KOMBINATORIKA A GRAFY I

- NAVAZUJEME NA DISKRÉTNÍ MATEMATIKU
→ WEBOVÁ STRÁNKA

ODHADY FAKTORIÁLŮ A KOMBINAČNÍCH ČÍSEL

$$n^{\frac{n}{2}} \leq n! \leq n^n$$

$$n! = 1\cdot 2\cdot 3\cdot \ldots \cdot n \leq \underbrace{n\cdot n\cdot \cdots \cdot n}_{n} = n^n$$

DRUHÝ ODHAD: CHCEME

$$(n!)^2 \geq n^n$$

$$(n\cdot 1)\cdot\big((n-1)\cdot 2\big)\cdot\big((n-2)\cdot 3\big)\cdots\cdot(1\cdot n) \geq n^n$$

STAČÍ $(n-i+1)\cdot i \geq n \iff (n-i+1)\cdot i - n \geq 0$ ... KVADR. FUNKCE V i,

NEBO

$$(n-i+1)\cdot i \geq 2\cdot\frac{n}{2}.$$

LEPŠÍ ODHAD

VĚTA (STIRLINGOVA FORMULE)

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n,$$ PŘESNĚJI $$\lim_{n\to\infty}\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1.$$

BEZ DŮKAZU

SLABŠÍ ODHADY (ALE BEZ LIMITY):

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \leq n! \leq e\sqrt{n}\left(\frac{n}{e}\right)^n$$

DOKÁŽEME JEŠTĚ SLABŠÍ ODHAD: $e\left(\frac{n}{e}\right)^n \leq n! \leq e\cdot n\cdot\left(\frac{n}{e}\right)^n$

DŮKAZ:

1. POMOCÍ FAKTU $1+x \leq e^x \quad \forall x \in \mathbb{R}$

$$\Updownarrow$$

$$1-x \leq e^{-x} \quad \forall x \in \mathbb{R}$$

$e^x - x \geq 1$

$\uparrow$ JE KONVEXNÍ

HORNÍ ODHAD MI

$n=1: \quad 1! \leq e\cdot 1\cdot\left(\frac{1}{e}\right)^1$

DÁLE

$$n! = n(n-1)! \leq n\cdot e\cdot(n-1)\cdot\left(\frac{n-1}{e}\right)^{n-1} =$$

$$= e\cdot n\cdot\left(\frac{1}{e}\right)^n\cdot\left(\frac{n-1}{n}\right)^n\cdot\frac{1}{e}$$

$\left(\frac{n-1}{n}\right)^n = \left(1-\frac{1}{n}\right)^n$

$$\leq e\cdot n\cdot\left(\frac{n}{e}\right)^n\cdot\left(e^{-\frac{1}{n}}\right)^n\cdot\frac{1}{e} = e\cdot n\cdot\left(\frac{n}{e}\right)^n.$$

DOLNÍ ODHAD OPĚT MI:

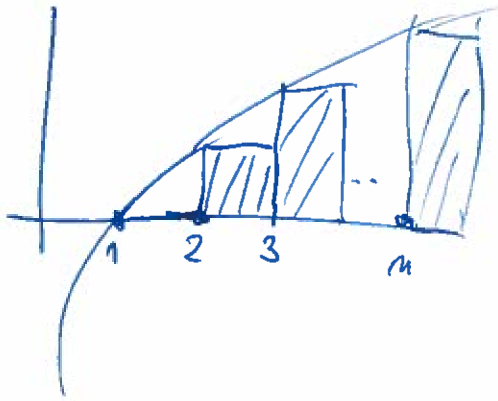
$$n! = n\cdot(n-1)! \geq n\cdot e\cdot\left(\frac{n-1}{e}\right)^{n-1} = e\cdot\left(\frac{n}{e}\right)^n\cdot\left(\frac{n-1}{n}\right)^{n-1}\cdot e$$

CHCEME $\left(\frac{n-1}{n}\right)^{n-1}\cdot e \geq 1 \Leftrightarrow \frac{1}{e}\left(\frac{n}{n-1}\right)^{n-1} \leq 1$

OPĚT Z FAKTU.

DŮKAZ 2: PŘES INTEGRÁL:

$$\ln n! = \ln 1 + \ln 2 + \dots + \ln n$$



$$\ln n! \leq \int_1^{n+1} \ln x \, dx$$

$$\int \ln x \, dx = \int 1 \cdot \ln x \, dx = x \cdot \ln x - \int x \cdot \frac{1}{x} \, dx = x \cdot \ln x - x + C$$

TEDY

$$\ln n! \leq \left[x \cdot \ln x - x \right]_1^{n+1} = (n+1) \ln (n+1) - (n+1) - (1 \cdot 0 - 1) =$$

$$= (n+1) \ln (n+1) - n$$

TEDY

$$n! = n \cdot (n-1)! \leq n \cdot e^{n \cdot \ln n - (n-1)} =$$

$$= n \cdot n^n \cdot e^{n-1} = n \cdot e \cdot \left(\frac{n}{e}\right)^n$$

PODOBNĚ DRUHÝ ODHAD

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n!}{(k-1)!\,(n-k+1)!} \cdot \frac{n-k+1}{k} = \binom{n}{k-1} \cdot \frac{n+1-k}{k}$$

PŘIČEMŽ $\frac{n+1-k}{k} \begin{cases} > 1 & \text{PRO } k < \frac{n+1}{2} \\ < 1 & \text{PRO } k > \frac{n+1}{2} \end{cases}$

TEDY NEJVĚTŠÍ KOMBINAČNÍ ČÍSLO MEZI KOMB. ČÍSLY $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}$ JE $\binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}$.

BIN. VĚTA: $\sum_{i=0}^{n} \binom{n}{i} = 2^n \Rightarrow \binom{n}{\lceil \frac{n}{2} \rceil} \leq 2^n$

ALE TAKÉ $\binom{n}{\lceil \frac{n}{2} \rceil} \geq \frac{2^n}{n+1}$

POMOCÍ STIRLINGOVY FORMULE:

$$\binom{2m}{m} = \frac{(2m)!}{m!\,m!} \approx \frac{\left(\frac{2m}{e}\right)^{2m} \sqrt{2\pi\, 2m}}{\left(\left(\frac{m}{e}\right)^{m} \sqrt{2\pi m}\right)^2} = \frac{2^{2m}}{\sqrt{\pi m}}$$

<u>VĚTA</u>: PRO $m \geq 1$ PLATÍ $\frac{2^{2m}}{2\sqrt{m}} \leq \binom{2m}{m} \leq \frac{2^{2m}}{\sqrt{2m}}$

DŮKAZ:

$$\binom{2m}{m} = \frac{2m!}{m!\,m!} = \frac{1\cdot 3\cdot \dots \cdot(2m-1)}{m!}\cdot\frac{2\cdot 4\cdot \dots \cdot 2m}{m!} =$$

$$= \frac{1\cdot 3 \cdots (2m-1)}{m!}\cdot 2^m = \frac{1\cdot 3\cdots(2m-1)}{2\cdot 4\cdots 2m}\cdot 2^{2m}$$

TEDY CHCEME DOKÁZAT

$$\frac{1}{2\sqrt{m}} \leq \underbrace{\frac{1\cdot 3\cdots(2m-1)}{2\cdot 4\cdots 2m}}_{V} \leq \frac{1}{\sqrt{2m}}$$

$$V^2 = \frac{(1\cdot 3\cdots(2m-1))^2}{(2\cdot 4\cdots 2m)^2} = 1\cdot\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot \dots \cdot\frac{(2m-3)(2m-1)}{(2m-2)^2}\cdot\frac{(2m-1)}{(2m)^2} \leq$$

< 1

$$\boxed{\frac{(x-1)(x+1)}{x^2} = \frac{x^2-1}{x^2} < 1}$$

$$\leq \frac{2m-1}{(2m)^2} < \frac{2m}{(2m)^2} = \frac{1}{2m}, \quad \text{TEDY } V \leq \frac{1}{\sqrt{2m}}$$

A TAKÉ

$$V^2 = \frac{1\cdot 1}{2}\cdot\frac{3\cdot 3}{2\cdot 4}\cdot\frac{5\cdot 5}{4\cdot 6}\cdot \dots \cdot\frac{(2m-1)(2m-1)}{(2m-2)\,2m}\cdot\frac{1}{2m} \geq \frac{1}{4m}$$

TEDY $V \geq \dfrac{1}{2\sqrt{m}}$

$\blacksquare$