INTRO TO APPROXIMATION, CLASS 1

probability, a computer scientist's tool

Exercise one We flip a fair coin eight times; for a fair coin we get heads with probability exactly 0.5. Find the probability of the following events (an expression containing powers, sums, binomial coefficient etc. is enough):

a) The number of flips in which we get heads is the same as the number of tails.
b) There are more heads than tails.
c) The ith flip and the $(9-i)$th flip are the same for $i = 1, \ldots, 4$.
d) We flip at least four consecutive heads. (Hint: first try a lower number of flips, that is, $4, 5, \ldots$ and finally 8 flips.)
e) * How can we count the probability of having four consecutive heads in n flips?

Exercise two Alice and Bob want to determine who will pay the tickets to the cinema. Alice has three weird dices (the dices are fair, so each of the six sides comes up with probability 1/6):

I with numbers 1, 1, 6, 6, 8, 8,
II with numbers 2, 2, 4, 4, 9, 9,
III with numbers 3, 3, 5, 5, 7, 7,

Alice suggests the following: Both pick one die (not the same one) and roll it, the one who gets the lowest number looses and will buy the tickets. Moreover, she lets Bob to choose a die first.

Help Bob to show that Alice has a higher probability to win in this game. In particular, show the following:

a) Suppose that Bob chooses die I, then Alice gets die II and the probability that Alice gets the biggest number is more than 0.5. Try to list all possible events that can occur.
b) If Bob chooses die II and Alice die III, the probability that Alice wins is again more than 0.5.
c) Finally, suppose that Bob picks die III and then Alice die I. Then Alice gets a higher number again with probability more than 0.5.

In conclusion, Bob cannot choose a die that is not "dominated" by another die.

Exercise three Suppose you have a biased coin for which heads come up with an *unknown* probability p. Simulate a flip of a fair coin using a few flips of the biased coin. What is the expectation of the number of coin-flips?

Exercise four Now we want to do the contrary: Simulate a biased coin with the probability of heads equal to p using a fair coin. In each of the following cases try to calculate the expected number of coin-flips (or an upper bound on it).

a) Start with the case in which $p = \ell/2^k$ for some $k, \ell \in \mathbb{N}$.
b) Then extend the previous algorithm to work for any rational p.
c) Finally, come up with a very efficient algorithm for an arbitrary p, even irrational. You can start by trying to decrease the number of flips in the first case.

Exercise five A bonus exercise: **Card tricks.** We have 52 cards, half red, half black, randomly shuffled (a uniformly random permutation). We now reveal one card after another. You (as the algorithm) have two options: either say "I want the top card" – you win if it is red, you lose if it is black, and the game ends after one guess – or say "Keep showing me cards" – and then we reveal the next top card in the deck, allowing you to guess on the next card.

We are interested in the best algorithm, that is one maximizing the probability that it wins.

1. What is the probability of winning for the algorithm $Fi \equiv$ "always guess on the first card"? And what is the probability for $La \equiv$ "always guess on the last card"?
2. Show that the probability of the next card being red is not always the same value – that the sequence of already revealed cards actually changes the probability of the next card being red.
3. Our main and final task: find an algorithm which chooses the red card with probability strictly more than $1/2$ – or prove that no such algorithm exists.