Mějme příprava

$$f_0 = \frac{1}{\sqrt[4]{\pi}} e^{-x^2/2}$$

$$f_1 = \sqrt{\frac{2}{\sqrt{\pi}}}\, x\, e^{-x^2/2}$$

$$f_2 = N_2 x^2 e^{-x^2/2}$$

- Nakreslete fce
- Skalární součin definujeme jako $\int_{-\infty}^{\infty} dx$.
- Tvoří ON bázi? (Bez počítání) ~~které jsou kolmé a proč?~~ jelen je sudý/lichý?

$N_2 = ?$ $\quad 1 = \int_{-\infty}^{\infty} |f_2(x)|^2 dx = N_2^2 \int_{-\infty}^{\infty} x^4 e^{-x^2} dx = \frac{3}{4}\sqrt{\pi}\, N_2^2$

$$\to N_2^2 = \frac{4}{3\sqrt{\pi}} \to N_2 = \frac{2}{\sqrt{3\sqrt{\pi}}}$$

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skalární součin
ON báze - G-S OG

$|0\rangle = f_0$

$|1\rangle = f_1$

$|2\rangle = f_2 - f_0 \langle f_0 | f_2 \rangle$

$$\langle f_0 | f_2 \rangle = \int_{-\infty}^{\infty} \frac{1}{\sqrt[4]{\pi}} \cdot \frac{2}{\sqrt{3}\sqrt[4]{\pi}} x^2 e^{-x^2} dx = \frac{2}{\sqrt{3}\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2} dx =$$

($\int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$)

$$= \frac{\cancel{2}}{\sqrt{3}\cancel{\sqrt{\pi}}} \frac{1}{\cancel{2}} \cancel{\sqrt{\pi}} = \frac{1}{\sqrt{3}}$$

$$|2\rangle = \frac{2}{\sqrt{3\sqrt{\pi}}} x^2 e^{-x^2/2} - \frac{1}{\sqrt[4]{\pi}} e^{-x^2/2} \cdot \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3\sqrt{\pi}}} (2x^2 - 1) e^{-x^2/2}$$

Nyní normalizovat!

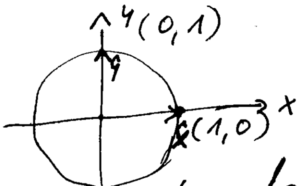
$$N^2 \int_{-\infty}^{\infty} \frac{1}{3\sqrt{\pi}} [4x^4 - 4x^2 + 1] e^{-x^2} dx = \frac{1}{3\sqrt{\pi}} \left[4 \cdot \frac{3}{4}\sqrt{\pi} - 4\frac{\sqrt{\pi}}{2} + \sqrt{\pi}\right] N^2 =$$

($4x^4 \to \frac{3}{4}\sqrt{\pi}$, $x^2 \to \frac{\sqrt{\pi}}{2}$, $1 \to \sqrt{\pi}$)

$$= \frac{1}{3}[3 - 2 + 1] N^2 = \frac{2}{3} N^2 = 1$$

$$\to N = \sqrt{\frac{3}{2}}$$

$$|2\rangle = \frac{1}{\sqrt{2}\sqrt[4]{\pi}} (2x^2 - 1) e^{-x^2/2}$$

• (x,y) : 

← částice (bod) na kružnici

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IGNORUJTE I/2

$\hat{x}, \hat{y}$ – 2 normované stavy báze

stav $a = c_x \hat{x} + c_y \hat{y}$; $c_x^2 + c_y^2 = 1$

— · — · — · — · —

• Spin 1/2: Dva stavy (vlastní) $|\uparrow\rangle, |\downarrow\rangle$; obecný stav $a = \begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix}$

• měření podél libovolné osy, ale preferujeme z

→ operátor je 2×2 matice, neboť působí na vektor 2D.

— · — · — · —

• Mějme matice

$$A = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad B = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad C = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

· jaké jsou vl. vektory?

$$C: \begin{vmatrix} 1/2 - \lambda & 0 \\ 0 & -1/2 - \lambda \end{vmatrix} = 0 = (1/2 - \lambda)(-1/2 - \lambda)$$

$$\underset{\uparrow}{\lambda = 1/2} \qquad \underset{\downarrow}{\lambda = -1/2}$$

$$\lambda = 1/2: \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow a = 1,\ b = 0 \rightarrow c_{\lambda+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda = -1/2: \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow a = 0,\ b = 1 \rightarrow c_{\lambda-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$A: \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} - 1\lambda \Rightarrow \begin{vmatrix} -\lambda & 1/2 \\ 1/2 & -\lambda \end{vmatrix} = 0$$

$$A(a_+) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \frac{1}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda_+(a_+)$$

$$\Rightarrow \lambda^2 - 1/4 = 0 \rightarrow$$

$$(\lambda + 1/2)(\lambda - 1/2) = 0$$

$$\lambda = -1/2: \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\rightarrow A_{\lambda-} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} ✓, \text{ normalizace } A_{\lambda-} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda = 1/2: \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \ ; \Rightarrow a_{\lambda+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

při výpočtu vl. vektorů B je třeba myslet na komplex. sdružení

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I - 2a

Matice $A = 1/2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad B = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$C = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$A^2 : 1/4 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 1/4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$B^2 : 1/4 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 1/4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$C^2 = 1/4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, neboť diagonální $\left[f(M) = \begin{pmatrix} f(\varepsilon_1) & 0 \\ 0 & f(\varepsilon_2) & \\ & & \ddots \end{pmatrix} \right]$

$|c_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad |b_+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}$

$|b_-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$

$\langle b_+ | c_+ \rangle = \frac{1}{\sqrt{2}} (i \quad 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{i}{\sqrt{2}} = k_+$

$\langle b_- | c_+ \rangle = \frac{1}{\sqrt{2}} (-i \quad 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{-i}{\sqrt{2}} = k_-$

$|k_+|^2 = \frac{i}{\sqrt{2}} \frac{-i}{\sqrt{2}} = \frac{1}{2}$

$|k_-|^2 = 1/2$

$\Big\}$ p.p. naměření je $1/2$ v obou případech

$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} + 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} + 1 - \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{2}} + 1 \\ 1 \end{pmatrix} = \sqrt{2} \begin{pmatrix} 1 + \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ ok

$$O = \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}$$

vl. vektory a čísla:

$$\text{Det}\begin{pmatrix} -\lambda & -i/2 \\ i/2 & -\lambda \end{pmatrix} = \lambda^2 - 1/4 = (\lambda - 1/2)(\lambda + 1/2)$$

$$\lambda_+ = 1/2 \qquad \lambda_- = -1/2$$

vektor:
pro λ_+ : $\begin{pmatrix} -1/2 & -i/2 \\ i/2 & -1/2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow \cancel{1/2a} \; -\frac{a}{2} - \frac{ib}{2} = 0$

$$a + ib = 0$$

$$\Rightarrow a = -ib$$

$$\Rightarrow \begin{pmatrix} -ib \\ b \end{pmatrix}$$

Normalizace

$$(ib \;\; b)\begin{pmatrix} -ib \\ b \end{pmatrix} = b^2 + b^2 = 2b^2 = 1$$

$$\Rightarrow b = \frac{1}{\sqrt{2}}$$

$$\rightarrow |b_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

podobně $|b_-\rangle$: $\begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow \frac{a}{2} - \frac{ib}{2} = 0$

$$a = ib \quad \Rightarrow \begin{pmatrix} ib \\ b \end{pmatrix}$$

po normalizaci: $|b_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$

-- vl. vektory:

- kolmé, ON báze: $\langle a_+|a_+\rangle = \frac{1}{2}(1 \;\; -1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1$

$$\langle a_+|a_-\rangle = \frac{1}{2}(1 \;\; -1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

$$\langle b_+|b_-\rangle = \frac{1}{2}(i \;\; 1)\begin{pmatrix} i \\ 1 \end{pmatrix} = 0$$

- platí $\hat{O} = \sum_i \lambda_i \, |i\rangle\langle i|$ ← vl. vektor
  ← vl. číslo
  ↑ operátor (matice)

např $C = \sum_i \lambda_i \, |i\rangle\langle i| = \frac{1}{2}|c+\rangle\langle c+| + \frac{1}{2}|c-\rangle\langle c-|$

$$= \frac{1}{2}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}(1 \;\; 0) - \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0 \;\; 1)\right]$$

$$= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{O} = \sum_i \lambda_i |i\rangle\langle i|$$

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pro $\hat{A} = 1/2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $|a+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $|a-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_{a+} = 1/2$ $\qquad$ $\lambda_{a-} = -1/2$

$$\hat{A} = \frac{1}{2}\left[\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}(1 \;\; 1) - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}(1 \;\; -1)\right]$$

$$= 1/4\left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right] = 1/4\begin{pmatrix} 0 & +2 \\ 2 & 0 \end{pmatrix} = 1/2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ OK

Také platí $O_{diag} = U^{+} O\, U$ ← matice z vl. vektorů

↑ transp. + c.c. $\quad$ ↑ originál matice

– rotace báze do báze vl. vektorů $\quad$ → další

– matice A, B, C jsou vyjádřeny v bázi vl. vektorů matice C, tj. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ a $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

· ~~Můžeme tázin~~

· Matice A a B ~~jsou~~ je možné získat jako

$$A = \begin{pmatrix} \langle c+|\hat{A}|c+\rangle & \langle c_+|\hat{A}|c_-\rangle \\ \langle c_-|\hat{A}|c_+\rangle & \langle c_-|\hat{A}|c_-\rangle \end{pmatrix}$$

$$\langle c+|\hat{A}|c+\rangle = \langle c+|\left[\sum_i \lambda_i |a_i\rangle\langle a_i|\right]|c_+\rangle =$$

$$= \langle c_+|\left[1/2\, |a+\rangle\langle a+| - 1/2\, |a-\rangle\langle a-|\right]|c_+\rangle$$

$$= 1/2\, \langle c_+|a_+\rangle\langle a_+|c_+\rangle - 1/2\, \langle c_+|a_-\rangle\langle a_-|c_+\rangle$$

$$= 1/2 \cdot \tfrac{1}{\sqrt{2}}(1 \;\; 0)\begin{pmatrix} 1 \\ 1 \end{pmatrix}\tfrac{1}{\sqrt{2}}(1 \;\; 1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1/2 \; 1/2\, (1 \;\; 0)\begin{pmatrix} 1 \\ -1 \end{pmatrix}(1 \;\; -1)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= 1/4\, (1) \cdot 1 - 1/4 \cdot 1 \cdot 1 = 0$$

⋮

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$$O_{diag} = U^{+} O U$$

$$\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \rightarrow U^{+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$A_{diag} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} =$$

$$= \tfrac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \tfrac{1}{4}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

funguje to!

A, B, C odpovídají měření spinu podle os x, y, z, resp. proto mají všechny vl. stavy $\pm 1/2$ [s $\hbar = 1$]

- měření do libovolné osy $\vec{n}$: $S_{\vec{n}} = \vec{n} \cdot (A, B, C)$

napr. $\vec{n} = \frac{1}{\sqrt{2}}(1, 0, 1)$ $\qquad \vec{n}\cdot\vec{v} = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}}+1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$$S_{\vec{n}} = \frac{1}{\sqrt{2}}(A + C) = \frac{1}{\sqrt{2}}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

vl. hodnoty: $\cancel{\frac{1}{2\sqrt{2}}\begin{pmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix} \Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} =}$

$$\cancel{= (1-\lambda)(-1-\lambda) - 1 = -(1-\lambda^2) -}$$

vl. hodnoty
$$\begin{vmatrix} \frac{1}{2\sqrt{2}} - \lambda & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} - \lambda \end{vmatrix} = -\left(\frac{1}{2\sqrt{2}} + \lambda\right)\left(\frac{1}{2\sqrt{2}} - \lambda\right) - \left(\frac{1}{2\sqrt{2}}\right)^2$$

$$= -\frac{1}{8} + \lambda^2 - \frac{1}{8} = -\tfrac{1}{4} + \lambda^2 = -(\lambda + 1/2)(\lambda - 1/2)$$

$$\lambda = -1/2 \qquad \lambda = 1/2$$

OK

vl. vektory: $|n+\rangle$: $\begin{pmatrix} \frac{1}{2\sqrt{2}} - \frac{1}{2} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} - \frac{1}{2} \end{pmatrix} \Rightarrow \tfrac{1}{2}\begin{pmatrix} \frac{1}{\sqrt{2}} - 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} - 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$

$$\left(\frac{1}{\sqrt{2}} - 1\right)a + \frac{b}{\sqrt{2}} = 0 \Rightarrow b = -(\sqrt{2} - 1)a$$

$$\Rightarrow N\begin{pmatrix} a \\ (\sqrt{2}-1)a \end{pmatrix} \rightarrow \text{normal}: N^2(a^2 + (\sqrt{2}-1)^2 a^2) = a^2(1 + 2 + 1 - 2\sqrt{2})N^2 =$$

$$= a^2(4 - 2\sqrt{2})N^2 = 1$$