3^{RD} TUTORIAL ON RANDOMIZED ALGORITHMS

Complexity classes and eigenvalues

1. *Classes of randomized algorithms.* You have seen that ZPP = RP $\cap$ co-RP.

a) Show that RP $\subseteq$ NP (and thus co-RP $\subseteq$ co-NP).
b) Show that if NP $\subseteq$ BPP then NP=RP.

2. Let $A \in \mathbb{R}^{n\times n}$ be a matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. Show that the matrix $A + dI_n$ has eigenvalues $d + \lambda_1, \dots, d + \lambda_n$.

3. Show Courant-Fisher: Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix ($A^T = A$). Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be its eigenvalues. Show that $\lambda_1 = \max_{x\in\mathbb{R}^n, \|x\|=1} x^T Ax$.
(Similarly, $\lambda_n = \min_{x\in\mathbb{R}^n, \|x\|=1} x^T Ax$ and for example, $\lambda_2 = \max_{x\in\mathbb{R}^n, \|x\|=1, x^T u_1=0} x^T Ax$, where u_1 is the eigenvector corresponding to λ_1.)

4. Show that a connected d-regular graph is bipartite iff the least eigenvalue of its adjacency matrix is $-d$.

5. Compute the eigenvalues and eigenvectors of the following graphs:

a) K_n, the complete graph on n vertices.
b) $K_{n,n}$, the complete bipartite graph with partites of size n each.