Stability for Erdős-Stone theorem

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Theorem 1 (Erdős-Stone). *Let F be a graph of chromatic number $r+1$. For every $\varepsilon > 0$, there exists n_0 such that every graph with $n \geq n_0$ vertices and at least $(1 - 1/r + \varepsilon)\frac{n^2}{2}$ edges contains F as a subgraph.*

The complete multipartite graph $T_r(n)$ does not contain F as a subgraph and has about $(1 - 1/r)\frac{n^2}{2}$ edges. By stability for Theorem 1, we mean that even the graphs without F that have only slightly fewer edges than that have a structure resembling $T_r(n)$.

Lemma 2. *Let F be a graph of chromatic number $r+1$. For every $\varepsilon > 0$ there exists $\beta > 0$ such that the following claim holds for every sufficiently large n. If G is an n-vertex graph of minimum degree at least $(1 - 1/r - \beta)n$ not containing F as a subgraph, then there exists a partition of $V(G)$ to parts A_1, ..., A_r such that*

$$\sum_{i=1}^{r} \|G[A_i]\| \leq \varepsilon n^2.$$

Proof. Let $t = |F|$, $s = \lceil \frac{32}{\varepsilon} t^2 \rceil + t$ and $\beta = \min\left(\frac{t}{rs}, \frac{1}{2r(r-1)}\right)$. Since $\beta < \frac{1}{r-1} - \frac{1}{r}$, Theorem 1 for sufficiently large n implies that G contains $T_r(rs)$ as a subgraph. Let $B_1, \ldots, B_r \subseteq V(G)$ be disjoint sets of size s such that G contains all edges between these sets, and let $B = B_1 \cup \ldots \cup B_r$. For sufficiently large n, we have $|B| \leq \frac{t}{rs} n \leq \frac{\varepsilon}{4} n$.

Let us partition $V(G) \setminus B$ into parts T, A'_1, ..., A'_r, and S such that

- T contains the vertices that have at least t neighbors in each of B_1, ..., B_r, and
- A'_i for $i = 1, \ldots, r$ contains the vertices, that have less than t neighbors in B_i and at least $s - \frac{16}{\varepsilon} t$ neighbors in every other part.

Note that $|T| < t\binom{s}{t}^r$, since otherwise $G \supseteq T_{r+1}((r+1)t) \supseteq F$. For sufficiently large n, this implies $|T| \leq \frac{t}{rs}n \leq \frac{\varepsilon}{4}n$.

Since G has minimum degree at least $(1-1/r-\beta)n$, the number of edges between B and $A'_1 \cup \ldots \cup A'_r \cup S$ is at least $((1-1/r-\beta)n - |B| - |T|)rs \geq (r-1)sn - 3tn$. On the other hand, the vertices in S have less than t neighbors in one of the parts and less than $s - \frac{16}{\varepsilon}t$ neighbors in another part, and thus

$$(r-1)sn - 3tn \leq (n-|S|)((r-1)s+t) + |S|((r-1)s+t-16t/\varepsilon) = (r-1)sn + tn - 16t|S|/\varepsilon.$$

Consequently, $|S| \leq \frac{\varepsilon}{4}n$.

If there exists $i \in \{1, \ldots, r\}$ such that $\|G[A'_i]\| \geq \frac{\varepsilon}{4r}n^2$, then for sufficiently large n, Theorem 1 implies that $G[A'_i]$ contains $K_{t,t}$ as a subgraph. The vertices of this $K_{t,t}$ have at least $s - 2t\frac{16}{\varepsilon}t \geq t$ common neighbors in each part B_j such that $j \neq i$, and thus G would contain $T_{r+1}((r+1)t) \supseteq F$ as a subgraph, which is a contradiction. Therefore, we have $\|G[A'_i]\| \leq \frac{\varepsilon}{4r}n^2$ for each i.

Let $A_i = A'_i$ for $i = 1, \ldots, r-1$ and $A_r = A'_r \cup (B \cup T \cup S)$. Then

$$\sum_{i=1}^{r} \|G[A_i]\| \leq r\frac{\varepsilon}{4r}n^2 + \frac{3\varepsilon}{4}n^2 = \varepsilon n^2.$$

□

For a graph G, let $m_r(G) = \|G\| - (1-1/r)|G|^2/2$.

Observation 3. *Let G be a graph, r and integer, and β a positive real number. If v is a vertex of G of degree less than $(1-1/r-\beta)|G|$, then $m_r(G-v) > m_r(G) + \beta|G| - 1$.*

Proof. We have

$$\begin{aligned} m_r(G-v) - m_r(G) &= (1-1/r)\left[|G|^2 - (|G|-1)^2\right]/2 - (\|G\| - \|G-v\|) \\ &> (1-1/r)(|G|-1/2) - (1-1/r-\beta)|G| > \beta|G| - 1. \end{aligned}$$

□

We are now ready to prove the stability version of Erdős-Stone theorem.

Theorem 4. *Let F be a graph of chromatic number $r+1$. For every $\varepsilon > 0$, there exists $\beta' > 0$ such that for sufficiently large n, if G is an n-vertex graph with at least $(1-1/r-\beta')n^2/2$ edges and $F \not\subseteq G$, then there exists a partition of $V(G)$ into parts $A_1, \ldots, A_r$ such that*

$$\sum_{i=1}^{r} \|G[A_i]\| \leq \varepsilon n^2.$$

Proof. We can assume $\varepsilon < 1/2$, as otherwise the claim trivially holds. Let β be the constant from Lemma 2 for $\varepsilon/2$. Let $\beta' = \beta\varepsilon/7$. Repeatedly remove from G vertices of degree less than $(1 - 1/r - \beta)$ times the current number of vertices, as long as such vertices exist or until we removed at least $\varepsilon n/2$ vertices, and let G' be the resulting graph. If we deleted at least $\lceil \varepsilon n/2 \rceil$ vertices, then Observation 3 implies

$$\begin{aligned} m_r(G') &\geq m_r(G) + (\beta|G'| - 1)\lceil \varepsilon n/2 \rceil \\ &\geq m_r(G) + \tfrac{\beta\varepsilon}{6} n^2 \geq (\beta\varepsilon/6 - \beta')n^2 \\ &\geq (\beta\varepsilon/6 - \beta')|G'|^2 = \tfrac{\beta'}{6}|G'|^2. \end{aligned}$$

By Theorem 1, if n (and thus also $|G'| \geq n - \lceil \varepsilon n/2 \rceil \geq \lfloor 3n/4 \rfloor$) is sufficiently large, then $F \subseteq G'$, which is a contradiction. Therefore, $|G| - |G'| < \varepsilon n/2$, and G' has minimum degree at least $(1 - 1/r - \beta)|G'|$. Let $A'_1, \ldots, A'_r$ be the partition of $V(G')$ obtained by Lemma 2 for $\varepsilon/2$. Suppose $A_i = A'_i$ for $i = 1, \ldots, r-1$ and $A_r = A'_r \cup (V(G) \setminus V(G'))$. Then

$$\sum_{i=1}^{r} \|G[A_i]\| \leq \tfrac{\varepsilon}{2} n^2 + \sum_{i=1}^{r} \|G[A'_i]\| \leq \varepsilon n^2.$$

□

Observation 5. *Let G be the complete r-partite graph with n vertices and parts $A_1, \ldots, A_r$. Then*

$$\|G\| = \Big(1 - 1/r - \sum_{i=1}^{r} (1/r - |A_i|/n)^2\Big) n^2/2.$$

Proof. For $i = 1, \ldots, r$, let $d_i = |A_i| - n/r$; we have $\sum_{i=1}^{r} d_i = 0$. Note that

$$\|G\| = \frac{1}{2} \sum_{i=1}^{r} |A_i|(n - |A_i|) = \Big(1 - \sum_{i=1}^{r} (|A_i|/n)^2\Big) n^2/2$$

and

$$\sum_{i=1}^{r} |A_i|^2 = \sum_{i=1}^{r} (d_i + n/r)^2 = \sum_{i=1}^{r} d_i^2 + n^2/r.$$

□

We now give our first application of Theorem 4.

Corollary 6. *Let F be a graph of chromatic number $r+1$, and let γ be a positive real number. Let G be an n-vertex graph with exactly $\mathrm{ex}(n; F)$ edges not containing F as a subgraph. If n is sufficiently large, then G has minimum degree at least $(1 - 1/r - \gamma)n$.*

Proof. Let $t = |F|$, $\varepsilon = \min\left(\frac{1}{8r^2}, \frac{\gamma}{5rt}\right)$ and let β' be the constant from Theorem 4 for $\varepsilon/2$; without loss of generality, we can assume $\beta' \le \varepsilon$.

For sufficiently large n, we have $\mathrm{ex}(n; F) \ge t_r(n) \ge (1 - 1/r - \beta')n^2/2$, and thus Theorem 4 implies that there exists a partition $V(G)$ to parts A_1, …, A_r such that $\sum_{i=1}^{r} \|G[A_i]\| \le \varepsilon n^2/2$. Denoting by q the number of edges of the complete r-partite graph with parts A_1, …, A_r, Observation 5 gives

$$\|G\| \le q + \varepsilon n^2/2 = \Big(1 - 1/r + \varepsilon - \sum_{i=1}^{r}(1/r - |A_i|/n)^2\Big)n^2/2,$$

and since $\|G\| = \mathrm{ex}(n; F) \ge (1 - 1/r - \beta')n^2/2$, it follows that $\sum_{i=1}^{r}(1/r - |A_i|/n)^2 \le \beta' + \varepsilon \le 2\varepsilon$. Hence, for each i we have $||A_i| - n/r| \le \sqrt{2\varepsilon}n \le \frac{n}{2r}$ and $|A_i| \ge \frac{n}{2r}$. Similarly, the total number of non-edges between different parts of the partition is at most $(\beta' + \varepsilon)n^2/2 \le \varepsilon n^2$, and thus G contains at most $\frac{n}{4r}$ vertices incident with more that $4r\varepsilon n$ such non-edges. Let A_1 be the smallest part of the partition. For sufficiently large n, there exist t vertices $x_1, \ldots, x_t \in A_1$ incident together with at most $4rt\varepsilon n$ non-edges to other parts. The set S of their common neighbors therefore has size at least $n - |A_1| - 4rt\varepsilon n \ge (1 - 1/r - \gamma)n + 1$.

If G contains a vertex v of degree less than $(1 - 1/r - \gamma)n$, consider the graph G' obtained from $G - v$ by adding a vertex u adjacent exactly to $S - v$. Then $\|G'\| > \|G\| = \mathrm{ex}(n; F)$, and thus G' contains F as a subgraph. Clearly, u belongs to this subgraph. However, at least one of x_1, …, x_t, say x_1, does not belong to this subgraph, and replacing u by x_1 gives us a subgraph of F in G. This is a contradiction, and thus G has minimum degree at least $(1 - 1/r - \gamma)n$. □