Density of critical graphs and critical graphs on surfaces

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1 Density of critical graphs

Recall a *Gallai tree* is a connected graph whose 2-connected blocks are cliques and odd cycles. A *Gallai forest* is a graph whose components are Gallai trees. From the last lecture:

Theorem 1. *Let $c \ge 3$ be an integer. If G is a $(c+1)$-critical graph, then $\delta(G) \ge c$ and the vertices of degree c induce a Gallai forest in G.*

We also need a bound on the number of edges of a Gallai forest.

Lemma 2. *Let $c \ge 3$ be an integer. If H is a Gallai forest of maximum degree at most c not containing K_{c+1}, then*

$$\|H\| \le \Big(\frac{c-1}{2} + \frac{1}{c}\Big)|H|.$$

Proof. We prove the claim by induction on the number of vertices of H. Clearly, we can assume that H is connected. If H is 2-connected, then H is either an odd cycle or a clique of size at most c, and thus its average degree is at most $c-1$, implying the inequality.

Hence, suppose that H is not 2-connected. Let H_1' be a leaf block of H. If $c \ge 4$ and H_1' is an odd cycle, or if H_1' is a clique of size at most $c-1$, then let $H_1 = H_1'$. Otherwise, H_1' is $(c-1)$-regular, and since the maximum degree of H is at most c and H is a leaf block, there exists exactly one block H_1'' intersecting H_1' and $H_1'' = K_2$; let $H_1 = H_1' \cup H_1''$. In either case,

$$\|H_1\| \le \Big(\frac{c-1}{2} + \frac{1}{c}\Big)(|H_1| - 1). \tag{1}$$

Let H_2 be the subgraph of H such that $H = H_1 \cup H_2$ and H_1 intersects H_2 in exactly one vertex. Then H_2 is a Gallai forest and by induction, we have

$$\|H_2\| \leq \Big(\frac{c-1}{2} + \frac{1}{c}\Big)|H_2|. \tag{2}$$

Since $|H| = (|H_1| - 1) + |H_2|$ and $\|H\| = \|H_1\| + \|H_2\|$, the inequality of the lemma follows by summing (1) and (2). □

We now can give a lower bound on the density of critical graphs.

Theorem 3. *Let $c \geq 3$ be an integer. If G is a $(c+1)$-critical graph and $G \neq K_{c+1}$, then G has average degree at least*

$$c + \frac{c-2}{c^2 + 2c - 2}.$$

Proof. Let S be the set of vertices of G of degree c. By Theorem 1 and Lemma 2, $G[S]$ has at most

$$\Big(\frac{c-1}{2} + \frac{1}{c}\Big)|S|$$

edges. Note that $c|S|$ is the number of edges of G incident with vertices of S, counting those in $G[S]$ twice. Hence,

$$\|G\| \geq c|S| - \|G[S]\| \geq \Big(\frac{c+1}{2} - \frac{1}{c}\Big)|S|. \tag{3}$$

All vertices of G not in S have degree at least $c+1$, and thus

$$2\|G\| \geq (c+1)|G| - |S|. \tag{4}$$

Multiplying (4) by $\big(\frac{c+1}{2} - \frac{1}{c}\big)$ and adding it to (3) gives

$$\Big(c + 2 - \frac{2}{c}\Big)\|G\| \geq \Big(\frac{c+1}{2} - \frac{1}{c}\Big)(c+1)|G|,$$

and thus

$$\frac{c^2 + 2c - 2}{c}\|G\| \geq \frac{c^3 + 2c^2 - c - 2}{2c}|G|,$$

and the average degree of G is

$$\frac{2\|G\|}{|G|} \geq \frac{c^3 + 2c^2 - c - 2}{c^2 + 2c - 2} = c + \frac{c-2}{c^2 + 2c - 2}.$$

□

2 Graphs on surfaces

Recall the bound on the average degree of embedded graphs.

Lemma 4. *For every surface Σ and integer $k \geq 3$, there exists a constant $C_{k,\Sigma} \geq 0$ such that any graph G of girth at least k embedded in Σ has average degree at most*

$$\frac{2k}{k-2} + C_{k,\Sigma}/|G|.$$

Combining Lemma 4 and Theorem 3, we have the following.

Corollary 5. *Let Σ be a surface and let $k \geq 3$ and $c \geq \max\left(3, \frac{2k}{k-2}\right)$ be integers. If a graph G is $(c+1)$-critical, $G \neq K_{c+1}$, G has girth at least k and G can be embedded in Σ, then*

$$|G| \leq \frac{c^2 + 2c - 2}{c - 2} C_{k,\Sigma}.$$

Recall c-colorability of a graph of girth at least k embedded in Σ can be verified by testing the presence of all $(c+1)$-critical graphs of girth at least k embedded in Σ. Corollary 5 shows that the number of such critical graphs is bounded when $c \geq 6$, or when $c, k \geq 4$, or when $c \geq 3$ and $k \geq 6$, newly giving us the polynomial-time algorithm marked in red in the following table.

girth \ colors	3	4	5	6	≥7
3 (general)	NP-hard	?	P	P	P
4 (triangle-free)	P	P	P	P	P
5	P	P	P	P	P
≥6	P	P	P	P	P

The blue entries (5-colorablility of embedded graphs and 3-colorability of embedded graphs of girth at least 5) follow by a much more complicated argument showing that the number of relevant critical graphs is bounded. The green entry (3-colorability of triangle-free embedded graphs) is even more complicated thanks to the fact that the number of critical graphs is not bounded (but an argument that sufficiently restricts their structure is known). The last entry (4-colorability of embedded graphs) is open; it is known there are infinitely many 5-critical graphs embeddable in any surface other than the sphere.

3 Triangulations

Let G be a triangulation of an orientable surface, and let φ be its 4-coloring. For distinct $i, j, k \in [4]$, let n_{ijk} denote the number of faces of G whose vertices have colors i, j, k in clockwise cyclic order. Let $t_{ijk} = n_{ijk} - n_{jik}$. For a face f of G, let $\delta_{ijk}(f) = 1$ if vertices of f have colors i, j, k in clockwise cyclic order, $\delta_{ijk}(f) = -1$ if vertices of f have colors j, i, k in clockwise cyclic order, and $\delta_{ijk}(f) = 0$ otherwise. Clearly,

$$t_{ijk} = \sum_{f \in F(G)} \delta_{ijk}(f).$$

For a set F' of faces of G, let

$$t_{ijk}(F') = \sum_{f \in F'} \delta_{ijk}(f).$$

Lemma 6. *Let $i, j, k, l \in [4]$ be distinct colors and let $F' \subseteq F(G)$ be a set of faces of G such that every edge of G whose ends have colors i and j is incident with either 0 or 2 faces of F'. Then $t_{ijk}(F') = t_{jil}(F')$.*

Proof. By symmetry, it suffices to consider only the case $i = 1$, $j = 2$, $k = 3$. Let E_{12} be the set of edges of G with one vertex colored 1 and the other one 2 that are incident with two faces of F'. For $e \in E_{12}$, let f_e and f'_e be the two faces of F' incident with e. Since each face of F' on which colors 1, 2, and 3 appear (in any order) is incident with exactly one edge of E_{12}, we have

$$t_{123}(F') = \sum_{e \in E_{12}} (\delta_{123}(f_e) + \delta_{123}(f'_e)),$$

and similarly

$$t_{214}(F') = \sum_{e \in E_{12}} (\delta_{214}(f_e) + \delta_{214}(f'_e)).$$

Hence, to prove $t_{123}(F') = t_{214}(F')$, it suffices to show that

$$\delta_{123}(f_e) + \delta_{123}(f'_e) = \delta_{214}(f_e) + \delta_{214}(f'_e)$$

for every edge $e \in E_{12}$. This follows by a straightforward case analysis. □

Note that applying Lemma 6 with $F' = F(G)$ gives $t_{123} = t_{142} = t_{134} = t_{243}$.

Lemma 7. *For distinct $i, j, k \in [4]$, the parity of t_{ijk} is the same as the parity of the number of vertices of G of odd degree colored by i.*

Proof. By symmetry, it suffices to prove the claim for t_{123} and the color 1. Let V_1 be the set of vertices of G of color 1. For $v \in V_1$, let $F(v)$ denote the set of faces incident with v. Since each face on which colors 1, 2, and 3 appear (in any order) is incident with exactly one vertex of V_1, we have

$$t_{123} = \sum_{v \in V_1} t_{123}(F(v)).$$

Hence, it suffices to show that the parity of $t_{123}(F(v))$ is the same as the parity of $\deg(v)$ for all $v \in V_1$. This is the case, since

$$\begin{aligned}\deg(v) &= \sum_{f \in F(v)} |\delta_{123}(f)| + |\delta_{142}(f)| + |\delta_{134}(f)| \\ &\equiv t_{123}(F(v)) + t_{142}(F(v)) + t_{134}(F(v)) \pmod 2,\end{aligned}$$

and by Lemma 6 we have $t_{123}(F(v)) = t_{142}(F(v)) = t_{134}(F(v))$. □

Corollary 8. *If G is a triangulation of an orientable surface containing exactly two vertices u and v of odd degree, then u and v have the same color in any 4-coloring of G. In particular, if u is adjacent to v, then G is not 4-colorable.*

Proof. If say u had color 1 and v color 2, then t_{123} is odd by Lemma 7 with $i = 1$, $j = 2$, $k = 3$, and even by the same lemma with $i = 3$, $j = 1$, $k = 2$, which is a contradiction. □

4 Quadrangulations

Let G be a quadrangulation of a non-orientable surface. Choose an orientation of each facial cycle of G (independently—this cannot be done consistently, by the non-orientability of the surface). Let D denote the directed graph with vertex set $V(G)$ and uv being an edge of D if and only if uv is an edge of G oriented towards v in both facial cycles of G that contain it. Let $p(G) = |E(D)| \bmod 2$. Note that $p(G)$ is independent of the choice of the orientations, since reversing an orientation of a 4-face with d edges belonging to D changes $|E(D)|$ by $4 - 2d \equiv 0 \pmod 2$.

Consider a 3-coloring of G. For an edge uv and distinct colors $i, j \in [3]$, let $\omega_{ij}(uv) = 1$ if u has color i and v has color j, $\omega_{ij}(uv) = -1$ if u has color j and v has color i, and $\omega_{ij}(uv) = 0$ otherwise. For an oriented cycle $C = v_1 \ldots v_k$ of G, let $w_{ij}(C) = \sum_{t=1}^{k} \omega_{ij}(v_t v_{t+1})$, where v_{k+1} stands for v_1. Note that similarly to Lemma 6, we have $w_{12}(C) = w_{23}(C) = w_{31}(C)$. For a face f of G, let $w(f)$ be defined as $w_{ij}(C_f)$ for the chosen orientation of the facial cycle C_f of f, for any $(i,j) \in \{(1,2),(2,3),(3,1)\}$.

By considering all 3-colorings of a 4-cycle, we conclude that $w(f) = 0$ for all $f \in F(G)$, and thus

$$\sum_{f \in F(G)} w(f) = 0.$$

On the other hand,

$$\begin{aligned}\sum_{f \in F(G)} w(f) &= \sum_{f \in F(G)} \sum_{e \in E(C_f)} \omega_{ij}(e) \\ &= 2 \sum_{e \in E(D)} \omega_{ij}(e).\end{aligned}$$

We conclude that

$$\sum_{e \in E(D)} (\omega_{12}(e) + \omega_{23}(e) + \omega_{31}(e)) = 0.$$

Since $|\omega_{12}(e) + \omega_{23}(e) + \omega_{31}(e)| = 1$ for all $e \in E(D)$, we have

$$\sum_{e \in E(D)} (\omega_{12}(e) + \omega_{23}(e) + \omega_{31}(e)) \equiv |E(D)| \equiv p(G) \pmod 2.$$

Consequently, we have the following.

Lemma 9. *If G is a quadrangulation of a non-orientable surface and $p(G) = 1$, then G is not 3-colorable.*

Corollary 10. *If G is a non-bipartite quadrangulation of the projective plane, then G is not 3-colorable.*

Proof. Let C be an odd cycle in G. Since G is a quadrangulation, C is non-orientable, and cutting along C turns the projective plane into a disk. Orient the faces of G clockwise in this disk. Defining D as before, we have $E(D) = E(C)$, and thus $p(G) = 1$. Lemma 9 shows that G is not 3-colorable. $\square$