$, $v$ has at most $(2\varepsilon + \beta)n$ non-neighbors in any other parts.

Suppose now that for some $i$ (say $i = 1$), the graph $G[A_i \setminus U]$ contains a matching $M$ of size $k - |U|$. For $j = 2, \ldots, r$, from $A_j \setminus U$ choose for each edge $e \in M$ a vertex $v_{e,j}$ such that $v_{e,j}$ is adjacent to both ends of $e$ and to the vertices $v_{e,2}, \ldots, v_{e,j-1}$, and moreover the vertices $v_{e,j}$ and $v_{e',j}$ for distinct edges $e, e' \in E(M)$ are distinct. This is possible, since $r(2\varepsilon + \beta)n + k \leq n/r - \varepsilon n \leq |A_j|$ for every $j$. This gives $k - |U|$ disjoint cliques of size $r+1$. Now we apply the claim from the second paragraph with $Z$ being the union of these cliques. This gives $|U|$ additional cliques of size $K_{r+1}$ disjoint from $Z$, and thus $kK_{r+1} \subseteq G$, a contradiction.

Therefore, every matching in $G[A_i \setminus U]$ for $i = 1, \ldots, r$ has size at most $k - 1 - |U|$. Let $X$ be a set such that $A_i \cap X$ is the vertex set of a largest matching in $G[A_i \setminus U]$ for every $i$. Then $|X| \leq 2(k-1-|U|)r$ and $A_i \setminus (U \cup X)$ is an independent set in $G$ for every $i$. Consequently

$$\|G\| \leq t_r(n - |U|) + 2(k - 1 - |U|)r\beta n + |U|(n - |U|) + \binom{|U|}{2}$$
$$\leq t_r(n - k + 1) + (k-1)(n - k + 1) + \binom{k-1}{2},$$

where the equality holds iff $|U| = k - 1$, $G - U$ is $T_r(n - k + 1)$ and each vertex of $U$ is adjacent to all other vertices of $G$. $\square$