Complex numbers: Component form

Definition: $\mathbb{C} = \{a + b\,\mathbf{i} \mid a, b \in \mathbb{R}\}$.

Addition: $(a + b\,\mathbf{i}) \pm (p + q\,\mathbf{i}) = (a \pm p) + (b \pm q)\,\mathbf{i}$.

Multiplication: $(a + b\,\mathbf{i}) \cdot (p + q\,\mathbf{i}) = ap + (aq + bp)\,\mathbf{i} + bq\,\mathbf{i}^2 =$
$= (ap - bq) + (aq + bp)\,\mathbf{i}$.

For $\alpha \in \mathbb{R}$: $\alpha(a + b\,\mathbf{i}) = \alpha a + \alpha b\,\mathbf{i}$.

Conjugation: $\overline{a + b\,\mathbf{i}} = a - b\,\mathbf{i}$.

Properties: $\overline{\overline{x}} = x$, $\overline{x \pm y} = \overline{x} \pm \overline{y}$, $\overline{x \cdot y} = \overline{x} \cdot \overline{y}$, $x \cdot \overline{x} \in \mathbb{R}$.

Absolute value: $|x| = \sqrt{x \cdot \overline{x}}$, so $|a + b\,\mathbf{i}| = \sqrt{a^2 + b^2}$.

For $\alpha \in \mathbb{R} : |\alpha x| = |\alpha| \cdot |x|$.

Division: $x/y = (x \cdot \overline{y})/(y \cdot \overline{y})$.

Complex numbers: Gauss plane and goniometric form

A geometric view of $\mathbb{C}$:

- Numbers correspond to points in $\mathbb{R}^2$: $a + b\,\mathbf{i} \leftrightarrow (a, b)$.
- $|x|$ is the distance from point $(0, 0)$.
- $|x| = 1$ for numbers on the unit circle (*complex units*).

Goniometric form:

- For complex units: $x = \cos\varphi + \mathbf{i}\sin\varphi$ for some $\varphi \in [0, 2\pi)$.
- Generally: $x = |x| \cdot (\cos\varphi(x) + \mathbf{i}\sin\varphi(x))$.

The number $\varphi(x) \in [0, 2\pi)$ is called the *argument* of the number x (denoted by $\arg x$).

Furthermore $\varphi(\overline{x}) = -\varphi(x)$.

Complex numbers: Exponential form

Euler's formula: $\mathrm{e}^{i\varphi} = \cos\varphi + \mathbf{i}\sin\varphi$.

Each $x \in \mathbb{C}$ can be expressed as $|x| \cdot \mathrm{e}^{\mathbf{i}\cdot\varphi(x)}$.

Multiplication:
$xy = \left(|x| \cdot \mathrm{e}^{\mathbf{i}\cdot\varphi(x)}\right) \cdot \left(|y| \cdot \mathrm{e}^{\mathbf{i}\cdot\varphi(y)}\right) = |x| \cdot |y| \cdot \mathrm{e}^{\mathbf{i}\cdot(\varphi(x)+\varphi(y))}$.
(absolute values are multiplied, arguments added)

Powers: $x^\alpha = \left(|x| \cdot \mathrm{e}^{\mathbf{i}\cdot\varphi(x)}\right)^\alpha = |x|^\alpha \cdot \mathrm{e}^{\mathbf{i}\alpha\varphi(x)}$.

Roots: $\sqrt[n]{x} = |x|^{1/n} \cdot \mathrm{e}^{\mathbf{i}\cdot\varphi(x)/n}$.
Roots are not unique: $1^4 = (-1)^4 = \mathbf{i}^4 = (-\mathbf{i})^4 = 1$.

Complex numbers: Roots of one

If $x \in \mathbb{C}$ is a n-th root of one, we have:

- $|x| = 1$, so $x = \mathrm{e}^{\mathbf{i}\varphi}$ for some φ,
- $\mathrm{e}^{\mathbf{i}\varphi n} = \cos \varphi n + \mathbf{i} \sin \varphi n = 1$,
  which happens everytime $\varphi n = 2k\pi$ for $k \in \mathbb{Z}$.
- We get n different n-th roots:
  $\varphi = 2k\pi/n$ for $k = 0, \ldots, n-1$.

General roots: $\sqrt[n]{x} = |x|^{1/n} \cdot \mathrm{e}^{\mathbf{i}\varphi(x)/n} \cdot \sqrt[n]{1}$.