Nowhere-zero flows

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1 Introduction

Let A be a finite Abelian group. A function to A is *nowhere-zero* if its range is a subset of $A \setminus \{0\}$.

Definition 1. *An A-*flow *in a graph G is a function f assigning to directed edges of G elements of A such that $f(uv) = -f(vu)$ for all $uv \in E(G)$ and for every $v \in V(G)$, we have*

$$\sum_{uv \in E(G)} f(uv) = 0.$$

For $S \subseteq V(G)$, let $f(S) = \sum_{u \in S, v \in V(G) \setminus S} f(uv)$.

Observation 1. *If f is an A-flow, then $f(S) = 0$ for all $S \subseteq V(G)$.*

Proof. We have

$$f(S) = \sum_{u \in S, uv \in E(G)} f(uv) = 0.$$

□

Corollary 2. *If G has a nowhere-zero A-flow, then G is bridgeless.*

For a walk $W = v_1 v_2 \ldots v_k$ and a function t from directed edges to A, let

$$t(W) = \sum_{i=1}^{k-1} t(v_i v_{i+1}).$$

Definition 2. *An A-*tension *in a graph G is a function t assigning to directed edges of G elements of A such that $t(uv) = -t(vu)$ for all $uv \in E(G)$ and $t(C) = 0$ for every cycle $C \subseteq G$.*

Lemma 3. *If t is an A-tension in a graph G and W is a closed walk, then $t(W) = 0$.*

Proof. Let H be the directed graph whose edges are exactly the edges of W directed along this closed walk, taken with multiplicities. The graph H is Eulerian, and thus it can be expressed as union of edge-disjoint directed cycles $C_1, \ldots, C_k$. Then $t(W) = t(C_1) + \ldots + t(C_k) = 0$. □

Lemma 4. *Let G be a connected graph. For a proper coloring φ of G by elements of A, let t_φ be defined by $t_\varphi(uv) = \varphi(v) - \varphi(u)$ for each $uv \in E(G)$. Then t_φ is a nowhere-zero A-tension. Conversely, if t is a nowhere-zero A-tension, then there exist exactly $|A|$ proper colorings ψ of G by elements of A such that $t = t_\psi$.*

Proof. For any closed walk $W = v_1 \ldots v_k$ (with $v_k = v_1$), we have

$$t_\varphi(W) = \sum_{i=1}^{k-1} (\varphi(v_{i+1}) - \varphi(v_i)) = 0,$$

since the contributions of the consecutive terms cancel out. Hence, t_φ is an A-tension, and it is nowhere-zero since φ is proper.

Conversely, fix any vertex $v_0 \in V(G)$, and for each $v \in V(G)$, let W_v denote any walk from v_0 to v in G. Let a be any element of A, and define $\psi_a(v) = a + t(W_v)$. For any edge $uv \in E(G)$, consider the closed walk consisting of W_v, the edge vu, and the reversal of W_u; we have

$$\psi_a(v) - \psi_a(u) = t(W_v) - t(W_u) = t(W) - t(vu) = t(uv),$$

and thus ψ_a is a proper coloring of G by elements of A such that $t = t_{\psi_a}$. Furthermore, if ψ is a proper coloring of G by elements of A and $t = t_\psi$, it is easy to see that $\psi = \psi_a$ for $a = \psi(v_0)$. □

Definition 3. *Let G be a plane graph, let uv be an edge of G, and let gh be the corresponding edge of the dual $G^\star$ of G, such that in the drawing of G, g is drawn to the left of uv (when looking from u in the direction of this edge). For any function f assigning values to directed edges of G, let us define $f^\star(gh) = f(uv)$.*

Lemma 5. *Let G be a connected plane graph. A function t assigning elements of A to directed edges of G is an A-tension iff $t^\star$ is an A-flow in $G^\star$.*

Proof. Consider any vertex g of $G^\star$. The incident edges of $G^\star$ correspond to the facial walk W_g of the face g of G, and thus if t is an A-tension, then $t^\star(\{g\}) = t(W_g) = 0$ by Lemma 3, and thus $t^\star$ is an A-flow.

Consider any cycle C in G, and let S be the set of faces G drawn inside C. If $t^\star$ is an A-flow, then $t(C) = t^\star(S) = 0$ by Observation 1, and thus t is an A-tension. □

Corollary 6. *The number of proper A-colorings of a connected plane graph G is equal to $|A|$ times the number of nowhere-zero A-flows in $G^\star$.*

Lemma 7. *Let G be a plane triangulation with no loops, and let $G^\star$ be its dual (a plane 3-regular bridgeless graph). Then G is 4-colorable iff $G^\star$ is 3-edge-colorable.*

Proof. The graph G is 4-colorable iff it has a proper coloring by elements of Z_2^2. By Corollary 6, this is the case iff $G^\star$ has a nowhere-zero Z_2^2-flow. However, a nowhere-zero function $f : E(G^\star) \to Z_2^2$ is a Z_2^2-flow iff for each $g \in V(G^\star)$, the three edges incident with g have different values, i.e., iff $G^\star$ has a proper edge coloring by the non-zero elements of Z_2^2. □

Corollary 8. *The following claims are equivalent:*

- *Every planar graph is 4-colorable.*
- *Every planar 3-regular bridgeless graph is 3-edge-colorable.*

2 Basic properties of nowhere-zero flows

Let $\chi^\star(G, A)$ denote the number of nowhere-zero A-flows of G.

Lemma 9. *Let e be an edge of G. If e is a loop, then $\chi^\star(G, A) = (|A| - 1)\chi^\star(G - e, A)$. If e is not a loop, then $\chi^\star(G, A) = \chi^\star(G/e, A) - \chi^\star(G - e, A)$.*

Proof. If e is a loop, then a nowhere-zero A-flow in $G - e$ extends to a nowhere-zero A-flow in G by setting its value on e to an arbitrary non-zero element of A, and conversely the restriction of a nowhere-zero A-flow in G to $E(G) \setminus \{e\}$ is a nowhere-zero A-flow in $G - e$, justifying the first claim.

If e is not a loop, then note that any A-flow f' in G/e extends to an A-flow f in G in unique way by setting the value on e so that the flow conservation law holds on both ends of e; and conversely, restriction of an A-flow in G to $E(G) \setminus \{e\}$ is an A-flow in G/e. Furthermore, if f' is nowhere-zero, then f is nowhere-zero everywhere except possibly on e. Finally, note that the A-flows in G whose value is 0 exactly on e are in 1-to-1 correspondence with nowhere-zero flows in $G - e$. Consequently, $\chi^\star(G, A) = \chi^\star(G/e, A) - \chi^\star(G - e, A)$. □

From this, we get the following by induction on the number of edges (and noting that an edgeless graph has exactly one nowhere-zero A-flow).

Corollary 10. *If A_1 and A_2 are finite Abelian groups of the same size, then $\chi^\star(G, A_1) = \chi^\star(G, A_2)$ for every graph G. In particular, a graph has a nowhere-zero A_1-flow iff it has a nowhere-zero A_2-flow.*

Hence, we will say that G has a *nowhere-zero k-flow* if it has a nowhere-zero A-flow for some Abelian group of size k.

Corollary 11. *Let G be a graph and $\{e_1, e_2\}$ be an edge-cut in G. Then $\chi^\star(G, A) = \chi^\star(G/e_1, A)$.*

Proof. By Lemma 9, we have $\chi^\star(G, A) = \chi^\star(G/e_1, A) - \chi^\star(G - e_1, A)$. However, $G - e_1$ has a bridge e_2, and thus $\chi^\star(G - e_1, A) = 0$. □

Let f be an A-flow, a an element of A, and C a directed cycle. Let $f + aC$ denote the flow obtained from f by increasing the value on edges of C by a, i.e., $(f + aC)(uv) = f(uv)$ if $uv \notin E(C)$, $(f + aC)(uv) = f(uv) + a$ if $uv \in E(C)$, and $(f + aC)(uv) = f(uv) - a$ if $vu \in E(C)$.

Lemma 12. *If T is a spanning tree of a connected graph G, then G has an A-flow which is zero only on a subset of edges of T.*

Proof. For every $e \in E(G) \setminus E(T)$, let C_e be directed cycle consisting of e and the path in T joining the ends of e. Let a be a non-zero element of A. Then $\sum_{e \in E(G) \setminus E(T)} eC_e$ is an A-flow in G and $f(e) = a \neq 0$ for every $e \in E(G) \setminus E(T)$. □

3 Existence of nowhere-zero flows

Since the Petersen graph is 3-regular and not 3-edge-colorable, it has no Z_2^2-flow. Tutte gave the following conjectures (the second one implies the Four Color Theorem, the third one implies Grötzsch' theorem).

Conjecture 1. 5-flow conjecture *Every bridgeless graph has a nowhere-zero 5-flow.*

4-flow conjecture *Every bridgeless graph not containing the Petersen graph as a minor has a nowhere-zero 4-flow.*

3-flow conjecture *Every 4-edge-connected graph has a nowhere-zero 3-flow.*

We use the following well-known result.

Theorem 13 (Nash-Williams)**.** *For any integer k, an $2k$-edge-connected graph has k pairwise edge-disjoint spanning trees.*

Theorem 14. *Every 4-edge-connected graph has a nowhere-zero 4-flow.*

Proof. A 4-edge-connected graph G has two edge-disjoint spanning trees T_1 and T_2. For $i = 1, 2$, let f_i be a Z_2-flow in G which is zero only on a subset of $E(T_i)$. Then $f(uv) = (f_1(uv), f_2(uv))$ is a nowhere-zero Z_2^2-flow in G. $\square$

Theorem 15. *Every bridgeless graph has a nowhere-zero 8-flow.*

Proof. By Corollary 11, it suffices to prove this is the case for a 3-edge-connected graph G. Let G' be obtained from G by doubling each edge; then G' is 6-edge-connected, and thus it has three pairwise edge-disjoint spanning trees T_1, T_2, and T_3. Each edge of G is contained in at most two of these spanning trees. For $i = 1, 2, 3$, let f_i be a Z_2-flow in G which is zero only on a subset of $E(T_i)$. Then $f(uv) = (f_1(uv), f_2(uv), f_3(uv))$ is a nowhere-zero Z_2^3-flow in G. $\square$

Lemma 16. *Let G be a 3-connected graph. Then there exists a partition V_1, …, V_k of vertices of G such that for $i = 1, \ldots, k$,*

- *either $|V_i| = 1$ or $G[V_i]$ has a Hamiltonian cycle, and*
- *if $i \geq 2$, then there exist at least two edges with one end in V_i and the other end in $V_1 \cup \ldots \cup V_{i-1}$.*

Proof. Choose V_1 consisting of an arbitrary vertex of G. For $i \geq 2$, let B be a leaf 2-connected block of $G_i = G - (V_1 \cup \ldots \cup V_{i-1})$. Since G is 3-connected and at most one vertex separates B from the rest of G_i, there exist at least two edges e_1 and e_2 from B to $V_1 \cup \ldots \cup V_{i-1}$. If e_1 and e_2 are incident with the same vertex v of B, we set $V_i = \{v\}$. Otherwise, since B is 2-connected, there exists a cycle C in B containing the endpoints of these two edges, and we set $V_i = V(C)$. $\square$

Theorem 17. *Every bridgeless graph has a nowhere-zero 6-flow.*

Proof. By Corollary 11, we can assume that G is 3-edge-connected. Furthermore, we can assume that G has maximum degree at most three (a vertex v with neighbors v_1, …, v_k can be replaced by a cycle $w_1w_2 \ldots w_k$ and edges w_iv_i for $1 \leq i \leq k$, and a flow in the resulting graph can be transformed into a flow in G by contracting the cycle back into a single vertex). Hence, G is 3-connected. Let V_1, …, V_k be a partition of the vertex set of G as in Lemma 16. For $m = 1, \ldots, k$, let Q_m be the union of Hamiltonian cycles of graphs $G[V_i]$ such that $1 \le i \le m$ and $|V_i| > 1$. For $m = 2, \ldots, k$, let H_m be the subgraph of G with vertex set $V_1 \cup \ldots \cup V_m$ that for $2 \le i \le m$ contains exactly two edges with one end in V_i and the other end in $V_1 \cup \ldots \cup V_{i-1}$. Observe that $H_m \cup Q_m$ is connected and there exists a cycle $C_m \subseteq H_m \cup Q_m$ containing both edges from V_m to $V_1 \cup \ldots \cup V_{m-1}$. Let f_0 be a Z_2-flow in G whose value is 1 on edges of Q_k and zero everywhere else.

Let T be a spanning tree of $Q_k \cup H_k$; note that T is also a spanning tree of G. Let f_k be a Z_3-flow in obtained by Lemma 12, whose values are zero only on a subset of edges of $H_k \cup Q_k$. We now define Z_3-flows $f_{k-1}, \ldots, f_1$, such that f_i can only have value zero on edges of $H_i \cup Q_k$. Assuming f_{i+1} was already constructed, we consider the three flows f_{i+1}, $f_{i+1} + C_i$, $f_{i+1} + 2C_i$. Note that one of these three flows is non-zero on both edges of H_{i+1} between V_{i+1} and $V_1 \cup \ldots \cup V_i$; we select this flow as f_i. Since all the edges whose values differ in f_{i+1} and f_i belong to $H_{i+1} \cup Q_k$, we conclude inductively that f_i can only have zeros on dges of $H_i \cup Q_k$.

Consequently, the Z_3-flow f_1 has only zeros on the edges of Q_k, and thus (f_0, f_1) is a nowhere-zero $(Z_2 \times Z_3)$-flow in G. □

4 3-colorings of quadrangulations

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