Relative Error Streaming Quantiles

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Joint work in progress with Graham Cormode (Warwick), Zohar Karnin (Amazon), Edo Liberty (HyperCube), and Justin Thaler (Georgetown)

Selection Problem & Streaming

- Input: N numbers
- Goal: find the k-th smallest
  - e.g.: the median, 99th percentile
- $\mathcal{O}(N)$ time offline algorithm [Blum *et al.* '73]

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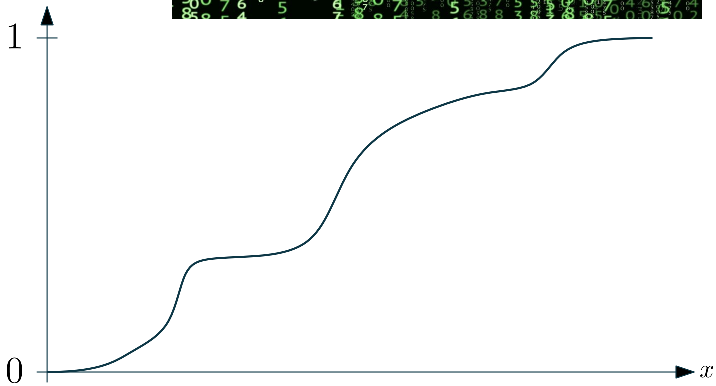

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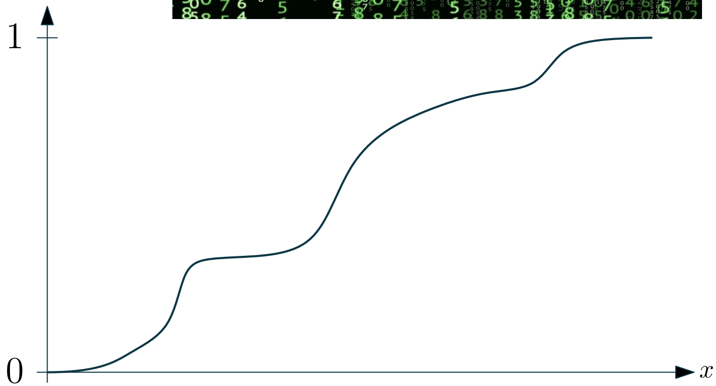

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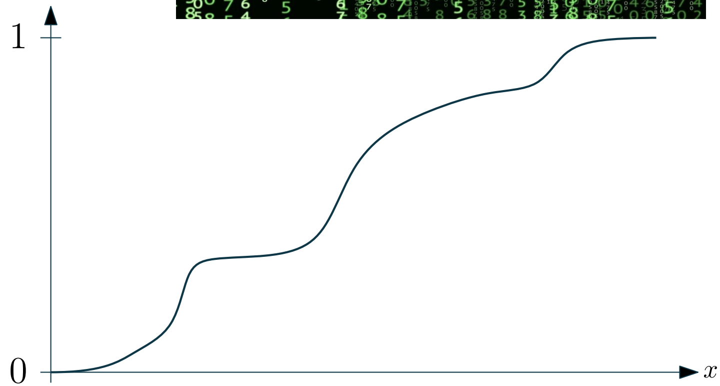

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ϕ-quantile = $\lceil \phi \cdot N \rceil$-th smallest element ($\phi \in [0, 1]$)

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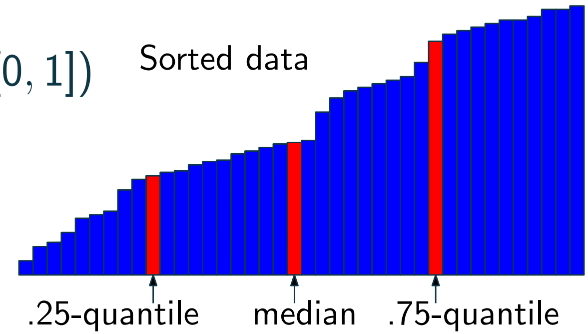

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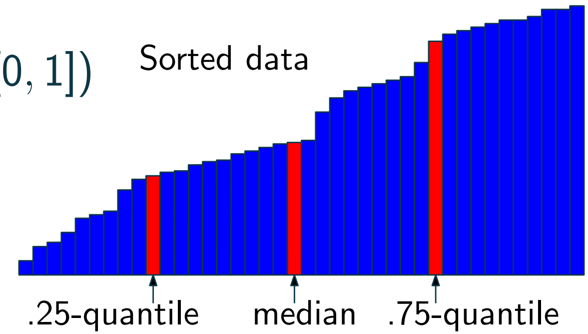

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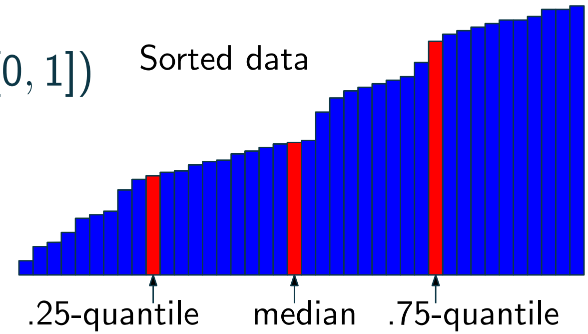

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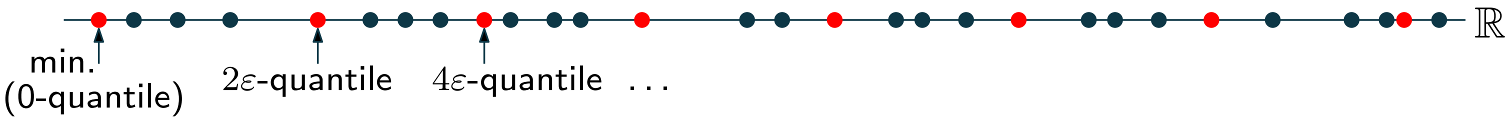

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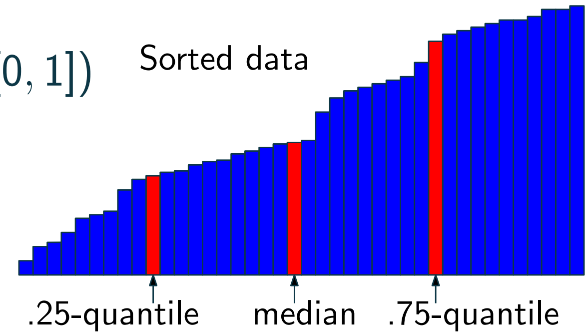

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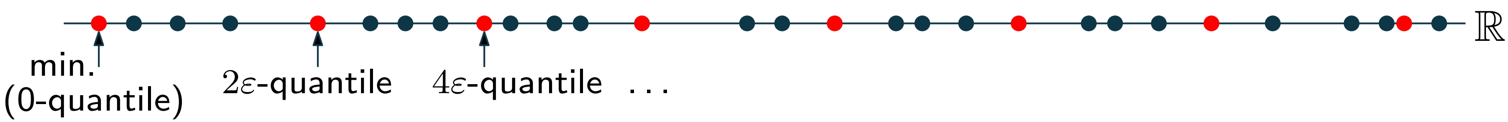

Very well-solved both in theory & practice:

- Deterministic algs.: space $\Theta\left(\frac{1}{\varepsilon} \cdot \log \varepsilon N\right)$ optimal [Greenwald & Khanna '01, Cormode, **V.** '20]
- Randomized algs.: space $\Theta\left(\frac{1}{\varepsilon}\right)$ optimal (w/ const. probability of too high error) [Karnin *et al.* '16]

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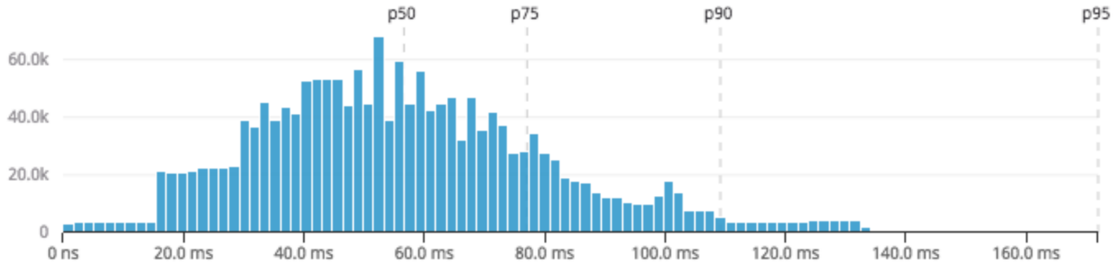

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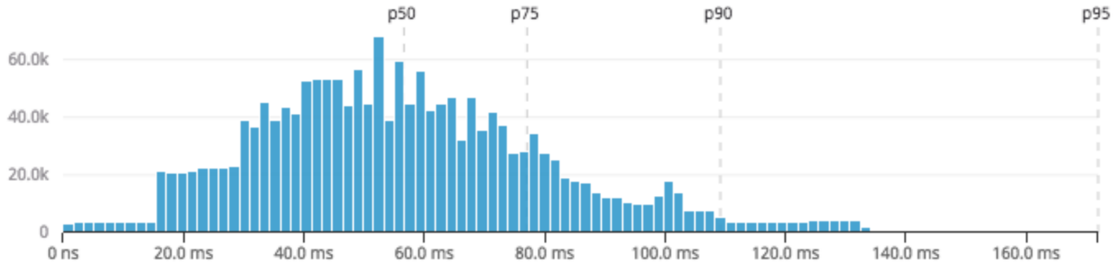

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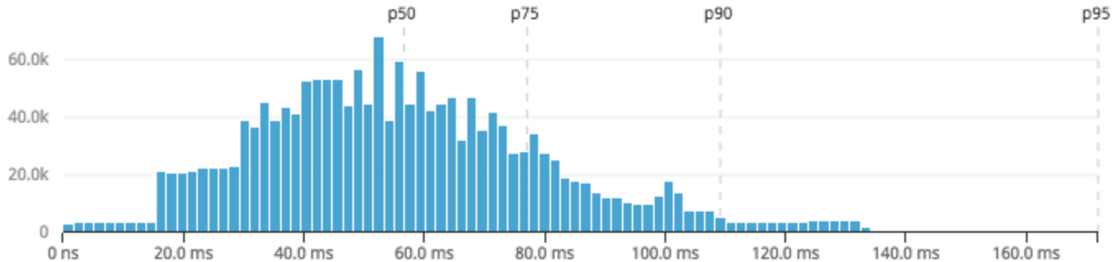

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Can we have a stronger error guarantee?

Can we understand the **tails** of the distribution better?

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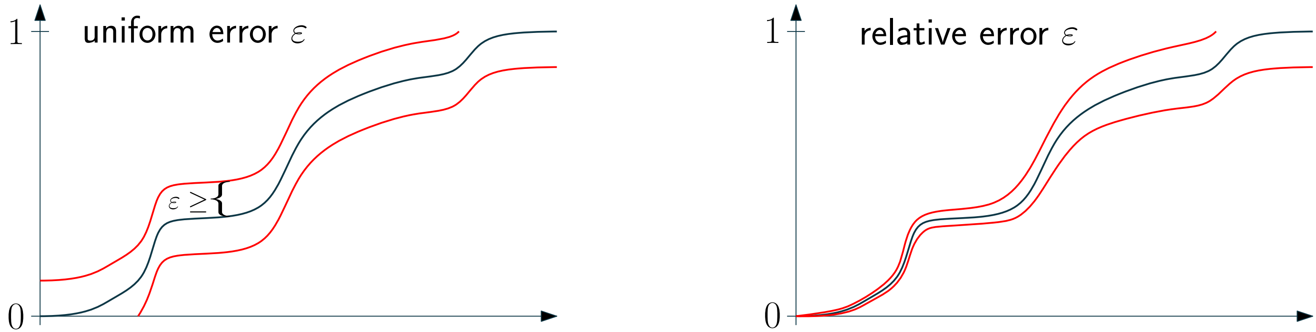

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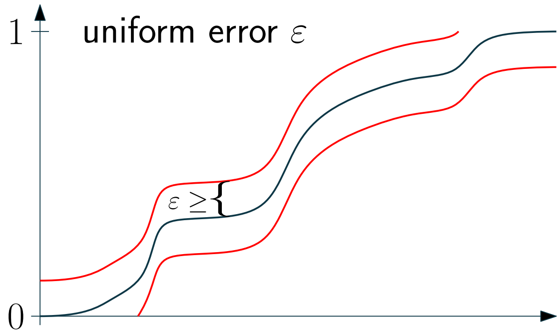

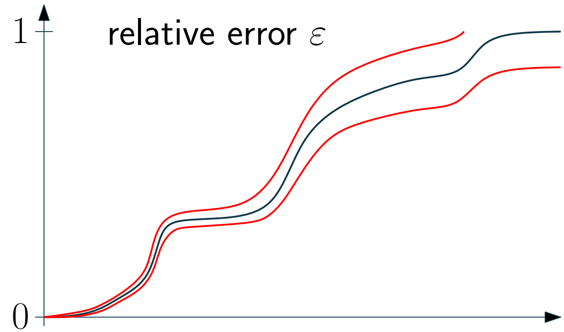

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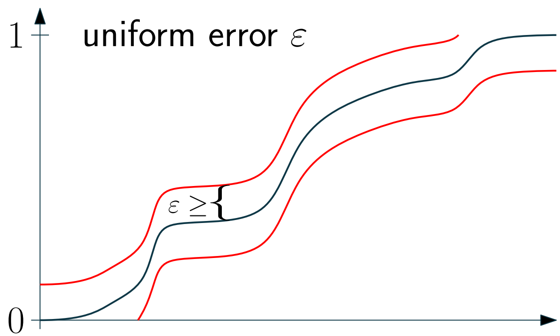

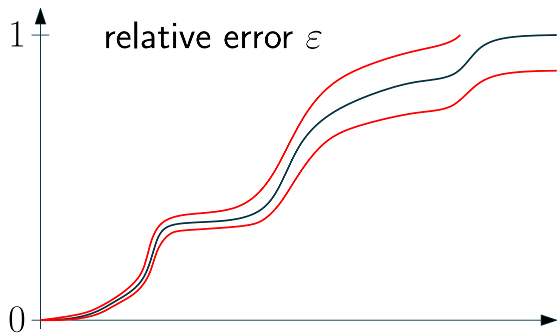

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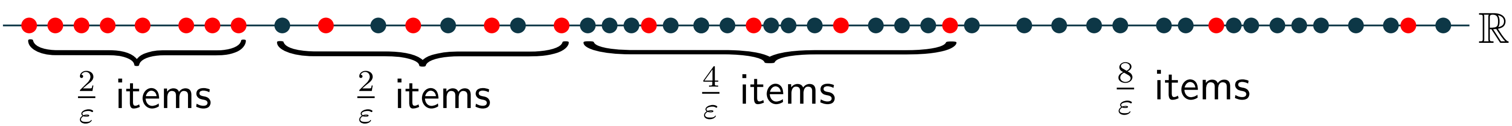
- example for $\varepsilon = 0.25$:



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Used in: [Manku *et al.* '99, Agarwal *et al.* '12, Wang *et al.* '13, Karnin *et al.* '16]

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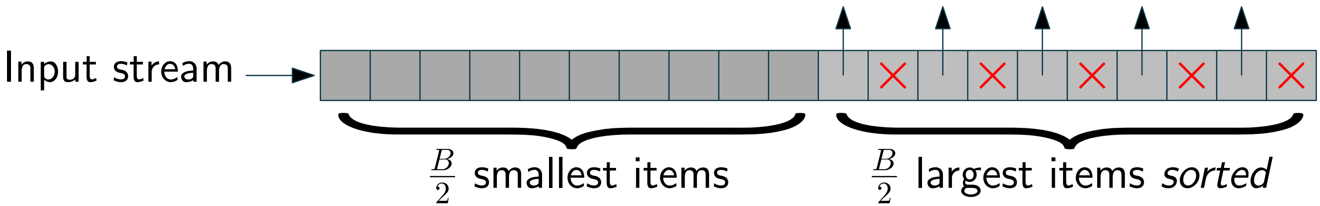

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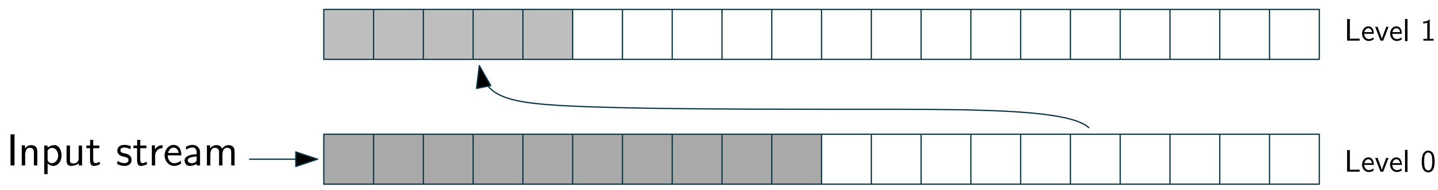

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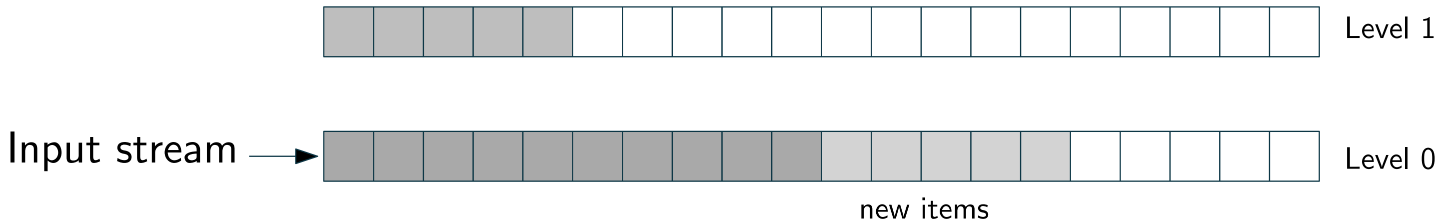

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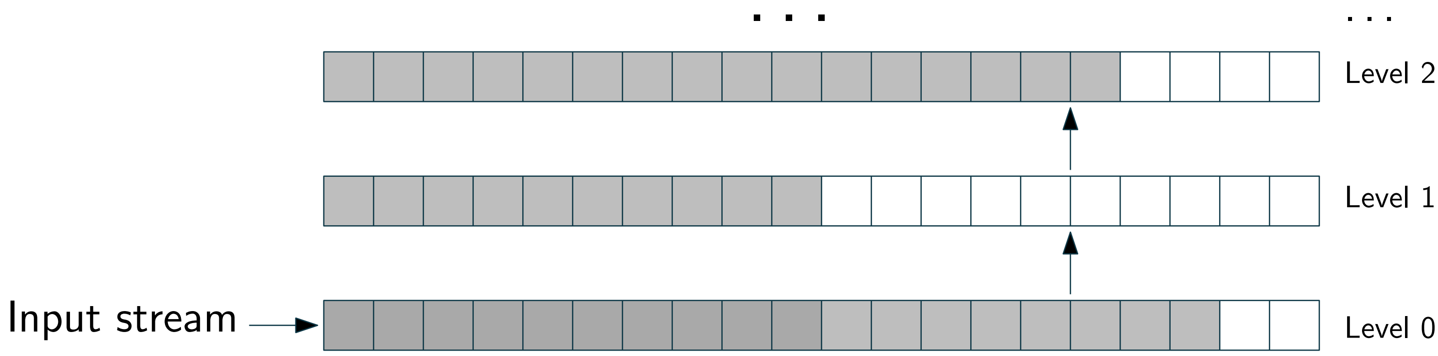

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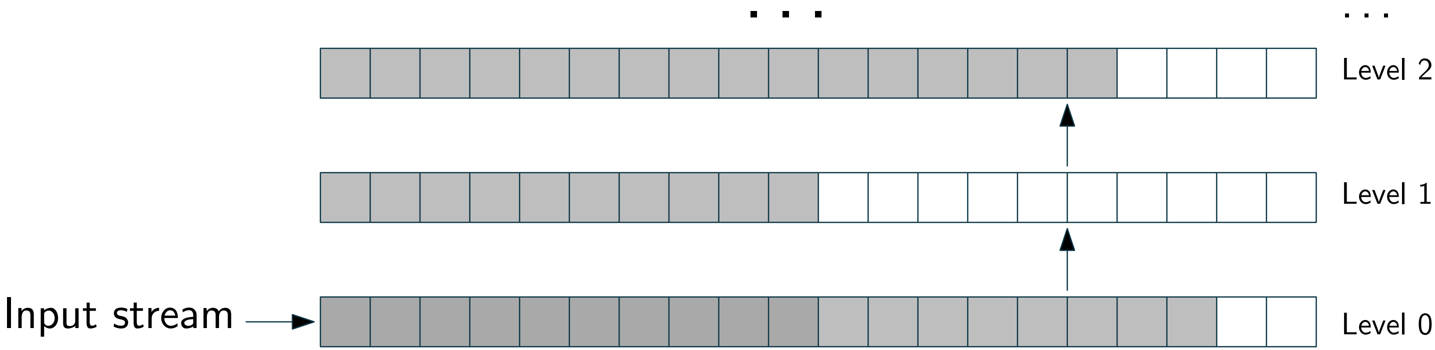

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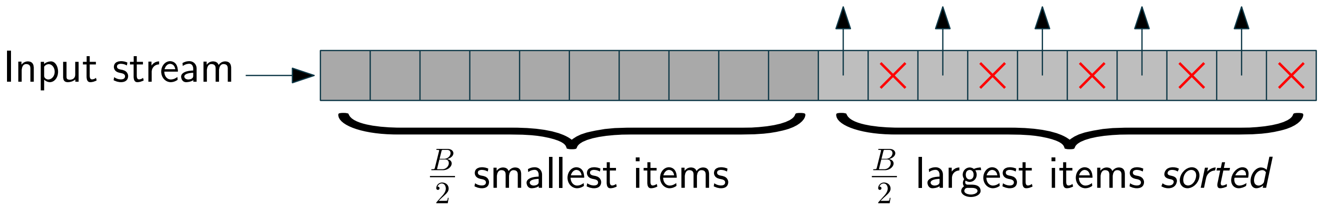

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- Highest level $H(y)$ affecting the error for y satisfies $2^{H(y)} \leq \mathcal{O}(R(y)/B)$

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- A compaction at level h adds $\pm 2^h$ to the error

  iff odd number of items $x \leq y$ compacted
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  - rank of y decreases by factor of ~ 2 at every level
- Highest level $H(y)$ affecting the error for y satisfies $2^{H(y)} \leq \mathcal{O}(R(y)/B)$

Variance of $\mathrm{Err}(y) \leq \sum_{h=0}^{H(y)} 2^{2h} \frac{R(y)}{2^h} \leq 2^{H(y)} R(y) \leq \frac{R(y)^2}{B}$ (up to constant factors)

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For $\text{Err}(y) \leq \varepsilon \cdot R(y)$ w/ const. probability, we need $\text{Var}[\text{Err}(y)] \leq \varepsilon^2 R(y)^2$

$\Rightarrow$ need to choose $B \sim \dfrac{1}{\varepsilon^{\mathbf{2}}}$ 

Relative compactor

Compaction affecting the error should remove k items $x \leq y$ on average

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	6	5	4	3	2	1	0

$B/2$ slots (never compacted) $\lceil \log_2(n/k) \rceil$ sections with k slots each

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- Choose $k = \dfrac{1}{\varepsilon \cdot \sqrt{\log(\varepsilon N)}}$, so that $\mathrm{Var}(\mathrm{Err}(y)) \leq \varepsilon^2 R(y)^2$
- Then $B = \mathcal{O}\left(\frac{1}{\varepsilon} \cdot \sqrt{\log(\varepsilon N)}\right)$ and $O(\log(\varepsilon N))$ levels $\Rightarrow$ space $\mathcal{O}\left(\frac{1}{\varepsilon} \cdot \log^{\mathbf{1.5}} \varepsilon N\right)$

Relative Error: Conclusions

Randomized sketch of size $\mathcal{O}\left(\frac{1}{\varepsilon} \cdot \log^{\mathbf{1.5}} \varepsilon N\right)$ (const. probability of error)

- $\sqrt{\log(1/\delta)}$ dependence on failure probability δ

Lower bound $\Omega\left(\frac{1}{\varepsilon} \cdot \log \varepsilon N\right) \Rightarrow$ gap $\sqrt{\log(\varepsilon N)}$

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- Handling unknown stream lengths
- Mergeability, and more
- Python code at GitHub

More: paper *Relative Error Streaming Quantiles* at arXiv (to be updated till WOLA)

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Questions welcomed at Slack or `pavel.vesely@warwick.ac.uk`