Even cycles

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From the last lecture:

Theorem 1. *If F is a bipartite graph such that all vertices in one of the parts of its bipartition have degree at most a, then*

$$\mathrm{ex}(n; F) = O(n^{2-1/a}).$$

For even cycles, this gives $\mathrm{ex}(n; C_{2k}) = O(n^{3/2})$. On the other hand, the straightforward lower bound is much lower.

Lemma 2. *For every integer $k \geq 2$,*

$$\mathrm{ex}(n; C_{2k}) = \Omega(n^{1+1/(2k-1)}).$$

Proof. Let $c = 6^{1/(1-2k)}$. Consider a random n-vertex graph G, where each pair of vertices forms an edge independently at random with probability

$$p = cn^{-\frac{2k-2}{2k-1}}.$$

For $n \geq 3$, we have

$$E[\|G\|] = p\binom{n}{2} \geq \frac{p}{3}n^2 = \frac{c}{3}n^{1+1/(2k-1)}$$

and

$$E[\text{number of } 2k\text{-cycles}] \leq n^{2k}p^{2k} = c^{2k}n^{1+1/(2k-1)} = \frac{c}{6}n^{1+1/(2k-1)}.$$

After deleting an edge from each $2k$-cycle, the graph still has $\Omega(n^{1+1/(2k-1)})$ edges left. □

Let us remark that there exist slightly better (and more explicit) constructions. Our goal is to prove a better upper bound (Bondy-Simonovits theorem). Let us start with a few lemmas.

Lemma 3. *Let H be a graph consisting of a cycle with a chord and let (A, B) be a partition of its vertices to non-empty parts such that $E(H[A]) \cup E(H[B]) \neq 0$. Then for every integer ℓ such that $1 \leq \ell \leq |H| - 1$, there exists a path in H from A to B of length exactly ℓ.*

Proof. Let $n = |H|$. Let us label the vertices of H by the elements of $\mathbb{Z}_n$ in order along the cycle and let $a : \mathbb{Z}_n \to \{0, 1\}$ be the characteristic function of the set A. Let e be the chord of the cycle of H, without loss of generality incident with the vertex 0. Let v denote the other end of e; by symmetry, we can assume $v \leq n - v$.

If the cycle $H - e$ contains paths of all lengths between 1 and $n - 1$ from A to B, then we are done. Otherwise, consider the smallest integer t such that $1 \leq t \leq n-1$ and $H-e$ does not contain a path of length t from A to B. Then $a(x) = a(x + t)$ for every $x \in \mathbb{Z}_n$, and consequently $a(x) = a(x + mt)$ for every integer m. Let $q = \gcd(t, n)$; then there exist integers m and r such that $q = mt + rn$, and thus $a(x+q) = a(x+mt+rn) = a(x+mt) = a(x)$ for every $x \in \mathbb{Z}_n$. Hence, $H - e$ does not contain a path of length q from A to B, and the minimality of t implies $t = q$. Hence, $t = \gcd(t, n)$, and t divides n. Since both A and B are non-empty, we have $t \geq 2$.

The minimality of t implies that for every $t' \in \{1, \ldots, t - 1\}$, there exists a path of length t' in $H - e$ from A to B, and thus for some $x \in \mathbb{Z}_n$ we have $a(x) \neq a(x + t')$. Since $a(x) = a(x + mt)$ for every integer m, the following claim holds.

($\star$) For every $t' \in \{1, \ldots, t - 1\}$ and any set K of t consecutive vertices of $H - e$, there exists $x \in K$ such that $a(x) \neq a(x + s)$ for every s such that $s \equiv t' \pmod{t}$.

In particular, $H - e$ contains a path from A to B of length ℓ for every $\ell \in \{1, \ldots, n - 1\}$ not divisible by t.

Consider now any $\ell \in \{1, \ldots, n - 1\}$ divisible by t. We now consider the paths containing the chord e. First, let us consider the case that $v \leq t$; we have $v \geq 2$, since e is a chord of the cycle $H - e$. By ($\star$) there exists $x \in \{0, 1, \ldots, t - 1\}$ such that $a(n - x) \neq a(n - x + s)$ for every $s \equiv v - 1 \pmod{t}$. Then $(n - x)(n - x + 1) \ldots 0v(v + 1) \ldots (\ell + v - x - 1)$ is a path from A to B of length ℓ (it is indeed a path, i.e., the vertices in the described sequence do not repeat, since $\ell + v - 1 < n$).

Therefore, we can assume $t < v < n - t$. Let us say that a path in H containing the edge e is *bent* if it contains at most one of the edges $(n - 1)0$ and $v(v+1)$ and at most one of the edges 01 and $(v-1)v$. Suppose now that H contains a bent path P of length t from A to B. By symmetry, we can assume P does not contain the edges $(n - 1)0$ and $(v - 1)v$. Let $w \in \{0, \ldots, t - 1\}$

and $z \in \{v, \ldots, v+t-1\}$ be the ends of P. If $w + \ell - t \leq v - 1$, then the concatenation of P with the path $w \ldots (w + \ell - t)$ is a path of length ℓ from A to B. Otherwise, let w' be the largest integer smaller than v such that $w' \equiv w \pmod{t}$; then the concatenation of P with the paths $w \ldots w'$ and $v \ldots (v + \ell + w - w' - t)$ is a path of length ℓ from A to B (it is indeed a path, i.e., the vertices in the described sequence do not repeat, since $v + \ell + w - w' - t = (v - w' - t) + \ell + w \leq \ell + w < n$).

Therefore, we can assume no such bent path exists, and thus

(a) for $w \in \{0, \ldots, t-1\}$ we have $a(w) = a(v + t - 1 - w)$, and

(b) for $w \in \{0, \ldots, t-1\}$ we have $a(-w) = a(v - t + 1 + w)$.

Therefore, for $w \in \{1, \ldots, t-1\}$ we have

$$a(v-1-w) = a(v+t-1-w) = a(w) = a(w-t) = a(v-t+1+(t-w)) = a(v+1-w).$$

Moreover (for $w = 0$) we have

$$a(v+1) = a(v-t+1) = a(0) = a(v+t-1).$$

Therefore, $a(x) = a(x+2)$ for $x \in \{v-t, \ldots, v-1\}$. Since this holds for t consecutive values of x, the periodicity of a implies that it holds for every $x \in \mathbb{Z}_n$. The minimality of t implies that $t = 2$ and (A, B) is a bipartition of the cycle $H - e$. By (a) we have $a(0) = a(v+1)$, and thus $a(0) \neq a(v)$ and $e \notin E(H[A]) \cup E(H[B])$. This contradicts the assumption $E(H[A]) \cup E(H[B]) \neq 0$. □

Lemma 4. *Let $k \geq 2$ be an integer. Let v be a vertex of a connected graph G. For $i \geq 0$, let V_i denote the set of vertices of G at distance exactly i from v, and let G_i denote the bipartite subgraph of G between V_i and V_{i+1}. If G does not contain a cycle of length $2k$ and $i \leq k-1$, then neither G_i nor $G[V_i]$ contains a bipartite subgraph isomorphic to a cycle of length at least $2k$ with a chord.*

Proof. Suppose for a contradiction F is such a bipartite subgraph in G_i or $G[V_i]$, and let (Y, Z) be its bipartition such that $Y \subseteq V_i$. Clearly $i \geq 1$ and $|Y| \geq 2$. For every vertex $x \in V_j$ such that $j \geq 1$, choose an arbitrary edge from x to V_{j-1} and let T denote the spanning tree of G consisting of these edges, rooted in v. Let y be the deepest vertex of T such that the subtree of T rooted in y contains Y. Let a be a child of y such that the subtree of a rooted in a contains at least one vertex of Y. Let A denote the set of vertices of Y contained in this subtree and let $B = V(F) \setminus A$. The choice of

y implies $Y \cap B \neq \emptyset$, and since (Y, Z) is a bipartition of F and $Z \subseteq B$, we have $E(F[B]) \neq \emptyset$.

Let t be the length of the paths from y to Y in T, and note that $1 \leq t \leq i \leq k-1$. By Lemma 3, F contains a path Q from A to B of length $2(k-t)$. Note that the end of Q in B belongs to Y, since Q has even length, (Y, Z) is a bipartition of G, and $A \subseteq F$. The union of Q with the paths in T from the ends of Q to y gives a cycle of length exactly $2k$ in G, which is a contradiction. □

Lemma 5. *Suppose $d \geq 3$ is an integer and G is a bipartite graph of average degree at least $2d$. Then G contains a cycle of length at least $2d$ with a chord.*

Proof. By repeatedly deleting vertices of degree less than d, we obtain a subgraph $G' \subseteq G$ of minimum degree at least d. Let $P = v_1 v_2 \dots v_m$ be a longest path in G'. Then all neighbors of v_1 belong to P and have even indices. Suppose v_a and v_b are such neighbors with the largest indices such that $a < b$. Note that $b \geq 2d$, and the cycle $v_1 \dots v_b$ has a chord $v_1 v_a$. □

Corollary 6. *Let $d \geq 3$ be an integer and let G be a graph of average degree at least $4d$. Then G contains a bipartite subgraph isomorphic to a cycle of length at least $2d$ with a chord.*

Proof. Observe G has a bipartite subgraph of average degree at least $2d$. We apply Lemma 5 to this subgraph. □

Theorem 7. *For every integer $k \geq 2$, we have*

$$\mathrm{ex}(n; C_{2k}) = O(n^{1+1/k}).$$

Proof. For $k = 2$ we know that $\mathrm{ex}(n; C_4) = \Theta(n^{3/2})$, and thus we can assume $k \geq 3$. Let H be an n-vertex graph without C_{2k} such that $\|H\| = \mathrm{ex}(n; C_{2k})$, and let $d = \frac{\mathrm{ex}(n; C_{2k})}{n}$. For a contradiction suppose that $d > 6k + 2kn^{1/k}$. The graph H has average degree $2d$, and thus it contains a connected subgraph G of minimum degree at least d.

Let v be an arbitrary vertex of G. For $i \geq 0$, let V_i denote the set of vertices of G at distance exactly i from v, and let G_i denote the bipartite subgraph of G between V_i and V_{i+1}.

For $0 \leq i \leq k-1$, Lemma 4 implies that neither G_i nor $G[V_i]$ contains a bipartite subgraph isomorphic to a cycle of length at least $2k$ with a chord. Lemma 5 and Corollary 6 imply that G_i and $G[V_i]$ have average degrees less than $2k$ and $4k$, respectively. We now prove by induction on i that for $0 \leq i \leq k-1$, we have

$$\|G_i\| < 2k|V_{i+1}|.$$

For $i = 0$, we have $\|G_i\| = \deg v = |V_{i+1}|$, and thus the claim holds. Suppose now that $i > 0$ and the claim is true for smaller values of i. Then

$$\begin{aligned} \|G_i\| &= \Big(\sum_{v \in V_i} \deg v\Big) - \|G_{i-1}\| - 2\|G[V_i]\| \\ &> d|V_i| - 2k|V_i| - 4k|V_i| = (d - 6k)|V_i| > 2k|V_i|. \end{aligned}$$

Consequently, the vertices of G_i belonging to V_i have average degree

$$\frac{\sum_{v \in V_i} \deg_{G_i}(v)}{|V_i|} = \frac{\|G_i\|}{|V_i|} > 2k.$$

Since G_i has average degree less than $2k$, the vertices of G_i belonging to V_{i+1} must have average degree less than $2k$, and thus $\|G_i\| < 2k|V_{i+1}|$.

For $i \in \{0, \dots, k-1\}$, this gives

$$(d - 6k)|V_i| < \|G_i\| < 2k|V_{i+1}|,$$

and thus

$$|V_{i+1}| > \frac{d - 6k}{2k}|V_i|,$$

and

$$n \geq |V_k| > \Big(\frac{d - 6k}{2k}\Big)^k.$$

Therefore $d \leq 6k + 2kn^{1/k}$, which is a contradiction. □