Turán problem for hypergraphs

Zdeněk Dvořák

January 2, 2021

By Turán's theorem, we know that $\overline{\mathrm{ex}}(\infty; K_{k+1}) = 1 - 1/k$. For hypergraphs, the situation is much more complicated, and we do not know the exact answer even for the very simplest cases. For $2 \le r \le k$, let $K_k^{(r)}$ denote the complete r-uniform hypergraph with k vertices. Let us start with giving a lower bound for $\overline{\mathrm{ex}}(\infty; K_4^{(3)})$. For a directed graph $\vec{G}$, let $H(\vec{G})$ be the 3-uniform hypergraph with the same vertex set such that a triple A is a hyperedge iff $\vec{G}[A]$ has no isolated vertices and maximum outdegree at most one.

Lemma 1. *If $\vec{G}$ does not contain an induced directed 4-cycle, then $K_4^{(3)} \not\subseteq H(\vec{G})$. Moreover, if $3|n$, the underlying undirected graph of $\vec{G}$ is $K_{n/3,n/3,n/3}$, and every vertex of $\vec{G}$ has indegree and outdegree $n/3$, then*

$$\|H(\vec{G})\| = \binom{n}{3} - 3\binom{n/3}{3} - n\binom{n/3}{2} = \frac{5}{9}\binom{n}{3} + O(n^2).$$

Proof. Consider any $X \subseteq V(\vec{G})$ of size four. If the underlying undirected graph of $\vec{G}[X]$ has a vertex v of degree at most one, then v and two of its non-neighbors form a non-hyperedge of $H(\vec{G})$. If $\vec{G}[X]$ has outdegree at least two, then v and two of its outneighbors form a non-hyperedge of $H(\vec{G})$. Otherwise, note that the sum of outdegrees of $\vec{G}[X]$ is equal to the number of edges of $\vec{G}[X]$, which in turn is equal to half the sum of degrees of the underlying undirected graph of $\vec{G}[X]$; hence, every vertex v of $\vec{G}[X]$ must be incident with two edges, with exactly one leaving v. However, this would imply $\vec{G}[X]$ is a directed 4-cycle, contradicting the assumptions. We conclude that $H(\vec{G})[X]$ is not $K_4^{(3)}$.

Suppose now that the underlying undirected graph of $\vec{G}$ is $K_{n/3,n/3,n/3}$ and every vertex of $\vec{G}$ has outdegree $n/3$. Consider a non-hyperedge A of $H(\vec{G})$. If a vertex of $\vec{G}[A]$ is isolated, then since the underlying undirected graph is complete multipartite, A is an independent set in $\vec{G}$. Otherwise, by the definition of $H(\vec{G})$, exactly one vertex of A has outdegree two in $\vec{G}[A]$. Hence,

$$\|H(\vec{G})\| = \binom{n}{3} - 3\binom{n/3}{3} - n\binom{n/3}{2} = \binom{n}{3} - \frac{1}{9}\binom{n}{3} - \frac{1}{3}\binom{n}{3} + O(n^2) = \frac{5}{9}\binom{n}{3} + O(n^2).$$

$\square$

Note such orientations exist (you can orient the edges cyclically between the three parts, but there are many other possible orientations). Hence, we have $\overline{\mathrm{ex}}(\infty; K_4^{(3)}) \geq \frac{5}{9}$. It has been conjectured that $\overline{\mathrm{ex}}(\infty; K_4^{(3)}) = \frac{5}{9}$. However, proving this is complicated by the fact that there is not just a unique hypergraph achieving this bound (Lemma 1 enables us to construct many such hypergraphs), and thus any proof would have to avoid describing a unique extremal hypergraph. This also poses a difficulty for the flag algebra method, as these examples have different densities of various induced subhypergraph, and all of them would have to give the same optimal value to the semidefinite program.

To obtain an upper bound, we can use Moon-Moser inequalities. For a hypergraph G and an integer s, let $N_s(G)$ denote the number of complete subhypergraphs of G with s vertices. Note that if G is r-uniform, then $N_r(G) = \|G\|$. Moreover, $N_{r-1}(G) = \binom{|G|}{r-1}$.

Lemma 2. *For any n-vertex r-uniform hypergraph G and an integer $s \geq r$, if $N_{s-1}(G) > 0$, then*

$$N_{s+1}(G) \geq \frac{s^2 N_s(G)}{(s-r+1)(s+1)} \left(\frac{N_s(G)}{N_{s-1}(G)} - \frac{(r-1)(n-s)+s}{s^2} \right).$$

Proof. For a set $S \subseteq V(G)$, let $d(S)$ denote the number of copies of $K^{(r)}_{|S|+1}$ in G that contain S.

Consider any $K \subseteq V(G)$ inducing a complete hypergraph of size s, and let p be the number of copies of $K_s^{(r)}$ in G with exactly $s-1$ vertices contained in K. Clearly

$$p = \sum_{K' \in \binom{K}{s-1}} (d(K') - 1) = -s + \sum_{K' \in \binom{K}{s-1}} d(K').$$

On the other hand, if for some $v \in V(G) \setminus K$, the subhypergraph induced by $K \cup \{v\}$ is not complete, then there exists $X \subseteq K$ of size $r-1$ such that $X \cup \{v\}$ is not a hyperedge, and any copy of $K_s^{(r)}$ including v and with $s-1$

vertices contained in K must be obtained from $K \cup \{v\}$ by deleting a vertex of X. Hence,

$$p \le (r-1)(n-s) + (s-r+1)d(K).$$

Combining these inequalities, we conclude that any $K \subseteq V(G)$ inducing a complete hypergraph of size s satisfies

$$d(K) \ge \frac{-((r-1)(n-s)+s) + \sum_{K' \in \binom{K}{s-1}} d(K')}{s-r+1}.$$

Therefore,

$$\begin{aligned}
N_{s+1}(G) &= \frac{\sum_{K \in \binom{V(G)}{s}, G[K] \text{ complete}} d(K)}{s+1} \\
&\ge \frac{\sum_{K \in \binom{V(G)}{s}, G[K] \text{ complete}} \Big(-((r-1)(n-s)+s) + \sum_{K' \in \binom{K}{s-1}} d(K')\Big)}{(s-r+1)(s+1)} \\
&= \frac{\Big(\sum_{K \in \binom{V(G)}{s}, G[K] \text{ complete}} \sum_{K' \in \binom{K}{s-1}} d(K')\Big) - ((r-1)(n-s)+s)N_s(G)}{(s-r+1)(s+1)} \\
&= \frac{\Big(\sum_{K' \in \binom{V(G)}{s-1}, G[K'] \text{ complete}} d^2(K')\Big) - ((r-1)(n-s)+s)N_s(G)}{(s-r+1)(s+1)} \\
&\ge \frac{\Big(\sum_{K' \in \binom{V(G)}{s-1}, G[K'] \text{ complete}} d(K')\Big)^2 / N_{s-1}(G) - ((r-1)(n-s)+s)N_s(G)}{(s-r+1)(s+1)} \\
&= \frac{\big(sN_s(G)\big)^2 / N_{s-1}(G) - ((r-1)(n-s)+s)N_s(G)}{(s-r+1)(s+1)} \\
&= \frac{s^2 N_s(G)}{(s-r+1)(s+1)} \Big(\frac{N_s(G)}{N_{s-1}(G)} - \frac{(r-1)(n-s)+s}{s^2}\Big).
\end{aligned}$$

□

For $n \ge s \ge r$, let

$$\begin{aligned}
F(n,s,r) &= \frac{1}{r}\Big(n-r+1-\frac{n-s+1}{\binom{s-1}{r-1}}\Big)\binom{n}{r-1} \\
&= \Big(1-\frac{(n-s+1)}{(n-r+1)\binom{s-1}{r-1}}\Big)\binom{n}{r} = \Big(1-\frac{1}{\binom{s-1}{r-1}}\Big)\binom{n}{r} + O(n^{r-1}).
\end{aligned}$$

Note that $F(n,r,r)=0$ and $F(n,s,r)$ is increasing in s.

Lemma 3. *Let G be an n-vertex r-uniform hypergraph. For $s \geq r$, if $N_{s-1}(G) > 0$, then*

$$N_s(G) \geq N_{s-1}(G)\frac{r^2\binom{s}{r}}{s^2\binom{n}{r-1}}(\|G\| - F(n,s,r)).$$

Proof. We prove the claim by induction on s. If $s = r$, we have

$$N_{s-1}(G)\frac{r^2\binom{s}{r}}{s^2\binom{n}{r-1}}(\|G\| - F(n,s,r)) = \|G\| = N_r(G).$$

Suppose now that the claim holds for $s \geq r$, and let us show it also holds for $s+1$. Indeed, using Lemma 2 and the induction hypothesis, we have

$$\begin{aligned}
\frac{N_{s+1}(G)}{N_s(G)} &\geq \frac{s^2}{(s-r+1)(s+1)}\Big(\frac{N_s(G)}{N_{s-1}(G)} - \frac{(r-1)(n-s)+s}{s^2}\Big) \\
&\geq \frac{s^2}{(s-r+1)(s+1)}\Big(\frac{r^2\binom{s}{r}}{s^2\binom{n}{r-1}}(\|G\| - F(n,s,r)) - \frac{(r-1)(n-s)+s}{s^2}\Big) \\
&= \frac{r^2\binom{s}{r}}{(s-r+1)(s+1)\binom{n}{r-1}}(\|G\| - F(n,s,r)) - \frac{(r-1)(n-s)+s}{(s-r+1)(s+1)} \\
&= \frac{r^2\binom{s+1}{r}}{(s+1)^2\binom{n}{r-1}}\Big(\|G\| - F(n,s,r) - \frac{((r-1)(n-s)+s)\binom{n}{r-1}}{r^2\binom{s}{r}}\Big).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&F(n,s,r) + \frac{((r-1)(n-s)+s)\binom{n}{r-1}}{r^2\binom{s}{r}} \\
&\quad= \Big(\frac{n-r+1}{r} - \frac{n-s+1}{r\binom{s-1}{r-1}} + \frac{(r-1)(n-s)+s}{r^2\binom{s}{r}}\Big)\binom{n}{r-1} \\
&\quad= \Big(\frac{n-r+1}{r} - \frac{s(n-s+1)}{r(s-r+1)\binom{s}{r-1}} + \frac{(r-1)(n-s)+s}{r(s-r+1)\binom{s}{r}}\Big)\binom{n}{r-1} \\
&\quad= \Big(\frac{n-r+1}{r} - \frac{s(n-s+1)-(r-1)(n-s)-s}{r(s-r+1)\binom{s}{r-1}}\Big)\binom{n}{r-1} \\
&\quad= \Big(\frac{n-r+1}{r} - \frac{n-s}{r\binom{s}{r-1}}\Big)\binom{n}{r-1} = F(n,s+1,r),
\end{aligned}$$

and thus

$$\frac{N_{s+1}(G)}{N_s(G)} \geq \frac{r^2\binom{s+1}{r}}{(s+1)^2\binom{n}{r-1}}(\|G\| - F(n,s+1,r)),$$

as required. □

If the number of hyperedges is larger than $F(n, s, r)$, this gives a lower bound on the number of appearances of $K_s^{(r)}$ in G.

Corollary 4. *Let G be an n-vertex r-uniform hypergraph. For $s \geq r$, if $\|G\| > F(n, s, r)$, then*

$$N_s(G) \geq \Big(\prod_{k=r}^{s} \frac{r^2\binom{k}{r}}{k^2}\Big) \frac{1}{\binom{n}{r-1}^{s-r}} \prod_{k=r}^{s} (\|G\| - F(n, k, r)) > 0.$$

Hence,

$$\overline{\mathrm{ex}}(\infty; K_s^{(r)}) \leq 1 - \frac{1}{\binom{s-1}{r-1}}.$$

For graphs (the case $r = 2$), this gives $\overline{\mathrm{ex}}(\infty; K_s) \leq 1 - \frac{1}{s-1}$, and thus we obtain another proof of Turán's theorem. For $K_4^{(3)}$, we obtain

$$\overline{\mathrm{ex}}(\infty; K_3^{(4)}) \leq \frac{2}{3} = \frac{5}{9} + \frac{1}{9},$$

and thus this bound is not tight. A better (but still not tight) bound

$$\overline{\mathrm{ex}}(\infty; K_3^{(4)}) \leq 0.561666$$

was obtained by Razborov using flag algebras.