Ackermann-type funcs

x \ y	0	1	2	3	4	5	6
0	0	2	4	6	8	10	12
1	0	2	4	8	16	32	64
2	0	2	4	2^{16}	$2^{2^{16}}$	…	
3	0	2	4				
4	0	2	4				
5	0	2	4				
⋮	⋮	⋮	⋮				

$2^{2^{2^{\cdot^{\cdot^{\cdot}}}}}\Big\}y$

$2 \uparrow y$

$A(x) = A(x,x)$
Diag. A.f.

Union-Find

$A(0,y) := 2y$

$A(x,y) := A(x-1, A(x,y-1))$

$A(1,y) = 2^y$
$A(2,y) = 2 \uparrow y$

$\lambda_i(n) := \min\{y \mid A(i,y) > \log n\}$

$\lambda_0(n) = \Theta(\log n)$
$\lambda_1(n) = \Theta(\log\log n)$
$\lambda_2(n) = \Theta(\log^* n)$

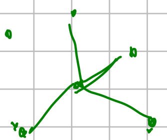
$\alpha(n) := \min\{y \mid A(y) > \log n\}$

$\alpha(m,n) := \min\{x \geq 1 \mid A(x, 4\lceil m/n \rceil) > \log n\}$

U.-F. $O(\alpha(m,n))$ — #op., #vertices

$m \geq n \cdot \lambda_i(n) \Rightarrow \alpha(m,n) \leq i$

for $m \geq n \cdot \log^* n \rightarrow \alpha(m,n) \in O(1)$



LCA on PM … $O(\underbrace{n + \overbrace{\#\text{queries}}^{q}}_{m})$

micro / macro decomp.
μ / M

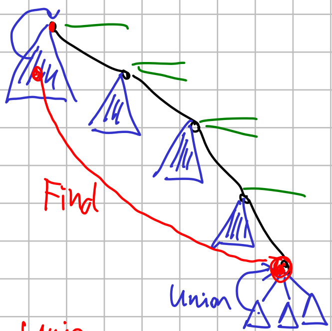
$\lambda_i(n)$

$g :=$ μ-tree size limit

can't on PM — on RAM: $O(n)$ preproc., $O(1)$ LCA queries

v is a μ-tree root $\equiv |T(v)| \leq g$ & $|T(p(v))| > g$

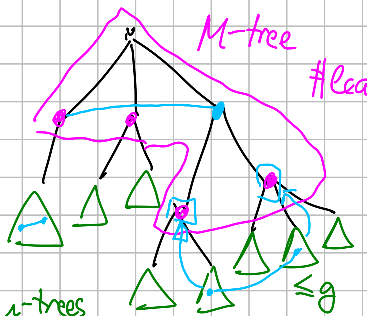
Run DFS



$\sim n$ Unions
q Finds

$O(m \cdot \alpha(m,n))$

$O(m \cdot \alpha(m, \frac{n}{g}))$
$\leq O(m)$

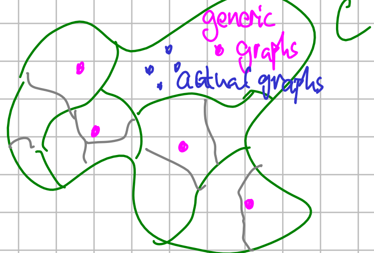


1 query on T → O(1) queries in M, μ

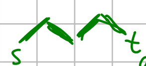
U-F with special elts.
n vertices
ℓ of them special
Every U. involves at least 1 special vertex
⇒ + ordinary comp. is trivial
U-F run in $O(\alpha(m,\ell))$ per op.
redirect ord. → spec.
& U-F on special

• Topological Graph Computations

C coll. of graphs
$\|C\|$



Input: Labelled graph of size g
Output: Diff. labelling of the same graph



$G \xrightarrow{\cong} H$
$\downarrow$ $\qquad$ $\downarrow$
$\mathrm{out}(G) \xrightarrow{f} \mathrm{out}(H)$

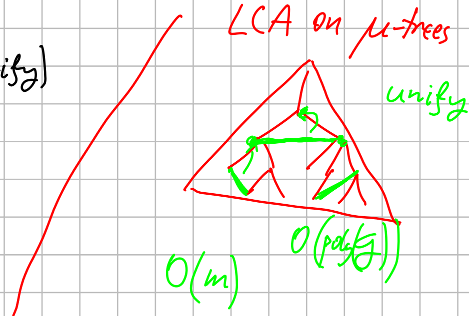
$\|C\| \ggg g$
$O(\|C\|)$

$A(i,g)$
$\sim 2^{2^g}$ $\quad 2 \uparrow g$

$$\begin{array}{ccc} \text{actual graph} & \xrightarrow{\text{canonical enc.}} & \text{generic graph (unify)} \\ \downarrow & & \downarrow \text{solve} \\ \text{actual sol.} & \longleftarrow & \text{generic sol.} \end{array}$$

$O(\|C\| + \cancel{f(g)})$
$O(n)$

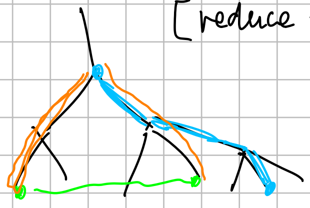
LCA on μ-trees



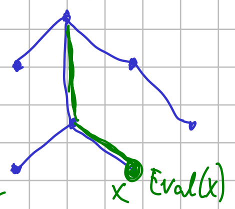
Thm: We solved offline LCA on PM in $O(n+q)$

Path Maxima $\xrightarrow{\text{Komlós's alg}}$ $O(n+q)$ comparisons

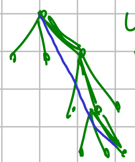
[reduce to "vertical" paths]



T a weighted tree (comparison oracle)

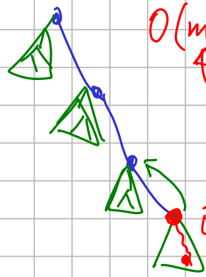


Macro-tree : Link-Eval



Union by rank
Path compr.

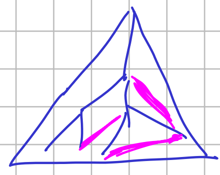
$O(\alpha(m,n))$



$O(m \cdot \alpha(m,n))$ ← $n+q$

μ-trees

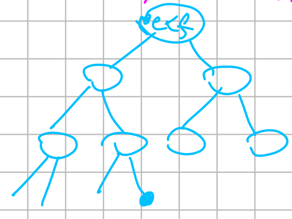
$\leq g$



$q \leq g^2$

Decision Tree for path maxima

$O(g^2)$



$g! \cdot \mathrm{poly}(g)$, $\leq g^g$, $\leq A(i,g)$

$O(g + q \text{ in } \mu\text{-tree})$

$O(n+q)$ for path maxima

even on PM { MST can be verified in $O(m)$ time
KKT finds the MST in expected $O(m)$ time