A. Lim 3004

Calculate the expectation values of L_x and L_x^2 for a state with angular momentum $l\hbar$ and a projection onto the z axis $m\hbar$.

Solution:

The expectation value of L_x can be obtained using the commutation relations for components of angular momentum: $i\hbar L_x = [L_y, L_z]$. Hence

$$\langle L_x \rangle = \langle lm|\frac{1}{i\hbar}[L_y, L_z]|lm\rangle = \frac{1}{i\hbar}\langle lm|L_y L_z - L_z L_y|lm\rangle$$

We now use $L_z|lm\rangle = \hbar m|lm\rangle$ (and its conjugate) to find

$$\langle L_x \rangle = \frac{1}{i\hbar}\langle lm|m\hbar L_y|lm\rangle - \frac{1}{i\hbar}\langle lm|m\hbar L_y|lm\rangle = 0\,.$$

Alternatively, we can make use of the rising and lowering operators for angular momentum

$$L_+ = L_x + iL_y L_- = L_x - iL_y\,.$$

Which gives $L_x = \frac{1}{2}(L_+ + L_-)$. As

$$L_\pm|lm\rangle = \hbar\sqrt{l(l+1) - m(m\pm 1)}|lm \pm 1\rangle\,,$$

we obtain

$$\begin{aligned}\langle lm|L_x|lm\rangle &= \frac{1}{2}\hbar\sqrt{l(l+1) - m(m+1)}\langle lm|lm+1\rangle \\ &+ \frac{1}{2}\hbar\sqrt{l(l+1) - m(m-1)}\langle lm|lm-1\rangle \\ &= 0\,,\end{aligned}$$

as the basis of the $|lm\rangle$ states is orthonormal.

The expectation value of the square of the L_x operator can be obtained using the operator of the magnitude of the momentum L^2 and of the projection into the z axis. As $L^2 = L_x^2 + L_y^2 + L_z^2$ and $\langle L_x^2\rangle = \langle L_y^2\rangle$, due to the symmetry of the problem, the expectation value is $\langle L_x^2\rangle = \frac{1}{2}\langle(L^2 - L_z^2)\rangle$. Using $L^2|lm\rangle = l(l+1)\hbar^2|lm\rangle$ and $L_z|lm\rangle = \hbar m|lm\rangle$, we obtain $\langle L_x^2\rangle = \frac{1}{2}\hbar^2(l(l+1) - m^2)$.

Using the raising and lowering operators, we have

$$L_x^2 = \frac{1}{4}(L_+ + L_-)(L_+ + L_-) = \frac{1}{4}(L_+L_+ + L_-L_- + L_-L_+ + L_+L_-)$$

The expectation value of the squares of the raising or lowering operators is zero (as $\langle lm|L_+^2|lm\rangle = c\langle lm|lm+2\rangle = 0$) and only the mixed terms remain. They can be evaluated as

$$\begin{aligned}
\langle L_x^2\rangle =&\frac{1}{4}\langle lm|L_-L_+ + L_+L_-|lm\rangle \\
=&\frac{1}{4}\hbar\langle lm|L_-\sqrt{l(l+1)-m(m+1)}|lm+1\rangle + \frac{1}{4}\hbar\langle lm|L_+\sqrt{l(l+1)-m(m-1)}|lm-1\rangle \\
=&\frac{1}{4}\hbar^2\langle lm|\sqrt{l(l+1)-(m+1)(m+1-1)}\sqrt{l(l+1)-m(m+1)}|lm\rangle \\
&+\frac{1}{4}\hbar^2\langle lm|\sqrt{l(l+1)-(m-1)(m-1+1)}\sqrt{l(l+1)-m(m-1)}|lm\rangle \\
=&\frac{1}{4}\hbar^2(l(l+1)-m(m+1)) + \frac{1}{4}\hbar^2(l(l+1)-m(m-1)) \\
=&\frac{1}{4}\hbar^2(2l(l+1)-m(m+1)-m(m-1)) \\
=&\frac{1}{2}\hbar^2(l(l+1)-m^2)\,.
\end{aligned}$$

Which agrees with the previous result.

B. Lim 3007

A particle with spin $S = 1$ is in a state with an angular momentum of $L = 2$. A spin-orbit Hamiltonian

$$H = AL\cdot S$$

describes the interaction between the particles. What are the possible energies and their degeneracies for this system.

Solution: The spin-orbit Hamiltonian does not commute with individual projections of the spin and angular momentum, i.e. $[L_z, L\cdot S] \neq 0$ and $[S_z, L\cdot S] \neq 0$. The Hamiltonian is, however, diagonal in the basis of the total momentum $J = L + S$ and its projections $J_z = L_z + S_z$.

The Hamiltonian can be rewritten using the operator of the magnitude of the total momentum $J^2 = L^2 + S^2 + 2L\cdot S$, from which we find

$$L\cdot S = \frac{1}{2}(J^2 - L^2 - S^2)\,.$$

The magnitude of the ($L = 2$) orbital momentum is $L^2|lm\rangle = l(l+1)\hbar^2|lm\rangle = 6\hbar^2|lm\rangle$ and for spin ($S = 1$) we have $S^2|ss_z\rangle = s(s+1)\hbar^2|ss_z\rangle = 2\hbar^2|ss_z\rangle$ and are thus identical for all the states.

The rules of combination of angular momenta give possible values for the total momentum $J = L+S, \dots, |L-S|$. Therefore, for $L = 2$ and $S = 1$ we have $J = 3, 2, 1$. The magnitude of the total momentum is $J^2|JJ_zLS\rangle = J(J+1)\hbar^2|JJ_zLS\rangle$, which, for the possible values of J gives $12\hbar^2$, $6\hbar^2$, and $2\hbar^2$, respectively. As the expectation values of the Hamiltonian are

$$\langle H\rangle = \langle JJ_zLS|\frac{A}{2}(J^2 - L^2 - S^2)|JJ_zLS\rangle \,,$$

We get

$$E(J=3) = 2A\hbar^2$$
$$E(J=2) = -A\hbar^2$$
$$E(J=1) = -3A\hbar^2$$

which are 7, 5, and 3-fold degenerate, respectively.

C. Two spins – Lim 3034

Consider a system with two non-interacting spins. The first is in a state $s_z^A = +1/2$, the second in a state $s_x^B = +1/2$. What's the probability that the total spin is zero?

Solution:

Two particles with spin one half lead to total spin one with three-fold degeneracy and a non-degenerate spin zero state. For the total spin equal to zero, the state is (taking z as the quantisation coordinate)

$$|S=0, S_z=0\rangle = \frac{1}{\sqrt{(2)}}(|\uparrow_A\downarrow_B\rangle - |\downarrow_A\uparrow_B\rangle)\,.$$

To be able to project on this state, we need to transform the $s_x^B = +1/2$ state into the s_z^B basis. The representation of the s_x^B states can be found by diagonalisation of the s_x operator matrix

$$s_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,. \tag{1}$$

The eigenvalues are $\pm 1\hbar/2$ and the states are $\frac{1}{\sqrt{2}}\binom{1}{1}$ for the $+\hbar/2$ state and $\frac{1}{\sqrt{2}}\binom{1}{-1}$ for the $-\hbar/2$ state. That is, $|s_x = +1/2\rangle = \frac{1}{\sqrt{2}}(|s_z = +1/2\rangle + |s_z = -1/2\rangle)$. Therefore,

$$|s_z^A = +1/2, s_x^B = +1/2\rangle = \frac{1}{\sqrt{2}}(|s_z^A = +1/2, s_z^B = +1/2\rangle + |s_z^A = +1/2, s_z^B = -1/2\rangle\,.$$

The projection on the zero spin state is then (in a short-hand notation)

$$\begin{aligned}\langle 00|s_z^A = +1/2, s_x^B = +1/2\rangle &= \frac{1}{2}\left(\langle\uparrow_A\downarrow_B| - \langle\downarrow_A\uparrow_B|\right)\left(|\uparrow_A\uparrow_B\rangle + |\uparrow_A\downarrow_B\rangle\right) \\ &= \frac{1}{2}(\langle\uparrow_A\downarrow_B|\uparrow_A\uparrow_B\rangle + \langle\uparrow_A\downarrow_B|\uparrow_A\downarrow_B\rangle - \langle\downarrow_A\uparrow_B|\uparrow_A\uparrow_B\rangle - \langle\downarrow_A\uparrow_B|\uparrow_A\downarrow_B\rangle) \\ &= \frac{1}{2}(0+1+0+0) = \frac{1}{2}\end{aligned} \tag{2}$$

The probability is

$$|\langle 00|s_z^A = +1/2, s_x^B = +1/2\rangle|^2 = \frac{1}{4}\,.$$

D. Lim 3017

An electron is prepared with projection of the spin $+\hbar/2$ along the z axis.

- What are the possible results of measurement of spin along the x axis?
- What is the probability of finding these results?
- If we measure the spin along axis restricted to the $x-z$ plane and rotated by an angle θ from the z axis, what are the probabilities of measuring the different results?
- What is the expectation value of spin measured along the rotated axis, given the initial projection along z?

Solution:

The measured electron spin will be $\pm\hbar/2$ along any axis. This can be shown by finding the eigenvalues of the s_x operator that describes the act of measuring the spin along x axis. Its eigenvalues are then the only possible results of measuring the spin (given an isolated system). We find the eigenvalues by diagonalising the matrix representation of the s_x operator in the basis of the states corresponding to measurement along the z axis. The s_x operator is given as

$$s_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,.$$

The eigenvalues are found by calculating the determinant of

$$\frac{\hbar}{2}\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1,$$

therefore the eigenvalues are indeed $\lambda = \pm\hbar/2$. The eigenvector corresponding to $\lambda = \hbar/2$ is $\frac{1}{\sqrt{2}}\binom{1}{1}$

E. Spin-spin Hamiltonian

Two particles (A and B) with spin 1/2 interact via Hamiltonian $H = Js^A \cdot s^B$. Find the eigenenergies of the Hamiltonian by rewriting it using the operators of the magnitudes of the spin.

Solution:

The problem is analogous to the problem of spin-orbit interaction. The dot product in the Hamiltonian does not commute with the operators of the projections of the individual spins into the z axis. We therefore introduce the total spin $S = s^A \otimes 1_B + 1_A \otimes s^B$ and its projection $S_z = s_z^A \otimes 1_B + 1_A \otimes s_z^B$, which communte with the Hamiltonian. By the rules of combination of angular momenta the total spin can be either $S = \frac{1}{2} + \frac{1}{2} = 1$ or $S = \frac{1}{2} - \frac{1}{2} = 0$. In the first case, three projections are possible, in the latter, only one.

To find the new eigenvalues we rewrite the dot product using the operator of the magnitude of the total spin

$$S^2 = (s_A + s_B)^2 = s_A^2 \otimes 1_B + 1_A \otimes s_B^2 + 2s^A \cdot s^B\,. \tag{3}$$

Hence

$$s^A \cdot s^B = \frac{1}{2}(S^2 - s_A^2 \otimes 1_B - 1_A \otimes s_B^2)\,. \tag{4}$$

The action of an operator of the magnitude of the spin is $s^2|ss_z\rangle = \hbar^2 s(s+1)|ss_z\rangle$. For particles with spin $s = \frac{1}{2}$, we obtain $\frac{3}{4}\hbar^2$. For the triplet states with total spin $S = 1$, we get $2\hbar^2$ and for the singlet state with spin $S = 0$ we get 0.

The energies are then for triplet

$$E(\mathrm{T}) = \frac{J}{2}(2\hbar^2 - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2) = \frac{J}{4}\hbar^2 \tag{5}$$

and for the singlet state

$$E(\mathrm{S}) = \frac{J}{2}(0 - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2) = -\frac{3J}{4}\hbar^2\,. \tag{6}$$

This agrees with the results obtained by exact diagonalisation.

F. Spin-spin Hamiltonian in matrix representation

Two particles (A and B) with spin 1/2 interact via Hamiltonian $H = Js^A \cdot s^B$. Write the matrix representation of the Hamiltonian in the direct basis. Find the eigenvectors and eigenvalues of the Hamiltonian.

Solution:

The Hamiltonian reads $As^A \cdot s^B$, where the dot product stands for

$$s^A \cdot s^B = s^A_x \otimes s^B_x + s^A_z \otimes s^B_z + s^A_z \otimes s^B_z\,.$$

The matrix representation of the spin operators (s_x etc.) in the standard quantisation along z axis uses the Pauli matrices

$$s_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$s_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$s_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The basis functions are spin up and down along the z axis: $|\uparrow\rangle = \binom{1}{0}$ and $|\downarrow\rangle = \binom{0}{1}$. Calculating the expectation values of the spin components using these vectors gives zeros for the x and y components and either $\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$ for the z component, as expected.

The Hamiltonian is written as a direct product of the spin matrices, it can be thus written as a 4×4 matrix. We will go from the direct product of two 2×2 matrices to the 4×4 matrix using the following scheme:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} A\alpha & A\beta & B\alpha & B\beta \\ A\gamma & A\delta & B\gamma & B\delta \\ C\alpha & C\beta & D\alpha & D\beta \\ C\gamma & C\delta & D\gamma & D\delta \end{pmatrix} \tag{7}$$

This corresponds to a basis set $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, and $|\downarrow\downarrow\rangle$, where the first spin is of the A particle and the latter of the B particle.

For example, the part of the Hamiltonian originating from the z components is

$$J\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\otimes\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = J\frac{\hbar^2}{4}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{8}$$

This shows that for $H = Js_z^A \otimes s_z^B$ there are two energy levels, one with aligned spins (H_{11} for $|\uparrow\uparrow\rangle$ state and H_{44} for $|\downarrow\downarrow\rangle$ state). In this case the energy is $E = \frac{\hbar^2}{4}J$. For the usual ferromagnetic ordering, $J < 0$ and this will be the (degenerate) ground state. The states with anti-parallel spins will be higher in energy. Anti-ferromagnetic order can be observed as well in some materials and then $J > 0$ and the anti-parallel states have lower energy than the states with parallel spins.

The parts of the Hamiltonian corresponding to the x and y components can be calculated analogously to the z component, for x we obtain

$$J\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\otimes\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = J\frac{\hbar^2}{4}\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix} \tag{9}$$

and for y

$$J\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\otimes\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = J\frac{\hbar^2}{4}\begin{pmatrix}0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix} \tag{10}$$

Overall, summing these three parts, we obtain the Hamiltonain

$$H = J\frac{\hbar^2}{4}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 2 & 0\\ 0 & 2 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{11}$$

We see that the Hamiltonian is not diagonal in the direct product basis, specifically, the states with anti-parallel spins are not eigenvectors of the new Hamiltonian. In contrast, the states with aligned spins are still eigenvectors. To find the new states with anti-parallel spins we need to diagonalise the part of the Hamiltonian corresponding to these two states

$$H_{\text{anti}} = J\frac{\hbar^2}{4}\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} . \tag{12}$$

For the eigenvalues we readily obtain $\epsilon_{1,2} = -1 \pm 2$ corresponding to vectors $\frac{1}{\sqrt{2}}\binom{1}{1} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ and $\frac{1}{\sqrt{2}}\binom{1}{-1} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$. The energies of these two states are $\frac{\hbar^2}{4}J$ for the first and $-\frac{3\hbar^2}{4}J$ for the latter. Therefore, the state with symmetric spin wavefunction has the same energy as the states with both spins up or both spins down. Together, they form the triplet states with total spin $S = 1$ and projections $S_z = 1, 0, -1$. The state with anti-symmetric spin wavefunction is the singlet state with $S = 0$ and only possible projection $S_z = 0$. This can be verified by explicitly applying the operator of the magnitude of the total spin S^2 (TODO) .

G. Coupling of momenta

A spin $\hbar/2$ particle is bound in a spherically symmetric potential and is in a state with orbital momentum $l = 1\hbar$. What are the possible values of the total momentum and the projections onto the z axis?

The particle is in a state with $j = \frac{3\hbar}{2}$ and $j = \frac{\hbar}{2}$ that can be written in the product basis as

$$|\psi\rangle = |j = 3/2, j_z = 1/2, l, s\rangle = \sqrt{\frac{2}{3}}|l, l_z = 0; s, s_z = +\frac{1}{2}\rangle + \sqrt{\frac{1}{3}}|l, l_z = 1; s, s_z = -\frac{1}{2}\rangle .$$

- What are the respective results when the operators j^2 and j_z act on the state $|\psi\rangle$?
- Verify, that the state $|\psi\rangle$ is also an eigenstate of the operator $j_z = l_z \otimes 1 + 1 \otimes s_z$ in the product basis of the original states.
- Rewrite the operator of the magnitude of the total momentum $j^2 = (\vec{l}\otimes 1 + 1 \otimes \vec{s})^2$ using the operators $l^2, l_z, l_-, l_+, s^2, s_z, s_-, s_+$ and verify that the state $|\psi\rangle$ is an eigenstate of j^2.

Solution:

The rules for combining angular momenta give two possible values for the total momentum: $j = \frac{3\hbar}{2}$ and $j = \frac{\hbar}{2}$. The possible projections are $j_z = \frac{3\hbar}{2}$, $\frac{\hbar}{2}$, $-\frac{\hbar}{2}$, and $-\frac{3\hbar}{2}$ for the first and $j_z = \frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ for the latter.

The state $|\psi\rangle$ corresponds to $j = \frac{3\hbar}{2}$ and $j_z = \frac{\hbar}{2}$, after acting with j^2 we obtain

$$j^2|j = 3/2, j_z = 1/2, l, s\rangle = \frac{3}{2}(\frac{3}{2}+1)\hbar^2|j = 3/2, j_z = 1/2, l, s\rangle = \frac{15}{4}\hbar^2|j = 3/2, j_z = 1/2, l, s\rangle \, .$$

For the projection onto the z axis we obtain

$$j_z|j = 3/2, j_z = 1/2, l, s\rangle = \frac{\hbar}{2}|j = 3/2, j_z = 1/2, l, s\rangle \, .$$

We will now use the product basis where operators of the individual momenta act. For the projection onto the z axis we obtain

$$\begin{aligned}
&(l_z \otimes 1 + 1 \otimes s_z)[\sqrt{\frac{2}{3}}|l, l_z = 0; s, s_z = +\frac{1}{2}\rangle + \sqrt{\frac{1}{3}}|l, l_z = 1; s, s_z = -\frac{1}{2}\rangle] \\
&\quad = \sqrt{\frac{2}{3}}(l_z \otimes 1)|l, l_z = 0; s, s_z = +\frac{1}{2}\rangle + \sqrt{\frac{2}{3}}(s_z \otimes 1)|l, l_z = 0; s, s_z = +\frac{1}{2}\rangle \\
&\quad + \sqrt{\frac{1}{3}}(l_z \otimes 1)|l, l_z = 1; s, s_z = -\frac{1}{2}\rangle + \sqrt{\frac{1}{3}}(s_z \otimes 1)|l, l_z = 1; s, s_z = -\frac{1}{2}\rangle \\
&\quad = \sqrt{\frac{2}{3}}0\hbar|l, l_z = 0; s, s_z = +\frac{1}{2}\rangle + \sqrt{\frac{2}{3}}\frac{\hbar}{2}|l, l_z = 0; s, s_z = +\frac{1}{2}\rangle \\
&\quad + \sqrt{\frac{1}{3}}\hbar|l, l_z = 1; s, s_z = -\frac{1}{2}\rangle - \sqrt{\frac{1}{3}}\frac{\hbar}{2}|l, l_z = 1; s, s_z = -\frac{1}{2}\rangle \\
&\quad = \frac{\hbar}{2}\sqrt{\frac{2}{3}}|l, l_z = 0; s, s_z = +\frac{1}{2}\rangle + \frac{\hbar}{2}\sqrt{\frac{1}{3}}|l, l_z = 1; s, s_z = -\frac{1}{2}\rangle \\
&\quad = \frac{\hbar}{2}|\psi\rangle \, .
\end{aligned}$$

That is, the results obatained in the product and coupled basis are identical, as they should.

The operator of the magnitude of the total momentum j^2 can be expressed as

$$j^2 = (l \otimes 1 + 1 \otimes s)^2 = (l^2 \otimes 1 + 1 \otimes s^2 + 2l_x \otimes s_x + 2l_y \otimes s_y + 2l_z \otimes s_z) \, .$$

Using $l_x = \frac{1}{2}(l_+ + l_-)$ a $l_x = \frac{1}{2i}(l_+ - l_-)$ and similarly for the spin component, we rewrite the x and y components as

$$\begin{aligned}
&l_x \otimes s_x + l_y \otimes s_y = \\
&\quad = \frac{1}{4}(l_+ + l_-) \otimes (s_+ + s_-) - \frac{1}{4}(l_+ - l_-) \otimes (s_+ - s_-) \\
&\quad = \frac{1}{2}(l_- \otimes s_+ + l_+ \otimes s_-) \, .
\end{aligned}$$

Overall, we have

$$j^2 = (l^2 \otimes 1 + 1 \otimes s^2 + l_- \otimes s_+ + l_+ \otimes s_- + 2l_z \otimes s_z) \, .$$

The state $|\psi\rangle$ is an eigenstate of both l^2 and s^2, with eigenvalues $2\hbar^2$ and $\frac{3\hbar^2}{4}$, respectively. The remaining contributions can be found by explicitly applying the operators on the states. For the first remaining term

$$l_- \otimes s_+[\sqrt{\frac{2}{3}}|l, l_z = 0; s, s_z = +\frac{1}{2}\rangle + \sqrt{\frac{1}{3}}|l, l_z = 1; s, s_z = -\frac{1}{2}\rangle]\,,$$

the action on the first component will give zero, $s_+|s, s_z = +\frac{1}{2}\rangle = 0$, as the spin projection onto the z axis is already maximal. We then have to evaluate

$$\begin{aligned}
&l_- \otimes s_+\sqrt{\frac{1}{3}}|l, l_z = 1; s, s_z = -\frac{1}{2}\rangle \\
&\quad = \hbar^2\sqrt{\frac{1}{3}}\sqrt{1(1+1) - 1(1-1)}\sqrt{\frac{1}{2}(\frac{1}{2}+1) - (-\frac{1}{2})(-\frac{1}{2}+1)}|l, l_z = 0; s, s_z = \frac{1}{2}\rangle \\
&\quad = \hbar^2\sqrt{\frac{1}{3}}\sqrt{2}\sqrt{1}|l, l_z = 0; s, s_z = \frac{1}{2}\rangle \\
&\quad = \hbar^2\sqrt{\frac{2}{3}}|l, l_z = 0; s, s_z = \frac{1}{2}\rangle
\end{aligned}$$

In the term

$$l_+ \otimes s_-[\sqrt{\frac{2}{3}}|l, l_z = 0; s, s_z = +\frac{1}{2}\rangle + \sqrt{\frac{1}{3}}|l, l_z = 1; s, s_z = -\frac{1}{2}\rangle]\,,$$

the second contribution will be zero as the l projection is the largest and the spin projection is the lowest possible. We therefore need to evaluate

$$\begin{aligned}
&l_+ \otimes s_-\sqrt{\frac{2}{3}}|l, l_z = 0; s, s_z = \frac{1}{2}\rangle \\
&\quad = \hbar^2\sqrt{\frac{2}{3}}\sqrt{1(1+1) - 0(0+1)}\sqrt{\frac{1}{2}(\frac{1}{2}+1) - (\frac{1}{2})(\frac{1}{2}-1)}|l, l_z = 1; s, s_z = -\frac{1}{2}\rangle \\
&\quad = \hbar^2\sqrt{\frac{2}{3}}\sqrt{2}\sqrt{1}|l, l_z = 1; s, s_z = -\frac{1}{2}\rangle \\
&\quad = \hbar^2\frac{2}{\sqrt{3}}|l, l_z = 1; s, s_z = -\frac{1}{2}\rangle\,.
\end{aligned}$$

Finally, the term containing the projections onto the z axis becomes

$$\begin{aligned}
&2l_z \otimes s_z[\sqrt{\frac{2}{3}}|l, l_z = 0; s, s_z = +\frac{1}{2}\rangle + \sqrt{\frac{1}{3}}|l, l_z = 1; s, s_z = -\frac{1}{2}\rangle] \\
&\quad = 2(0\hbar)(\frac{\hbar}{2})\sqrt{\frac{2}{3}}|l, l_z = 0; s, s_z = +\frac{1}{2}\rangle + 2(\hbar)(-\frac{\hbar}{2})\sqrt{\frac{1}{3}}|l, l_z = 1; s, s_z = -\frac{1}{2}\rangle \\
&\quad = -\hbar^2\sqrt{\frac{1}{3}}|l, l_z = 1; s, s_z = -\frac{1}{2}\rangle\,.
\end{aligned}$$

Summing the last three results we find

$$
\begin{aligned}
&(l_- \otimes s_+ + l_+ \otimes s_- + 2l_z \otimes s_z)|\psi\rangle = \\
&\quad =\hbar^2\sqrt{\frac{2}{3}}|l, l_z = 0; s, s_z = \frac{1}{2}\rangle + \hbar^2\frac{2}{\sqrt{3}}|l, l_z = 1; s, s_z = -\frac{1}{2}\rangle \\
&\quad\quad - \hbar^2\sqrt{\frac{1}{3}}|l, l_z = 1; s, s_z = -\frac{1}{2}\rangle \\
&\quad =\hbar^2\sqrt{\frac{2}{3}}|l, l_z = 0; s, s_z = \frac{1}{2}\rangle + \hbar^2\sqrt{\frac{1}{3}}|l, l_z = 1; s, s_z = -\frac{1}{2}\rangle \\
&\quad =\hbar^2[\sqrt{\frac{2}{3}}|l, l_z = 0; s, s_z = \frac{1}{2}\rangle + \sqrt{\frac{1}{3}}|l, l_z = 1; s, s_z = -\frac{1}{2}\rangle] \\
&\quad =\hbar^2|\psi\rangle\,.
\end{aligned}
$$

Therefore, $|\psi\rangle$ is an eigenstate of the operator $2l \cdot s$ with an eigenvalue of $\hbar^2$. Together with the contributions of l^2 and s^2 the total eigenvalue of j^2 is $(2 + \frac{3}{4} + 1)\hbar^2 = \frac{15}{4}\hbar^2$, as it should for $j = \frac{3}{2}\hbar$.