2022 - UKM - T2.1

- fce $\frac{1}{\sqrt{\pi}}\sin(n\phi)$, $\frac{1}{\sqrt{\pi}}\cos(n\phi)$, $\frac{1}{\sqrt{2\pi}}$ } eigenstates!
- $N=1 \to$ [2022 - UKM - T1.3]
- $\langle \ell | k \rangle = \delta_{\ell k} \to$ [2022 - UKM - T1.3]
- $\hat{A} = i\frac{d}{d\phi}$, $\hat{B} = \sin\phi$: akce ; komutátor ; $(AB)^2 \xrightarrow{\text{místo}} [A,B]$
- matrix repre $\hat{A}, \hat{B}$, $(AB)^2$, $[A,B]$

$$\hat{A} = i\frac{d}{d\phi}$$

$$\hat{A}\frac{1}{\sqrt{\pi}}\sin\phi = i\frac{d}{d\phi}\frac{1}{\sqrt{\pi}}\sin\phi = \frac{i}{\sqrt{\pi}}\cos\phi = i\,\psi_{c1}$$

$$\hat{A}\frac{1}{\sqrt{\pi}}\sin(n\phi) = i\frac{d}{d\phi}\frac{1}{\sqrt{\pi}}\sin(n\phi) = \frac{in}{\sqrt{\pi}}\cos(n\phi) = in\,\psi_{cn}$$

$$\hat{A}\frac{1}{\sqrt{2\pi}} = 0 = 0\,\psi_0$$

$$\hat{A}\frac{1}{\sqrt{\pi}}\cos(n\phi) = i\frac{d}{d\phi}\frac{1}{\sqrt{\pi}}\cos(n\phi) = -\frac{in}{\sqrt{\pi}}\sin(n\phi) = -in\,\psi_{sn}$$

NOTE - not an eigenstate, only $\frac{1}{\sqrt{2\pi}}$ with $\lambda = 0$

$$\hat{B} = \sin\phi$$

$$\hat{B}\frac{1}{\sqrt{\pi}}\sin\phi = \frac{1}{\sqrt{\pi}}\sin^2\phi = \frac{1}{\sqrt{\pi}}\left(\frac{1}{2i}\left(e^{i\phi}-e^{-i\phi}\right)\right)^2 = \frac{1}{\sqrt{\pi}}\left(-\frac{1}{4}\right)\left[e^{2i\phi}+e^{-2i\phi}-2\right]$$

$$= \frac{1}{4\sqrt{\pi}}\left[\left(e^{2i\phi}+e^{-2i\phi}\right)(-1)+\frac{1}{2}\right] = \frac{1}{2\sqrt{\pi}}\left[1-\cos(2\phi)\right] \to \frac{1}{\sqrt{2}}\psi_0 - \frac{1}{2}\psi_{c2}$$

$$\hat{B}\frac{1}{\sqrt{\pi}}\sin(n\phi) = \frac{1}{\sqrt{\pi}}\sin\phi\sin(n\phi) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{2i}\right)^2\left(e^{i\phi}-e^{-i\phi}\right)\left(e^{in\phi}-e^{-in\phi}\right) =$$

$$= -\frac{1}{4\sqrt{\pi}}\left[e^{i(n+1)\phi} \underbrace{- e^{-i(n-1)\phi} - e^{i(n-1)\phi}}_{-2\cos[(n-1)\phi]} + e^{-(n+1)\phi}\right], \quad e^{i(n+1)\phi}+e^{-(n+1)\phi} = 2\cos[(n+1)\phi]$$

$$= -\frac{1}{2\sqrt{\pi}}\left[\cos[(n+1)\phi] - \cos[(n-1)\phi]\right]$$

$$= -\frac{1}{2}\left[\psi_{c(N+1)} - \psi_{c(N-1)}\right]$$

$\hat{B} = \sin\phi$

$\hat{B}\frac{1}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}\sin\phi = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{\pi}}\sin\phi\right) = \frac{1}{\sqrt{2}}\psi_{s1}$

$\hat{B}\frac{1}{\sqrt{\pi}}\cos\phi = \frac{1}{\sqrt{\pi}}\cos\phi\sin\phi = \frac{1}{4i\sqrt{\pi}}\left(e^{i\phi}+e^{-i\phi}\right)\left(e^{i\phi}-e^{-i\phi}\right) =$

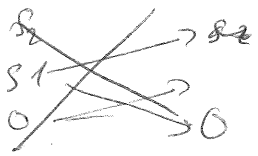
$= \frac{1}{4i\sqrt{\pi}}\left[e^{2i\phi}-e^{-2i\phi}\right] = \frac{1}{2\sqrt{\pi}}\sin(2\phi) = \frac{1}{2}\psi_{s2}$

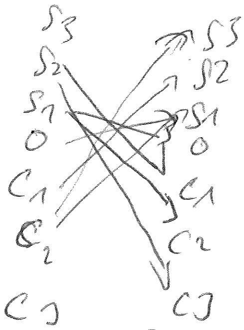
$\hat{B}\frac{1}{\sqrt{\pi}}\cos(n\phi) = \frac{1}{\sqrt{\pi}}\cos(n\phi)\sin\phi = \frac{1}{4i\sqrt{\pi}}\left(e^{in\phi}+e^{-in\phi}\right)\left(e^{i\phi}-e^{-i\phi}\right)$

$= \frac{1}{4i\sqrt{\pi}}\Big[\underbrace{e^{i(n+1)\phi}-e^{-i(n+1)\phi}}_{2i\sin[(n+1)\phi]} + \underbrace{e^{-i(n-1)\phi}-e^{i(n-1)\phi}}_{-2i\sin[(n-1)\phi]}\Big]$

$= \frac{1}{2\sqrt{\pi}}\Big[\sin[(n+1)\phi]-\sin[(n-1)\phi]\Big] = \frac{1}{2}\left[\psi_{s(N+1)}-\psi_{s(N-1)}\right]$

!





(! symmetry changes) does it! ?

! we don't end-up in the same state

commutator: $\hat{A}\hat{B}$ vs $\hat{B}\hat{A}$ [cf. ·, +, /, AB]

$\hat{A}\hat{B}\psi_{s1} = \hat{A}\hat{B}\frac{1}{\sqrt{\pi}}\sin\phi = i\frac{d}{d\phi}\frac{1}{\sqrt{\pi}}\sin^2\phi = i\frac{2}{\sqrt{\pi}}\sin\phi\cos\phi \to i2\cos\phi\,\psi_{s1}$

$= \frac{i}{\sqrt{\pi}}\sin(2\phi) = i\psi_{s2}$

$\hat{B}\hat{A}\psi_{s1} = \hat{B}\hat{A}\frac{1}{\sqrt{\pi}}\sin\phi = \sin\phi\, i\frac{d}{d\phi}\frac{1}{\sqrt{\pi}}\sin\phi = \frac{i}{\sqrt{\pi}}\sin\phi\cos\phi = \frac{i}{2}\psi_{s2}$

$\hookrightarrow \cos\phi\frac{1}{\sqrt{\pi}}\sin\phi - i\cos\phi\,\psi_{s1}$

$\Rightarrow [\hat{A}\hat{B} - \hat{B}\hat{A}]\psi = \Phi\psi \Rightarrow [\hat{A}\hat{B} - \hat{B}\hat{A}] \neq \Phi$

general?

$[\hat{A}\hat{B} - \hat{B}\hat{A}]\psi = \left[i\frac{d}{d\phi}\sin\phi - \sin\phi\, i\frac{d}{d\phi}\right]\psi = i\frac{d(\sin\phi\,\psi)}{d\phi} - \sin\phi\frac{i\,d\psi}{d\phi}$

$= \psi i\frac{d\sin\phi}{d\phi} + i\sin\phi\cancel{\frac{d\psi}{d\phi}} - i\sin\phi\cancel{\frac{d\psi}{d\phi}} = i\psi\cdot\cos\phi \Rightarrow$ general

$[\hat{A}\hat{B} - \hat{B}\hat{A}] = i\cos\phi\,\psi \to \hat{A}\hat{B}\psi = [\hat{B}\hat{A} + i\cos\phi]\psi$, OK

$(\hat{A}\hat{B})^2 = ?$ $\quad \hat{A} = i\frac{d}{d\phi} \quad \hat{B} = \cancel{\cos\phi}\ \sin\phi$ 2022-Ukm-72.3.

KO: $\left(i\frac{d}{d\phi} \sin\phi\ i\frac{d}{d\phi}\sin\phi \right) \overset{?}{=} \left(i\frac{d}{d\phi}\sin\phi\ i\cos\phi\right)$ NE! NE! KN!

OK: $\left(i\frac{d}{d\phi}\sin\phi\, i\frac{d}{d\phi}\sin\phi\right) f = -\frac{d}{d\phi}\sin\phi\frac{d\sin\phi f}{d\phi}$

$= -\frac{d}{d\phi}\sin\phi\left[\cos\phi f + \sin\phi f'\right] = -\frac{d}{d\phi}\left[\sin\phi\cos\phi f + \sin^2\phi f'\right]$

$= -\cancel{\frac{d}{d\phi}}\left[\cos^2\phi f - \sin^2\phi f + \cos\phi\sin\phi f' + 2\sin\phi\cos\phi f' + \sin^2\phi f''\right]$

$= \sin^2\phi f - \cos^2\phi f - 3\cos\phi\sin\phi f' - \sin^2\phi f''$

other way?

$(\hat{A}\hat{B})^2 = \hat{A}\hat{B}\hat{A}\hat{B}$

$\hat{A}\hat{B} = i\frac{d}{d\phi}\sin\phi$ ← not nice

$\hat{B}\hat{A} = i\sin\phi\frac{d}{d\phi}$ ← nicer

! $\hat{A}\hat{B} = \hat{B}\hat{A} + i\cos\phi$

$\hat{A}\hat{B}\hat{A}\hat{B} = (\hat{B}\hat{A} + i\cos\phi)(\hat{B}\hat{A} + i\cos\phi) = \hat{B}\hat{A}\hat{B}\hat{A} - \cos^2\phi + i\cos\phi\hat{B}\hat{A} + \hat{B}\hat{A}i\cos\phi$

$= \hat{B}(\hat{B}\hat{A} + i\cos\phi)\hat{A} - \cos^2\phi + i\cos\phi\hat{B}\hat{A} + \hat{B}\hat{A}i\cos\phi$

$= \hat{B}^2\hat{A}^2 + \hat{B}i\cos\phi\hat{A} - \cos^2\phi + i\cos\phi\hat{B}\hat{A} + \hat{B}\hat{A}i\cos\phi$

commute

$\left(i\frac{d}{d\phi}i\cos\phi\right)f = -\frac{d}{d\phi}(\cos\phi f) = \sin\phi f - \cos\phi f'$

$= \hat{B}f + i\cos\phi\hat{A}f = (\hat{B} + i\cos\hat{A})f$

$\to \hat{B}^2\hat{A}^2 + \cancel{\hat{B}i\cos}\ 2i\cos\phi\hat{B}\hat{A} - \cos^2\phi + \hat{B}(\hat{B} + i\cos\hat{A})f$

$-\sin^2\phi\frac{d^2}{d\phi^2} - 2\cos\phi\sin\phi\frac{d}{d\phi} - \cos^2\phi + \sin^2\phi + \cancel{\sin\phi} - \sin\phi\cos\phi\frac{d}{d\phi}$

• ! - general thing: put derivatives, lowering operators, etc. to the right so that their action is zero

$a^-|0\rangle = \phi$

$a^+a^-|0\rangle = \phi$

$a^-a^+|0\rangle = a^+a^-\cancel{|0\rangle} + [a^-,a^+]|0\rangle$ $\to \phi$ → leftover

2022-UCM-T2.7.

- $[\hat{A}\hat{B},\hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B} = \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} + \hat{A}\hat{C}\hat{B} - \hat{C}\hat{A}\hat{B}$

$= \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$

- $[\hat{A},\hat{B}^2] = \hat{A}\hat{B}\hat{B} - \hat{B}\hat{B}\hat{A} = \hat{A}\hat{B}\hat{B} - \cancel{\hat{A}\hat{B}+\hat{A}}\ \hat{B}\hat{A}\hat{B} + \hat{B}\hat{A}\hat{B} - \hat{B}\hat{B}\hat{A}$

$= \underbrace{[A,B]}_{1}B + B\underbrace{[A,B]}_{1} = 2B$

- $[\frac{d}{dx}, f(x)] \to [\frac{d}{dx}, f(x)]\psi = \frac{d}{dx}(f\psi) - f\frac{d\psi}{dx}$

$= \psi\frac{df}{dx} + f\frac{d\psi}{dx} - f\frac{d\psi}{dx} = \psi\frac{df}{dx}$

$[\frac{d}{dx}, f(x)] = \left(\frac{df}{dx}\right)$

- $[\frac{d}{dx}, x] \to [\frac{d}{dx}, x]f = \frac{d(xf)}{dx} - x\frac{df}{dx} = f + x\frac{df}{dx} - x\frac{df}{dx} = f$

$\Rightarrow [\frac{d}{dx}, x] = 1 \Rightarrow \frac{d}{dx}x - x\frac{d}{dx} = 1 \Rightarrow \frac{d}{dx}x = 1 + x\frac{d}{dx}$

- $\left(\frac{d}{dx}+x\right)^2 \to \left(\frac{d}{dx}+x\right)^2 f = \left(\frac{d}{dx}+x\right)\left(\frac{d}{dx}+x\right)f =$

$= \frac{d^2}{dx^2}f + x^2 f + \frac{d(xf)}{dx} + x\frac{df}{dx} = \frac{d^2}{dx^2}f + x^2 f + 2x\frac{df}{dx} + \frac{dx}{dx}f$

$\left(\frac{d}{dx}+x\right)^2 = \frac{d^2}{dx^2} + x^2 + 2x\frac{d}{dx} + 1$

$\left(\frac{d}{dx}+x\right)^2 = \left(\frac{d}{dx}+x\right)\left(\frac{d}{dx}+x\right) = \frac{d^2}{dx^2} + x^2 + \underbrace{\frac{d}{dx}x}_{1+x\frac{d}{dx}} + x\frac{d}{dx}$

$= \frac{d^2}{dx^2} + x^2 + 1 + 2x\frac{d}{dx}$ OK