INTRO TO APPROXIMATION, CLASS 5

LPs are back and **rounding** is coming together with them

EXERCISE ONE Consider the problem of GRAPH BALANCING, where you get an undirected graph G with weights on the edges $p\colon E(G) \to \mathbb{R}^+$. Our goal is to make the graph directed so that the most loaded vertex (the vertex with the most weight directed towards it) has minimum possible load. Formally we seek an orientation of the edges which minimizes the goal function $u = \max_{v \in V} \sum_{e \in E; e \text{ directed to } v} p(e)$.

Suggest and analyze a 2-approximation algorithm for the problem. (Tip: Linear programming might come in handy.)

EXERCISE TWO We now consider MAX DICUT. On the input we get a directed graph $G = (V, \vec{E})$ and a non-negative weight function on the edges. Our task is to find a subset of vertices S so that $\vec{E}(S, V \setminus S)$ (the edges directed from S to the rest) have maximum possible weight.

Suggest a probabilistic $\frac{1}{4}$-approximation algorithm for MAX DICUT. This should be fairly easy, and it will not require linear programming.

EXERCISE THREE Let us try to improve on our algorithm for MAX DICUT:

1. Suggest a natural $\{0, 1\}$-integer program solving MAX DICUT.
2. Choose each vertex v_i with probability $1/4 + x_i^*/2$, where x_i^* is the optimum of the linear relaxation of the previous integer program. Show that it is a 1/2-approximation.

EXERCISE FOUR Recall the integer program for MAX SAT and its linear relaxation. During the lecture you have seen a 3/4-approximation algorithm for MAX SAT based on choosing a better of two solutions, one of which was created by rounding a solution of this relaxation. By a better rounding one can avoid choosing a better of two solutions and still maintain the approximation ratio 3/4.

Find an instance, that is, a set of clauses, such that the optimum of the relaxation OPT_r and the optimum of the instance OPT satisfies $\mathrm{OPT} = (3/4)\mathrm{OPT}_r$.

This shows that using the linear relaxation one cannot obtain a better than 3/4-approximation algorithm. (The worst case ratio between OPT_f and OPT is called the *integer gap*.)

Hint: you can use, for example, just 2 variables and 4 clauses.

EXERCISE FIVE In the MAXIMUM COVERAGE problem you get as input a number n – the size of universe $U = \{1, 2, \ldots, n\}$, and also a list of m subsets $S_1, S_2, \ldots, S_m$ of the universe (thus $S_i \subseteq U$). You also get $k \leq m$.

The task is to choose k subsets S_i such that their union covers as many elements of U as possible.

1. Formulate MAXIMUM COVERAGE as integer linear program.
2. Design a randomized approximation algorithm for the problem based rounding the solution of the linear program. Be aware that rounding must choose at most k sets.
   Then analyze this rounding and get the approximation ratio. (This time you don't know what ratio should come out — which is mostly the case.)