$$\frac{d\langle A\rangle}{dt} = \frac{i}{\hbar}\langle [H,A]\rangle + \underbrace{\left\langle \frac{\partial A}{\partial t}\right\rangle}_{\to\, 0}$$

$$= \frac{i}{\hbar}\left\langle i m\omega^2 \hbar \frac{\alpha}{\sqrt{2}}(a+a^+)\right\rangle$$

$$= \frac{i}{\hbar} - m\omega^2\sqrt{\frac{\hbar}{m\omega}}\frac{1}{\sqrt{2}}\frac{1}{3}\left[\langle 0|e^{\frac{i\omega t}{2}} + \langle 1|e^{\frac{3i\omega t}{2}} + \langle 3|e^{\frac{7i\omega t}{2}}\right]$$

$$(a+a^+)\left[|0\rangle e^{-\frac{i\omega t}{2}} + |1\rangle e^{-\frac{3i\omega t}{2}} + |3\rangle e^{-\frac{7i\omega t}{2}}\right]$$

$$= -\frac{1}{3}\sqrt{\hbar m\omega}\frac{\omega}{\sqrt{2}}\underbrace{\left[\langle 0|a|1\rangle e^{-i\omega t} + \langle 1|a^+|0\rangle e^{i\omega t}\right]}_{2\cos(\omega t)}$$

$$= -\frac{\sqrt{2}}{3}\sqrt{\hbar m\omega}\,\omega\cos(\omega t)$$

$$\frac{dE_\lambda}{d\lambda} = \frac{d\langle \psi_\lambda|H_\lambda|\psi_\lambda\rangle}{d\lambda} =$$

$$= \left\langle\frac{d\psi_\lambda}{d\lambda}\right|H_\lambda|\psi_\lambda\rangle + \left\langle\psi_\lambda\right|\frac{dH_\lambda}{d\lambda}\left|\psi_\lambda\right\rangle + \left\langle\psi_\lambda\right|H_\lambda\left|\frac{d\psi_\lambda}{d\lambda}\right\rangle$$

↓ vl. stav

$$\left\langle\frac{d\psi_\lambda}{d\lambda}\right|E_\lambda|\psi_\lambda\rangle \qquad\qquad E_\lambda\left\langle\psi_\lambda\middle|\frac{d\psi_\lambda}{d\lambda}\right\rangle$$

$$E_\lambda\frac{d}{d\lambda}\langle\psi_\lambda|\psi_\lambda\rangle = 0$$

$$= \left\langle\psi_\lambda\right|\frac{dH_\lambda}{d\lambda}\left|\psi_\lambda\right\rangle$$

$$|0\rangle = \frac{1}{\sqrt{\alpha\sqrt{\pi}}}e^{-\frac{x^2}{2\alpha^2}} \qquad \alpha = \sqrt{\frac{\hbar}{m\omega}} \qquad \frac{d|0\rangle}{d\lambda} =$$

$$\frac{d|0\rangle}{d\omega} = \frac{d}{d\omega}\left[\frac{1}{\pi^{1/4}}\left(\frac{m\omega}{\hbar}\right)^{1/4}e^{-\frac{x^2 m\omega}{2\hbar}}\right] = \left(\frac{m}{\pi\hbar}\right)^{1/4}\left[\frac{1}{4}\omega^{-3/4} - \frac{x^2 m}{2\hbar}\omega^{1/4}\right]e^{-\frac{x^2 m\omega}{2\hbar}}$$

Snad pro $m = \hbar = 1$

$$\frac{\partial E_\lambda}{\partial \lambda} \stackrel{?}{=} \langle \psi_\lambda | \frac{dH_\lambda}{d\lambda} | \psi_\lambda \rangle$$ pokud ψ_λ je vl. stav H

pro $\lambda = \omega$ & LHO g.s.

$|0\rangle : \frac{1}{\sqrt{\alpha\sqrt{\pi}}} e^{-\frac{x^2}{2\alpha^2}}$; $E = \frac{1}{2}\hbar\omega$; $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$

$= \hbar\omega(a^+a + \frac{1}{2})$

$$\frac{\partial E_\omega}{\partial \omega} = \frac{1}{2}\hbar$$

$$\frac{dH_\omega}{d\omega} = \frac{d}{d\omega}\left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2\right)$$

$$= m\omega x^2$$

$$\langle 0 | m\omega x^2 | 0 \rangle = \langle 0 | m\omega \frac{\alpha^2}{2}(a^2 + a^{+2} + 2a^+a + 1) | 0 \rangle$$

MILE

$$= \langle 0 | m\omega \frac{\hbar}{m\omega} \frac{1}{2} \langle 0 | 1 | 0 \rangle$$

$\alpha = \sqrt{\frac{\hbar}{m\omega}}$

$$= \frac{\hbar}{2} \quad \text{OK}$$

pro $|1\rangle$? $\frac{\partial E_\omega}{\partial \omega} = \frac{3}{2}\hbar$ $a^2 + a^{+2}$ nepřispívají

$$\langle 1 | m\omega x^2 | 1 \rangle = \frac{\hbar}{2} \langle 1 | 2a^+a + 1 | 1 \rangle$$

$$= \frac{3\hbar}{2}$$

pro $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$?

$E = ?$ $\hbar\omega\langle\psi| a^+a + \frac{1}{2} |\psi\rangle =$

$$= \frac{1}{2}\left[\langle 0| + \langle 1|\right](a^+a + \tfrac{1}{2})\left[|0\rangle + |1\rangle\right]\hbar\omega$$

jen diag.

$$= \frac{1}{2}\left[\langle 0| a^+a + \tfrac{1}{2} |0\rangle + \langle 1| a^+a + \tfrac{1}{2} |1\rangle\right]\hbar\omega$$

$$= \frac{1}{2}\left[\tfrac{1}{2} + (1 + \tfrac{1}{2})\right]\hbar\omega = \hbar\omega \qquad \frac{\partial E}{\partial \omega} = \hbar$$

$$\langle \psi | m\omega x^2 | \psi \rangle = \cancel{\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)}$$

$$= \frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)\left[m\omega x^2\right]\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$= \frac{1}{2}\frac{\hbar}{2}(\langle 0| + \langle 1|)(a^2 + a^{+2} + 2a^+a + 1)(|0\rangle + |1\rangle)$$

$$= \frac{\hbar}{4}(1 + 3) = \hbar$$ ~~dobře~~ dobře, symetrie?

$$|x_0\rangle = \frac{1}{\sqrt{\alpha\sqrt{\pi}}}\, e^{-\frac{(x-x_0)^2}{2\alpha^2}}$$

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 (x - x_0)^2$$

$$[x^2 - 2x_0 x + x_0^2]$$

$$\frac{dH}{dx_0} = -m\omega^2 x + m\omega^2 x_0 = m\omega^2 (x_0 - x) = m\omega^2\left[x_0 - \frac{\alpha}{\sqrt{2}}(a + a^+)\right]$$

$\left\langle \frac{dH}{dx_0} \right\rangle = 0$ pro $|0\rangle$

$$\psi = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$\left\langle \frac{dH}{dx_0} \right\rangle = \frac{1}{2}\left[\langle 0| + \langle 1|\right] m\omega^2\left[x_0 - \frac{\alpha}{\sqrt{2}}(a + a^+)\right]\left[|0\rangle + |1\rangle\right]$$

$$= \frac{1}{2}\left[m\omega^2 x_0 (1+1) - m\omega^2\frac{\alpha}{\sqrt{2}}(1+1)\right]$$

$$= m\omega^2 x_0 - \sqrt{2}\, m\omega^2 \alpha \qquad \neq 0$$

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pro $\frac{1}{\sqrt{2}}(|0\rangle + |2\rangle)$

$$E = \frac{1}{2}\left(\frac{1}{2}\hbar\omega + \frac{5}{2}\hbar\omega\right) = \frac{3}{2}\hbar\omega \quad \rightarrow \frac{\partial E}{\partial \omega} = \frac{3}{2}\hbar$$

$$\langle\psi| m\omega x^2 |\psi\rangle = \frac{1}{2}(\langle 0| + \langle 2|)\left[\frac{\hbar}{2}(a^2 + a^{+2} + 2a^+a + 1)\right](|0\rangle + |2\rangle)$$

$$= \frac{\hbar}{4}\Big[\langle 0|2a^+a + 1|0\rangle + \langle 2|2a^+a + 1|2\rangle$$

$a^+a|2\rangle = a^+\sqrt{2}|1\rangle = 2|2\rangle$

$$+ \langle 0|a^2|2\rangle + \langle 2|a^{+2}|0\rangle\Big]$$

$\hookrightarrow = \langle 0|a\sqrt{2}|1\rangle = \langle 0|\sqrt{2}|0\rangle = \sqrt{2}$ $\quad = \langle 2|a^+\sqrt{2}|1\rangle = \langle 2|\sqrt{2}|2\rangle = \sqrt{2}$

$$= \frac{\hbar}{4}\left[1 + \cancel{3}\,5 + 2\sqrt{2}\right] = \frac{3}{2}\hbar + \frac{\hbar}{\sqrt{2}}$$ konečně špatně !?

Mějme LHO částici ve stavu $|\psi\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |3\rangle)$

Ověřte $\frac{d\langle A\rangle}{dt} = \frac{i}{\hbar}\langle[H, A]\rangle + \left\langle\frac{\partial A}{\partial t}\right\rangle$

pro $\hat{A} = \hat{p}$

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

$$[H, p] = \left[\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2, p\right]$$

$$= \left[\frac{p^2}{2m}, p\right] + \frac{1}{2}m\omega^2[x^2, p]$$

$$= 0 + \frac{1}{2}m\omega^2\left[x^2 p - p x^2\right]$$, známe $[x,p] = i\hbar$

$$x^2 p - p x^2 = xxp - pxx = \underbrace{xxp - xpx}_{x[x,p]} + \underbrace{xpx - pxx}_{[x,p]x}$$

$$= xi\hbar + xi\hbar = 2i\hbar x$$

$$[H, p] = \frac{1}{2}m\omega^2\, 2i\hbar x = im\omega^2\hbar x$$

$$= im\omega^2\hbar\,\frac{\alpha}{\sqrt{2}}(a + a^+)$$

$$p = \frac{\sqrt{\hbar m\omega}}{\sqrt{2}\,i}(a - a^+)$$

$$\langle p\rangle = \frac{1}{3}\left(\langle 0|e^{\frac{i\omega t}{2}} + \langle 1|e^{\frac{i3\omega t}{2}} + \langle 3|e^{\frac{i7\omega t}{2}}\right)\frac{\sqrt{\hbar m\omega}}{\sqrt{2}\,i}(a - a^+)\left(|0\rangle e^{-\frac{i\omega t}{2}} + |1\rangle e^{-\frac{3i\omega t}{2}} + |3\rangle e^{-\frac{7i\omega t}{2}}\right)$$

$$= \frac{1}{3}\frac{\sqrt{\hbar m\omega}}{\sqrt{2}\,i}\Big[\underbrace{\langle 0|a|1\rangle}_{1} e^{-i\omega t} - \underbrace{\langle 1|a^+|0\rangle}_{1} e^{i\omega t}\Big]$$

$$= \frac{1}{3}\sqrt{\hbar m\omega}\sqrt{2}\,\frac{1}{2i}(-1)\left[e^{i\omega t} - e^{-i\omega t}\right] = -\frac{\sqrt{2}}{3}\sqrt{\hbar m\omega}\,\sin(\omega t)$$

$$\frac{\partial\langle A\rangle}{\partial t} = -\frac{\sqrt{2}}{3}\sqrt{\hbar m\omega}\,\omega\cos(\omega t)$$

viriál věta

$$2\langle T\rangle = \left\langle x\frac{dV}{dx}\right\rangle$$

Máme $\frac{d\langle A\rangle}{dt} = \frac{i}{\hbar}\langle [H,A]\rangle + \left\langle\frac{\partial A}{\partial t}\right\rangle$

použijme $A = xp$ $\quad = 0$

$$[H, xp] \overset{LHO}{=} \left[\frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2,\ xp\right]$$

$$= \left[\frac{p^2}{2m}, xp\right] + \frac{1}{2}m\omega^2[x^2, xp]$$

$$[p^2, xp] = ppxp - xppp = [ppx - xpp]p$$

$$= [ppx - pxp + pxp - xpp]p$$

$$= p[p,x]p + [p,x]p^2 \qquad [x,p] = i\hbar$$

$$= -i\hbar p^2 - i\hbar p^2 = -2i\hbar p^2$$

$$[x^2, xp] = x^2xp - xpxx = xxxp - xxpx + xxpx - xpxx$$

$$= xx[x,p] + x[x,p]x$$

$$= 2i\hbar x^2$$

$$\to \cancel{\frac{i}{\hbar}}[H, xp] = \frac{1}{2m}(-2i\hbar p^2) + \frac{1}{2}m\omega^2\, 2i\hbar x^2$$

$$= (-2i\hbar)\frac{p^2}{2m} + (2i\hbar)\frac{1}{2}m\omega^2x^2$$

$$\frac{i}{\hbar}[H, xp] = 2\frac{p^2}{2m} - m\omega^2x^2$$

$$\to \langle 2\hat{T}\rangle = \left\langle \hat{x}\frac{d\hat{V}}{dx}\right\rangle$$

$[aa^+ + 1]$

$$p = \frac{\sqrt{\hbar m\omega}}{\sqrt{2}\,i}(a - a^+)\ ;\ p^2 = \frac{\hbar m\omega}{2}(-1)\left[a^2 + a^{+2} - aa^+ - a^+a\right]$$

$$= -\frac{\hbar m\omega}{2}\left[a^2 + a^{+2} - 2a^+a - 1\right]$$

$$x = \sqrt{\frac{\hbar}{m\omega}}\frac{1}{\sqrt{2}}(a + a^+)\ ;\ x^2 = \frac{\hbar}{2m\omega}\left[a^{+2} + a^2 + 2a^+a + 1\right]$$

$$\langle 2T \rangle = \left\langle \frac{2p^2}{2m} \right\rangle = \left\langle \frac{p^2}{m} \right\rangle \quad \& \quad \langle m\omega^2 x^2 \rangle$$

pro $|\psi\rangle = |0\rangle$

$$\langle 0| \frac{1}{m}\left(-\frac{\hbar m\omega}{2}\right)\left(\underset{\to 0}{a^2} + \underset{\to 0}{a^{+2}} - \underset{\to 0}{2a^+a} - \underset{\to 1}{1}\right)|0\rangle = \frac{\hbar\omega}{2}$$

$$\langle 0| m\omega^2 \frac{\hbar}{2m\omega}\left(\underset{\to 0}{a^{+2}} + \underset{\to 0}{a^2} + \underset{\to 0}{2a^+a} + \underset{\to 1}{1}\right)|0\rangle = \frac{\hbar\omega}{2}$$

pro $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |2\rangle\right)$

$$\frac{1}{2}\left[\langle 0| + \langle 2|\right] \frac{1}{m}\left(-\frac{\hbar m\omega}{2}\right)\left(a^2 + a^{+2} - 2a^+a - 1\right)\left[|0\rangle + |2\rangle\right]$$

$$= -\frac{\hbar\omega}{4}\left[\underset{\to -1}{\langle 0|-1|0\rangle} + \underset{\to -5}{\langle 2|-2a^+a-1|2\rangle} + \underset{\sqrt{2}}{\langle 0|a^2|2\rangle} + \underset{\sqrt{2}}{\langle 2|a^{+2}|0\rangle}\right]$$

$$= \frac{\hbar\omega}{4}\left[6 - 2\sqrt{2}\right] = \frac{\hbar\omega}{2}\left[3 - \sqrt{2}\right]$$

$$\frac{1}{2}\left[\langle 0| + \langle 2|\right] m\omega^2 \frac{\hbar}{2m\omega}\left(a^{+2} + a^2 + 2a^+a + 1\right)\left[|0\rangle + |2\rangle\right]$$

$$= \frac{\hbar\omega}{4}\left[\underset{\to 1}{\langle 0|1|0\rangle} + \underset{\to 5}{\langle 2|2a^+a+1|2\rangle} + \underset{\to \sqrt{2}}{\langle 0|a^2|2\rangle} + \underset{\to \sqrt{2}}{\langle 2|a^{+2}|0\rangle}\right]$$

$$= \frac{\hbar\omega}{4}\left[6 + 2\sqrt{2}\right]$$

$$= \frac{\hbar\omega}{2}\left[3 + \sqrt{2}\right]$$

• LHO - "pásma, vdW, x^2 čas vývoj ?, a, a^+ na x-stavy, a, a^+ na p-stav, OC, co na basics? x^3 v prvním order
HF theorem

• Mějme $\psi = \frac{2}{\sqrt{3\alpha\sqrt{\pi}}} \frac{x^2}{\alpha^2} e^{-\frac{x^2}{2\alpha^2}}$, jaký je časový vývoj? pokud se nachází v potenciálu $\frac{1}{2} m\omega^2 x^2$

• Normalizace:

$\frac{3}{2B}\frac{1}{2B}\sqrt{\frac{\pi}{B}}$ $\quad B = \frac{1}{\alpha^2}$

$$\int_{-\infty}^{\infty} \frac{4}{3\alpha\sqrt{\pi}} \frac{x^4}{\alpha^4} e^{-\frac{x^2}{\alpha^2}} dx = \frac{4}{3\alpha\sqrt{\pi}\,\alpha^4} \cdot \frac{3\alpha^2}{2}\frac{\alpha^2}{2}\sqrt{\pi}\,\alpha = 1 \quad \text{OK}$$

• koef.

$$\langle 0 | \psi \rangle = \frac{1}{\sqrt{\alpha\sqrt{\pi}}} \frac{2}{\sqrt{3\alpha\sqrt{\pi}}} \int_{-\infty}^{\infty} \frac{x^2}{\alpha^2} e^{-\frac{x^2}{\alpha^2}} dx$$

$$|0\rangle = \frac{1}{\sqrt{\alpha\sqrt{\pi}}} e^{-\frac{x^2}{2\alpha^2}}$$

$$= \frac{2}{\alpha\sqrt{\pi}\sqrt{3}\,\cancel{\alpha^2}} \sqrt{\pi}\,\alpha \frac{\alpha^2}{2} = \frac{1}{\sqrt{3}} = c_0$$

$c_2 = \sqrt{1 - c_0^2} = \sqrt{\frac{2}{3}}$ ... v principu $e^{i\delta}$ porovnáním s $|0\rangle$ zjistíme, že $\delta = 0$

$$\rightarrow |\psi\rangle = \frac{1}{\sqrt{3}} |0\rangle + \sqrt{\frac{2}{3}} |2\rangle$$

$$|\psi\rangle = \frac{1}{\sqrt{3}} |0\rangle e^{-\frac{i\omega t}{2}} + \sqrt{\frac{2}{3}} |2\rangle e^{-i\frac{5\omega t}{2}}$$

$$\rho = \psi^* \psi = \left[\frac{1}{\sqrt{3}} \frac{1}{\sqrt{\alpha\sqrt{\pi}}} e^{-\frac{x^2}{2\alpha^2}} e^{\frac{i\omega t}{2}} + \sqrt{\frac{2}{3}} \frac{1}{\sqrt{2\alpha\sqrt{\pi}}} \left[\frac{2x^2}{\alpha^2} - 1 \right] e^{-\frac{x^2}{2\alpha^2}} e^{\frac{i5\omega t}{2}} \right]$$

$$\left[\frac{1}{\sqrt{3}} \frac{1}{\sqrt{2\sqrt{\pi}}} e^{-\frac{x^2}{2\alpha^2}} e^{-\frac{i\omega t}{2}} + \sqrt{\frac{2}{3}} \frac{1}{\sqrt{2\alpha\sqrt{\pi}}} \left[\frac{2x^2}{\alpha^2} - 1 \right] e^{-\frac{x^2}{2\alpha^2}} e^{-\frac{i5\omega t}{2}} \right]$$

$$= \frac{1}{\alpha\sqrt{\pi}} \frac{1}{3} e^{-\frac{x^2}{\alpha^2}} + \frac{2}{3} \frac{1}{2\alpha\sqrt{\pi}} e^{-\frac{x^2}{\alpha^2}} \left(\frac{2x^2}{\alpha^2} - 1 \right)^2$$

$$+ \frac{\sqrt{2}}{3} \frac{1}{\alpha\sqrt{\pi}} \frac{1}{\sqrt{2}} e^{-\frac{x^2}{\alpha^2}} \underbrace{\left[e^{-2\omega t} + e^{2\omega t} \right]}_{2\cos(2\omega t)\ \sim\ \cos(\Delta E)} \left[\frac{2x^2}{\alpha^2} - 1 \right]$$